2000 Paper 7 Question 11

Types

The terms of the untyped lambda calculus, $M ::= x \mid \lambda x(M) \mid M\,M$, are to be assigned types of the form $\tau ::= \alpha \mid \tau \to \tau$, where α ranges over an infinite set of type variables. Give an inductive definition of a typing judgement of the form $A, \Delta \vdash M : \tau$, where Δ is a finite function from variables to types whose domain of definition contains the free variables of M, and where A is a finite set of type variables containing the type variables occurring in τ and Δ. [3 marks]

Write $\mathrm{Typ}(A)$ for the set of types involving only type variables in the set A. Let A, A', A'' be finite sets of type variables; S be a function from A to $\mathrm{Typ}(A')$ and T a function from A' to $\mathrm{Typ}(A'')$; τ_1, τ_2 be types in $\mathrm{Typ}(A)$; and τ' be a type in $\mathrm{Typ}(A')$. Give definitions of the following concepts:

(a) The type $S(\tau_1)$ resulting from simultaneously substituting the type $S(\alpha)$ for occurrences of α in τ_1, as α ranges over A. [2 marks]

(b) The composition $TS : A \to \mathrm{Typ}(A'')$ of the type substitutions S and T. [2 marks]

(c) S *unifies* τ_1 and τ_2. [2 marks]

(d) S is the *most general unifier* of τ_1 and τ_2. [2 marks]

(e) (S, τ') is a *typing* for a partial typing judgement $A, \Delta \vdash M : ?$. [2 marks]

(f) (S, τ') is a *principal typing* for a partial typing judgement $A, \Delta \vdash M : ?$. [2 marks]

Give examples, with proof, of closed lambda terms M_1 and M_2 for which $\emptyset, \emptyset \vdash M_1 : ?$ has a typing and $\emptyset, \emptyset \vdash M_2 : ?$ does not. [4 marks]

If a partial typing judgement has a typing, does it necessarily have a principal one? [1 mark]