2000 Paper 10 Question 10

Mathematics for Computation Theory

Let A be a set, R be a relation on A. What conditions must be satisfied for the following?

(i) R is a partial order on A [3 marks]

(ii) R is a total order on A [1 mark]

(iii) R is a well-founded relation on A [2 marks]

$x \in A$ is a *minimal* element for R if $y \in A, (y, x) \in R \Rightarrow y = x$.

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For each of the sets $A = \mathbb{N}$ (natural numbers) and $A = \mathbb{Z}$ (integers) we define relations:

(a) $R_1 = \leqslant$, the standard ordering

(b) $(a, b) \in R_2$ if and only if $\exists\, q \in A$ such that $aq = b$

(c) $(a, b) \in R_3$ if and only if $\exists\, p \in A$ such that $ap = b$, where $\mid p \mid \in \mathbb{N}$ is a prime

Explain with reasons which of conditions (i)–(iii) is satisfied when a relation R_j is defined on either $\mathbb{N}$ or $\mathbb{Z}$. Identify the maximal and minimal elements in each case. [14 marks]