1999 Paper 4 Question 8

Continuous Mathematics

Suppose we need to solve a linear second-order differential equation with constant coefficients A and B in which the combined derivatives of the solution we seek, $f(x)$, are only known to be related to another function, $g(x)$:

$$A\frac{d^2f(x)}{dx^2} + B\frac{df(x)}{dx} = g(x)$$

We know the function $g(x)$, and we can compute its Fourier Transform $G(\mu)$.

How can we use the properties of the Fourier Transform and its inverse in order to compute the solution $f(x)$ of this differential equation? Provide an expression for $F(\mu)$, the Fourier Transform of $f(x)$, in terms of $G(\mu)$, frequency variable μ, and the coefficients in the differential equation. [8 marks]

What final step is now required in order to compute the actual solution $f(x)$ of the differential equation, given your expression for $F(\mu)$? [2 marks]

In numerical computing, differential operators must always be represented by finite differences. Assume that a function has been sampled at uniform, closely spaced, intervals. How many consecutive sample points are necessary in order to compute the N^{th} derivative of the function at some point? [2 marks]

To compute the third derivative in a local region of a function, what set of weights would you use to multiply consecutive samples of the function? [3 marks]

What is the principal computational advantage of using orthogonal functions, over non-orthogonal ones, when representing a set of data as a linear combination of a universal set of basis functions? [2 marks]

If $\Psi_k(x)$ belongs to a set of orthonormal basis functions, and $f(x)$ is a function or a set of data that we wish to represent in terms of these basis functions, what is the basic computational operation we need to perform involving $\Psi_k(x)$ and $f(x)$? [3 marks]