1998 Paper 2 Question 5

Probability

The Gambler's Ruin Problem presupposes a game for two players, A and B, each of whom has a pile of £1 coins. The game proceeds by a sequence of turns and at each turn A wins with probability p and B wins with probability q (and $p+q=1$). When the outcome of a turn is known, £1 is transferred from the loser's pile to the winner's pile. Play continues until one player is ruined by having no money left.

At a particular stage in the game A has £n and B has £$(a-n)$. If u_n is the probability that A ultimately wins from this position the following difference equation holds:

$$u_n = p\,u_{n+1} + q\,u_{n-1} \qquad \text{provided } 0 < n < a$$

The right-hand side makes use of the multiplication theorem twice and the addition rule once. Explain why such usages are justified. [3 marks]

Provide suitable boundary conditions to reflect the two possible end-of-game outcomes. [2 marks]

Suppose that d_n represents the duration of play, the expected number of turns from the position described until the end of the game. Write down the related difference equation for d_n and provide suitable boundary conditions. [4 marks]

Solve the revised difference equation for the fair case where $p = q = \frac{1}{2}$. [8 marks]

Ten children sit in a circle to play pass the parcel. At each turn, the child who holds the parcel passes it one child to the left or right with equal probability. At the start of the game, at turn zero, a particular child (the leader) is handed the parcel by an outsider. At what turn number may the leader expect to receive the parcel again? The term "expect" is used in the probabilistic sense of expectation. [3 marks]