1997 Paper 1 Question 8

Discrete Mathematics

Let A be a set and R a relation on A; also write R^k for the usual k-fold composition of R, i.e. $R^1 = R, R^{k+1} = R \circ R^k$. Let $t(R)$ be the smallest relation which is transitive and has $R \subseteq t(R)$, similarly let $u(R) = \bigcup_{k=1}^{\infty} R^k$.

(a) Show (e.g. by induction on k) that $(\forall k \geqslant 1) R^k \subseteq t(R)$;

(b) deduce $u(R) \subseteq t(R)$.

(c) Show further that $u(R)$ is transitive;

(d) and hence argue that $u(R) = t(R)$.

[8 marks]

Now suppose A has n elements; give a sketch of an algorithm which takes as input an $n \times n$ boolean array V representing R above and produces as output a similar boolean array W representing $t(R)$ above. Give the running time of your algorithm in the form $O(f(n))$. [8 marks]

Find a set A and a relation R on A which is not reflexive, but where $t(R)$ is reflexive. [4 marks]