1996 Paper 1 Question 8

Discrete Mathematics

The fiercely logical inhabitants of planet Volcano use coins whose values are all powers of two, the smallest being the one-pfatz (1pf) coin. Designs for all non-negative integer powers of two exist, but in practice computers are used instead of physically minting very high value coins.

For entry to the moon-fleet academy a popular question is to enquire as to the number of different ways of representing sums of money (ignoring order of course) of the form $n$pf. Show that the number of ways $w(n, k)$ of representing $n$pf using coins of value up to 2^kpf satisfies the recurrence

$$w(n, k) = w(n - 2^k, k) + w(n, k - 1)$$

and add appropriate base case(s). [5 marks]

Show, assuming n is a multiple of 4, that

$$w(n, 2) = (n/4 + 1)^2.$$

[5 marks]

Show also, if n is a multiple of 2^k with $k > 0$, that

$$w(n, k) \leqslant \frac{n}{2^k} w(n, k - 1) + 1.$$

[5 marks]

Using the above or otherwise show further that

$$2^k \leqslant w(2^k, k) \leqslant 2^{k^2}.$$

[Hint: for the lower bound you might well consider how one can derive one way of representing 2^{k+1} from each way of representing 2^k but only by using coins of even value and use induction.] [5 marks]