1994 Paper 3 Question 9

Numerical Analysis I

The mid-point rule can be expressed in the form

$$I_n = \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} f(x)\,dx = f(n) + e_n$$

where

$$e_n = f''(\theta_n)/24$$

for some θ_n in the interval $(n - \frac{1}{2}, n + \frac{1}{2})$. Assuming that a formula for $\int f(x)\,dx$ is known, and using the notation

$$S_{p,q} = \sum_{n=p}^{q} f(n),$$

describe a method for estimating the sum of a slowly convergent series $S_{1,\infty}$, by summing only the first N terms and estimating the remainder by integration.
[7 marks]

Assuming that $f''(x)$ is a positive decreasing function, derive an estimate of the error $|E_N|$ in the method. [5 marks]

Given

$$\int \frac{dx}{1+x^2} = \tan^{-1} x,$$

apply the method to

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2}.$$

What is the integral remainder to be added to $S_{1,N}$? [4 marks]

To the nearest power of 10, how large should N be to achieve an absolute error of approximately 10^{-16}? [4 marks]