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Bigraphs whose names have multiple locality

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Bigraphs whose names have multiple locality

Robin Milner[1]

Abstract The previous definition of binding bigraphs is generalised so that local names may be located in more than one region, allowing more succinct and flexible presentation of bigraphical reactive systems. This report defines the generalisation, verifies that it retains relative pushouts, and introduces a new notion of bigraph *extension*; this admits a wider class of parametric reaction rules. Extension is shown to be well-behaved algebraically; one consequence is that —as in the original definition of bigraphs— discrete parameters are sufficient to generate all reactions.

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1 Binding bigraphs

We generalise the *binding bigraphs* defined in Section 11 of Jensen and Milner [1], by allowing the names in an interface to be local to more than one site. We assume familiarity with Sections 1–10 of [1], but not with Section 11. We also refer the reader to the bibliography of that paper.

Definition 1.1 (binding signature) A *binding signature* $\mathcal{K}$ is like a pure one except that the arity of a control $K\colon h \to k$ now consists of a pair of finite ordinals, the *binding arity* h and the *free arity* k, indexing respectively the *binding* and the *free* ports of any K-node. If K is atomic then $h = 0$.

We denote by $\mathcal{K}^{\mathsf{u}}$ the pure signature underlying $\mathcal{K}$; for each $K\colon h \to k$ in $\mathcal{K}$ it assigns a single arity $h + k$, forgetting the binding property of ports. ∎

We wish to define a binding bigraph G in terms of an underlying pure one, in which all points linked to a binding port of a node v lie inside v, i.e. the scope of the binding is (the interior of) v. Some of these points may be inner names, and we need to ensure that they transmit this scope discipline to another bigraph F in the composition GF. For this purpose we enrich interfaces as follows:

Definition 1.2 (binding interface) A *binding interface* takes the form $I = \langle m, loc, X\rangle$, where the *width* m and *name set* X are as in pure bigraphs, and $loc \subseteq m \times X$ is called the *locality* of I. If $(i, x) \in loc$ we say that i is a *place* of x (in I). We call $I^{\mathsf{u}} = \langle m, X\rangle$ the pure interface *underlying* I. ∎

We often write loc_I and nms_I for the locality and the names of I.

Definition 1.3 (binding bigraph) If I and J are binding interfaces, a *(concrete) binding bigraph* $G\colon I \to J$ consists of an *underlying* pure bigraph $G^{\mathsf{u}}\colon I^{\mathsf{u}} \to J^{\mathsf{u}}$ satisfying certain locality conditions. To state these conditions we first define a relation loc_G, assigning places to points and links of G^{u}, as the smallest relation such that:

POINTS
- if $(i, x) \in loc_I$ then $(i, x) \in loc_G$
- if p is a binding port of node v then $(v, p) \in loc_G$
- if p is a free port of node v then $(prnt_G(v), p) \in loc_G$.

LINKS
- if $(j, y) \in loc_J$ then $(j, y) \in loc_G$
- if an edge e contains a binding port of v then $(v, e) \in loc_G$.

We then impose two rules on G. In the scoping rule (illustrated in Figure 1) q and ℓ range over points and links respectively, and w, w' over places:

BINDING: A link has at most one binding port; an open link has none.

SCOPING: If $link_G(q) = \ell$ is a local link then q is also local, and whenever $(w, q) \in loc_G$ then there exists w' such that $w \leq_G w'$ and $(w', \ell) \in loc_G$. ∎

We shall often call I or J the inner or outer *face* of $G\colon I \to J$. Note that we are using the word ‘place’ in two ways. Its simple meaning is a root, node or site of a bigraph;

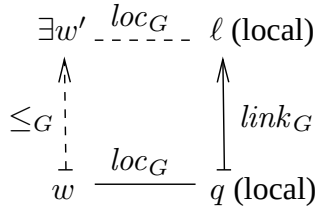

Figure 1: The scoping rule for a point q in a local link ℓ

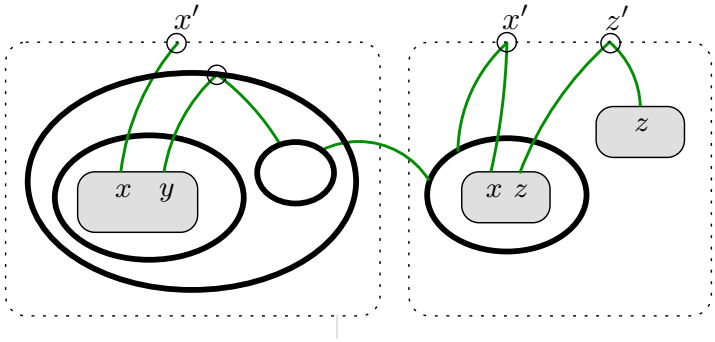

Figure 2: A binding bigraph $G\colon (\{xy\},\{xz\},\{z\}) \to (\{x'\},\{x'z'\})$

but we also talk of the place *of* a point or link, meaning a place assigned to a point or link by the interface or bigraph under consideration.

Consider what the scoping rule implies for a point q in a local link ℓ. If q is a port then it has a unique place; the rule insists that this lies inside a place of ℓ. If q is an inner name the rule is stronger; it insists not only that q is local, but that *each* of its places lies inside a place of ℓ. In either case, if ℓ is an edge its place is unique, but if it is an outer name it may have many places.

Figure 2 illustrates a binding bigraph (controls not shown). A local link is indicated by a small circle at each of its places. Note that the link x' has one point in the first region and two in the second. Note also that there are two closed links; one (containing the inner name y) is local, and the other is global and straddles two regions.

As for pure bigraphs, we can quotient concrete binding bigraphs to yield abstract ones. For now we consider only the concrete ones, and omit the word 'concrete'.

It is easy to check that both composition and tensor product preserve the binding and the scoping rules. This justifies the following:

Definition 1.4 (s-category of binding bigraphs) Given a binding signature $\mathcal{K}$, the wide monoidal s-category $\acute{}\textsc{Bbg}(\mathcal{K})$ of binding bigraphs is defined by taking composition and tensor product of the underlying pure bigraphs. Width and origin are as for pure bigraphs. The forgetful functor from binding to pure bigraphs is denoted by

$$\mathcal{U}\colon \acute{}\textsc{Bbg}(\mathcal{K}) \to \acute{}\textsc{Big}(\mathcal{K}^{\mathsf{u}})\ . \qquad \blacksquare$$

When $\mathcal{K}$ is understood we shall write ´BBG for ´BBG($\mathcal{K}$).

Binding bigraphs naturally inherit the properties *barren, sibling, active, passive, hard* from place graphs, and *idle, open, closed, peer, lean* from link graphs. In addition we say that a link is *bound* (by p) if it contains a binding port p, otherwise free. Note that every open link is free. We shall call a bigraph *prime* iff it has unit width.

Binding bigraphs also inherit other notions:

Proposition 1.5 (isos, epis, monos) *In* ´BBG *a bigraph is* iso *(resp.* epi, mono*) iff its underlying pure bigraph is iso (resp. epi, mono).*

A striking difference from pure bigraphs arises when we consider wirings. In the terminology of [1], Chapter 9, a wiring is described as a bigraph with interfaces of zero width (and hence a trivial place graph). But in the present context we need to 'wire up' local names as well as global ones; we are therefore forced to involve places in our wirings. So we shall adopt the following:

Definition 1.6 (wiring) A *wiring* is a binding bigraph without nodes. It is a *placing* if its link graph is an identity; it is a *linking* if its place graph is an identity. ■

We shall use ω and ζ to range over wirings, π over placings and λ over linkings respectively. In binding bigraphs a wiring cannot be expressed simply as a combination $\langle \pi, \lambda \rangle$ of a place graph with a link graph, since the interfaces have localities and the scoping rule must be respected. Thus placings and linkings, as defined above, will involve all three elements: places, links and locality. For example, in a placing $\langle 2, loc, X \rangle \rightarrow \langle 2, loc', X \rangle$ which swaps its two sites, the localities loc and loc' must normally differ in order to respect the scoping rule.

Exercise Show that $\omega \colon I \rightarrow J$ is both a placing and a linking iff its place map and link map are identities and $loc_I \subseteq loc_J$.

We shall postpone further taxonomy of wirings to the following section, where we explore it just for *local* wirings, i.e. those where every name is local.

A similar difference from pure bigraphs arises when we consider ions. In pure bigraphs, if $K \colon k$ is a non-atomic control with arity k, then for each k-vector $\vec{y}$ of names there is an *ion* $K_{\vec{y}} \colon \langle 1, Y \rangle$, where $Y = \{\vec{y}\}$, consisting of a K-node whose ports are named $\vec{y}$ in order. (The identity of the node is unspecified.) Within the ion we may place any ground bigraph $G \colon \epsilon \rightarrow \langle 1, Z \rangle$, where $Z \cap Y = \emptyset$; this creates the *molecule* $(K_{\vec{y}} \otimes \mathsf{id}_Z) \circ G$, with outer face $\langle 1, Y \uplus Z \rangle$. So the construction $K_{\vec{y}} \otimes \mathsf{id}_Z$ yields a family of ions, indexed by new names Z to be exported from inside the ion.

The same construction does not work for local names Z. Instead, we define the corresponding family directly:

Definition 1.7 (ion, molecule, atom) Let $K \colon h \rightarrow k$ be a control, with $h = 0$ if K is atomic. Let $\vec{x}$ be an h-vector and $\vec{y}$ a k-vector, the members of $\vec{x}$ being distinct, and let $X = \{\vec{x}\}$ and $Y = \{\vec{y}\}$. Then for any Z disjoint from $X \cup Y$, the *ion*

$$K^Z_{\vec{x},\vec{y}} \colon (X \uplus Z) \rightarrow (Y \uplus Z)$$

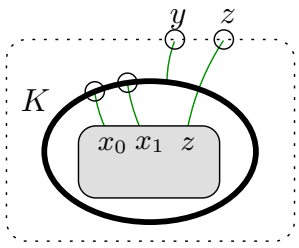

Figure 3: The ion $K^z_{x_0x_1,y}$ for a control $K\colon 2\to 1$

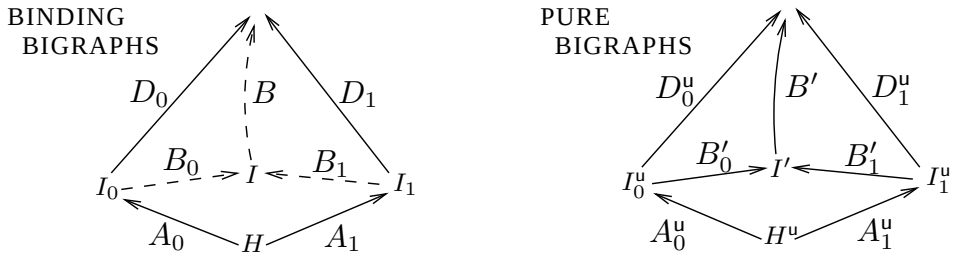

Figure 4: Constructing an RPO in binding bigraphs

has a single K-node, which is parent of its single site. It links the binding and free ports of the node respectively to the inner and outer names $\vec{x}$ and $\vec{y}$, in order, and its link map also includes the identity function on Z. Figure 3 shows an ion for $K\colon 2\to 1$.

A *molecule* then takes the form $(K^Z_{\vec{x},\vec{y}}\otimes \mathsf{id}_W)\circ G$ where W are the global outer names of G; in a *local* molecule $W=\emptyset$. The case $Z=W=\emptyset$ and $G=\epsilon$ yields an *atom*; this must be the case if K is atomic. ∎

We now have enough elements to construct all binding bigraphs:

Proposition 1.8 (enough elements) *Every binding bigraph can be constructed from ions and wirings using composition and tensor product.*

We turn now to the existence of relative pushouts (RPOs) in binding bigraphs. The way we construct them is, roughly, to pull the construction for pure bigraphs back along the forgetful functor $\mathcal{U}$. This is what was done for the more limited notion of binding bigraph in Section 11 of [1].

Construction 1.9 (building a binding RPO) Let the pair (A_0,A_1) have a bound (D_0,D_1) in ´BBG, where $A_i\colon H\to I_i$ and $D_i\colon I_i\to K$ $(i=0,1)$. We wish to build a binding RPO

$$(B_0,B_1,B)$$

for (A_0,A_1) relative to (D_0,D_1), as shown in the left-hand diagram of Figure 4. We start by building an RPO (B'_0,B'_1,B') for $(A^{\mathsf{u}}_0,A^{\mathsf{u}}_1)$ to $(D^{\mathsf{u}}_0,D^{\mathsf{u}}_1)$ in pure bigraphs,

following [1]; this is shown in the right-hand diagram, with mediating interface I'. Since a binding bigraph between known interfaces is determined by its underlying pure bigraph, our construction amounts to finding a binding interface I such that $I^{\mathsf{u}} = I'$, and then defining B_i and B so that $B_i^{\mathsf{u}} = B_i'$ and $B^{\mathsf{u}} = B'$. Of course, we must then check that the binding and scoping rules are satisfied.

Let $I_i = \langle m_i, loc_i, X_i \rangle$ $(i = 0, 1)$, and $I' = \langle m, X \rangle$. We proceed to construct $I = \langle m, loc, X \rangle$ with $I^{\mathsf{u}} = I'$; we need only define the locality relation loc. So for each $x \in X$ we must determine the places $r \in m$ for which $(r, x) \in loc$. Now x is linked to one or more names in I_0 or I_1 or both. Whenever x is linked to x_i in I_i, and $(s, x_i) \in loc_i$, let r be the unique place in I' such that $s \leq_{B_i'} r$; then declare that $(r, x) \in loc$. This concludes the definition of I, and our construction is complete. ∎

This construction is even easier than in the case where each name has at most one place [1]; for in that case we needed to verify this property when constructing I. However, we still have to check that our construction is valid in other ways:

Proposition 1.10 (binding RPOs) *A binding RPO for (A_0, A_1) to (D_0, D_1) is provided by Construction 1.9.*

Proof (outline) Our first task is to show that B_i and B, as constructed, obey the binding and scoping rules. Having done this, we have to show that, for any relative bound (C_0, C_1, C) for (A_0, A_1) to (D_0, D_1) with mediating interface J, there exists a unique mediator $E\colon I \to J$ with the required commutation properties.

Such a mediator has, as its underlying pure bigraph, the corresponding pure mediator between the pure RPO and the pure relative bound $(C_0^{\mathsf{u}}, C_1^{\mathsf{u}}, C^{\mathsf{u}})$; it can be found to obey the binding and scoping rules. Finally, the unicity of E follows from the unicity of E^{u} and the fact that the forgetful functor $\mathcal{U}$ is faithful.

There is no special difficulty in the details of these steps. ∎

2 Local bigraphs

From now on we confine our attention to a subclass of binding bigraphs. It seems that, once we allow a name to have many places, the *global* names (those with no place) are less necessary for modelling. We therefore define

Definition 2.1 (s-category of local bigraphs) A *local (binding) interface* is one in which every name is local. A *local (binding) bigraph* is one whose interfaces are local.

Given a binding signature $\mathcal{K}$, the wide monoidal s-category ´$\mathrm{LBG}(\mathcal{K})$ of local (binding) bigraphs is the full sub-s-category of ´$\mathrm{BBG}(\mathcal{K})$ whose interfaces are local. ∎

This definition is justified by the obvious fact that both composition and tensor product preserve the local property. Note that the scoping rule is slightly simpler for local bigraphs, because every point is local. Also every *open* link is local, but a *closed* link may still be non-local; indeed, the bigraph in Figure 2 is local but has a non-local link.

We now see that this smaller class of binding bigraphs still has RPOs:

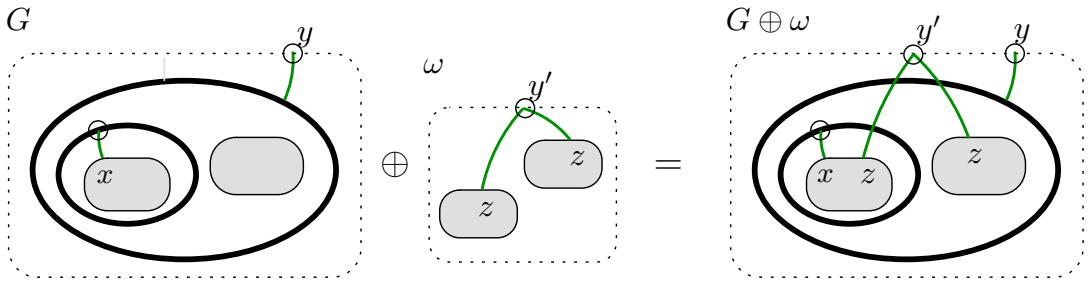

Figure 5: Extending a bigraph G with a wiring ω of equal width

Corollary 2.2 (local RPOs) *A local RPO for local (A_0, A_1) to local (D_0, D_1) is provided by Construction 1.9.*

Proof Given the proof of Proposition 1.10, we need only check that in this case the interface I in the RPO produced by the construction is local; this is immediate. ∎

Notation We shall often represent a local I by $\langle m, \vec{X} \rangle$, or even just $\vec{X}$, where the vector $\vec{X} = (X_0, \ldots, X_{m-1})$ specifies the names local to each $i \in m$. We shall call I a *partition* if the sets X_i are disjoint. In particular (X) is the interface of width 1 with local names X. This is in contrast with the interface X in pure bigraphs, which has width 0 and *global* names X.

As usual we write a ground bigraph in lower case, and write $a\colon I$ for $a\colon \epsilon \to I$. Also we shall write $G\colon \to J$ for $G\colon I \to J$ when we do not care about I. ∎

From now on we shall assume that every bigraph mentioned is local, unless otherwise stated, and we shall omit the adjective 'local'. We shall also write a singleton local interface (X) as X; there is no confusion, since we no longer admit a *global* interface called X.

We now proceed to a new operation which is essential for the treatment of wiring in local bigraphs, especially in handling parametric reaction rules. Recall the construction of ions in Definition 1.7; we defined a family of ions $K^Z_{\vec{x}\vec{y}}$, so that any bigraph with outer names $\vec{x}$ and Z can be inserted into an ion to form a K-molecule, exporting the bigraph's extra names Z at its outer face. To treat parametric reaction rules we need to generalise this construction from ions to an arbitrary bigraphs. Consider the bigraph $G\colon (x, \emptyset) \to y$ shown in Figure 5; the figure also shows a wiring ω of equal width, and shows the result of 'adding' this wiring to G without widening its interfaces. We denote this operation by $\oplus$, and call it *extension*.

In general, we want to be able to compose a context $G\colon I \to J$ with a parameter $a\colon I'$ whose outer face I' *extends* I, that is, it has the same width as I but possibly extra names, arbitrarily located. Then the composite should have an outer face J' similarly extending J. So we must define extension first of interfaces, then of bigraphs.

Definition 2.3 (extension) If two interfaces $I = \langle m, loc, X \rangle$ and $I' = \langle m, loc', X' \rangle$

have equal width they are *conformal*. If $X \cap X' = \emptyset$ then the *extension of I by I'* is

$$I \oplus I' \stackrel{\text{def}}{=} \langle m, loc \uplus loc', X \uplus X' \rangle \ .$$

If a bigraph $G\colon I \to J$ and a wiring $\omega\colon I' \to J'$ have equal width and disjoint supports[2] they are *conformal*. This implies that their inner faces I and I' are conformal, and so are their outer faces. Then if $I \oplus I'$ and $J \oplus J'$ are defined, the *extension of G by ω* is

$$G \oplus \omega\colon I \oplus I' \to J \oplus J' \ ;$$

it has the place graph of G, and the tensor product of the two link graphs. ∎

It is not possible to define the extension of G by an *arbitrary* bigraph —or even by one with equal width— since its place graph would would be ill-defined. We therefore admit $G \oplus F$ only when F is a wiring.

A useful form of wiring $\omega\colon m \to I$ has no inner names, and its link map is just the empty function $\emptyset_X$, where X are the names of I. We denote such a wiring just by I when its place map is determined by the context; for example the *idle* extension $G \oplus I$ simply extends the outer face of G with idle names X located by I.

It is easy to check that extension preserves the scoping rule. It also behaves well with composition:

Proposition 2.4 (composing extensions) *The following hold, with appropriate interfaces, when both sides are defined:*

$$\begin{array}{rcl} \omega \oplus \omega' & = & \omega' \oplus \omega \\ (G \oplus \omega) \oplus \omega' & = & G \oplus (\omega \oplus \omega') \\ (F \oplus \omega) \circ (G \oplus \omega') & = & (F \circ G) \oplus (\omega \circ \omega') \ . \end{array}$$

With the help of interface extension we now define two important properties of a bigraph.

Definition 2.5 (minimal, discrete) Let $G\colon \to I \oplus H$ be any bigraph.

G is *minimal for I* if, for all $(i, x) \in H$, the name x is linked to some point of G located below i.

G is *discrete for I* of it is open, and for all $(i, x) \in H$, the name x is linked to exactly one point of G located below i.

When $I = \epsilon$ we omit 'for I'. ∎

Thus discreteness implies minimality; it also generalises the notion of discreteness defined for binding bigraphs in [1].

We now define some linkings which, when decomposed from the outer face of a bigraph, reduce it to a bigraph that is open, minimal or discrete.

Definition 2.6 (closure, substitution) Define the following linkings:

A *closure* $/Z\colon I \to I'$ is the identity on I except that it maps every name $z \in Z \cap nms_I$ to an edge, and this name is omitted in forming I' from I. A closure may also contain idle edges.

A *substitution* $\sigma\colon \langle m, loc, X \rangle \to \langle m, loc', X' \rangle$ has a surjective link map; in forming loc' from loc, every name x is replaced by its image under the link map. ∎

[2] Since ω has no nodes, this merely excludes the possibility that G and ω share an edge.

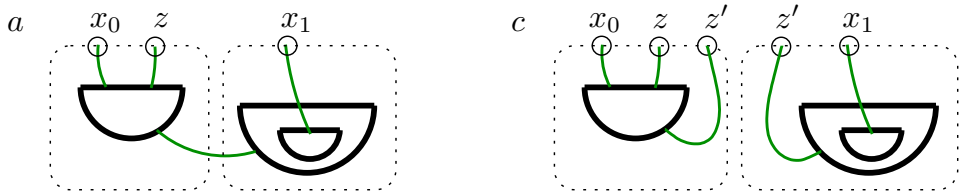

Figure 6: A non-open bigraph a as a closure $a = /z' \circ c$, with c open

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Recall that a *free* link is one without a binding port, that an *open* link is an outer name, and that a bigraph G is *open* iff every free link is open. For parametric reactions that replicate parts of their parameters, we need to express every bigraph in terms of an open one, in order to define replication unambiguously.

For example, the ground bigraph a of Figure 6 is not open because it has a closed link joining two free ports. (It is immaterial whether these ports lie in the same region, or –as here– in different regions.) However, a can easily be represented as a closure of an open bigraph c, as shown. In fact, we claim that in general:

Proposition 2.7 (open decomposition) *A bigraph $G\colon \to I$ may be expressed uniquely up to isomorphism as $G = /Z \circ F$, where $/Z\colon I \oplus H \to I$ is a closure and $F\colon \to I \oplus H$ is open and minimal for I.*

Proof Include in Z a distinct new name z for each free non-idle edge of G, and replace this edge by an open link z in F. Also include in $/Z$ all the idle edges of G. ∎

We now come to the most important property of open bigraphs:

Proposition 2.8 (open factorisation) *Any open ground bigraph $c\colon \vec{X}$ with outer width m can be uniquely factorised into primes, as*

$$c = c_0 \parallel \cdots \parallel c_{m-1}\ ,\ \textit{with}\ c_i\colon X_i\ .$$

Combining this with open decomposition, we can uniquely determine the prime parts of *any* ground bigraph; this allows us to define unambiguously its factorisation into prime parts. Let us begin with simple form of instantiation, in which all copies of the same prime part will share their free links.

Definition 2.9 (instantiation) Let $\vec{X}$ be of width m, and let $f\colon n \to m$ be a map of ordinals. Define $\vec{Y}$ of width n by setting $Y_j \stackrel{\text{def}}{=} X_{f(j)}$ for $j \in n$. We proceed to define the *instantiation* $\overline{f}$ induced by f, a map of ground homsets

$$\overline{f}\colon \mathsf{Gr}(\vec{X}) \to \mathsf{Gr}(\vec{Y})\ .$$

For any $a\colon \vec{X}$, by Propositions 2.7 and 2.8 we have $a = /Z \circ (c_0 \parallel \cdots \parallel c_{m-1})$, with $c_i\colon X_i \uplus Z_i$ and $Z = \bigcup_i Z_i$. Let $d_j \eqcirc c_{f(j)}$ $(j \in n)$ have disjoint supports, and define

$$\overline{f}(a)\colon \vec{Y} \stackrel{\text{def}}{=} /Z \circ (d_0 \parallel \cdots \parallel d_{n-1})\ .$$ ∎

The unicity results make instantiation well-defined up to support equivalence. Although instantiation is not well-defined for arbitrary bigraphs, nor even for arbitrary wirings, it extends naturally to linkings:

Definition 2.10 (link instantiation) Let $\lambda\colon \langle m, \vec{X}\rangle \to \langle m, \vec{X}'\rangle$ be a linking, and let $f\colon n \to m$ be a map of ordinals. Define $\vec{Y}$ and $\vec{Y}'$ by $Y_j = X_{f(j)}$ and $Y'_j = X'_{f(j)}$ ($j \in n$). Then define the *instance*

$$\overline{\lambda}\colon \langle n, \vec{Y}\rangle \to \langle n, \vec{Y}'\rangle$$

to have $link_\lambda \restriction Y$ as its link map, where $Y = \bigcup\{\vec{Y}\}$. ∎

It is easy to check that this respects the scoping rule. It follows directly that instantiation distributes over composition among linkings and ground bigraphs:

Proposition 2.11 (instantiation distributes) *For linkings λ, λ' and ground bigraphs a with appropriate interfaces we have*

$$\begin{aligned} \overline{f}(\lambda \circ \lambda') &= \overline{f}(\lambda) \circ \overline{f}(\lambda') \\ \overline{f}(\lambda \circ a) &= \overline{f}(\lambda) \circ \overline{f}(a)\,. \end{aligned}$$

Now, recalling how instantiation of ground bigraphs is defined in [1], we wish to refine our definition so that some of the names of each copy of a prime part are disjoined, and can therefore be bound differently in a context. This requires explicit name bijections:

Definition 2.12 (parametric instantiation) Let $I = \vec{X}$ and $J = \vec{Y}$ be partitions with widths m and n, and let $f\colon n \to m$ be a map of ordinals. Let $\vec{\iota}$ be bijections $\iota_j\colon X_{f(j)} \to Y_j$ ($j \in n$). Then the *parametric instantiation*

$$\overline{f}_{\vec{\iota}}\colon \mathsf{Gr}(\vec{X} \oplus \vec{X}') \to \mathsf{Gr}(\vec{Y} \oplus \vec{Y}')$$

is defined as in Definition 2.9, except that we have $c_i\colon X_i \oplus X'_i \oplus Z_i$ and we take $d_j \eqcirc (\iota_j \oplus \mathsf{id}) \circ c_{f(j)}$. ∎

Again we have distributivity, but of a refined form:

Proposition 2.13 (parametric instantiation distributes) *Let $a\colon \vec{X} \oplus \vec{X}'$ be a ground bigraph, with width m. Let $\lambda\colon \vec{X}' \to \vec{X}''$ be a linking and $f\colon n \to m$ a map of ordinals. Let $\vec{\iota}$ be isomorphisms from $X_{f(j)}$ to Y_j ($j \in n$). Then*

$$\overline{f}_{\vec{\iota}}((\mathsf{id}_{\vec{X}} \oplus \lambda) \circ a) \;=\; (\mathsf{id}_{\vec{Y}} \oplus \overline{f}(\lambda)) \circ \overline{f}_{\vec{\iota}}(a)\,.$$

This result will be important later when we wish to prove that, to generate a reaction relation from a given set of parametric rules, it is sufficient to consider only parameters that are appropriately discrete.

3 Reactions

We are now ready to define reactions, together with the reaction relation they induce between bigraphs.

Definition 3.1 (reaction rule, reaction relation) A *ground (reaction) rule* is a ground pair (r, r'), *redex* and *reactum*, with the same outer face. Given a set of ground rules, the *reaction relation* $\longrightarrow\!\!\!\!\triangleright$ over agents is the least, closed under support equivalence ($\eqsim$), such that $D \circ r \longrightarrow\!\!\!\!\triangleright D \circ r'$ for each active D and each ground rule (r, r'). ∎

Definition 3.2 (parametric reaction rule) A *parametric (reaction) rule* takes the form

$$(\, R\colon I \to K,\ R'\colon I' \to K,\ f,\ \vec{\iota}\,)$$

where R and R' are called the *(parametric) redex* and *reactum*, $I = \vec{X}$ and $I' = \vec{X}'$ are partitions with widths m and m', and $f\colon m' \to m$ is a map of ordinals. The fourth component is a vector of bijections $\iota_j\colon X_{f(j)} \to X'_j$, one for each $j \in m'$.

The parametric rule generates ground rule of the form

$$(\, (R \oplus \omega) \circ a,\ (R' \oplus \omega') \circ a'\,)$$

as follows. Let $I \oplus H$, $I' \oplus H'$ and $K \oplus L$ be interface extensions with $H' = \overline{f}(H)$. Let $\omega\colon H \to L$ and $\omega'\colon H' \to L$ be wirings that agree on the names of H'. Then for any $a\colon I \oplus H$, complete the ground rule by defining $a' = \overline{f}_{\vec{\iota}}(a)\colon I' \oplus H'$. ∎

The extension H in the outer face of the parameter a allows names to be exported from it via ω. The names of H' in the outer face of the instance a' are similarly exported; these will be among the names of H, and will coincide with them if f is surjective.

On the other hand, consider a rule which takes a prime parameter and discards it, so that $m = 1, m' = 0$. Then we may have $H = L$ and $\omega = \mathsf{id}_L$, while $H' = \epsilon$ and $\omega'\colon \epsilon \to L$ is simply an idle extension.

We have placed no constraints upon the parameter a of a rule. This constrasts with [1] where we required parameters to be discrete. We can now show that if we confine ourselves to suitably discrete parameters, then we still generate the same reaction relation. This will make it easier to analyse properties of the latter relation.

First, we need to show that every bigraph can be expressed simply in terms of a suitably discrete one. The following is closely analogous to Proposition 2.7:

Proposition 3.3 (discrete decomposition) *A bigraph $G\colon \to I \oplus H$ may be expressed uniquely up to isomorphism as $G = (\mathsf{id}_I \oplus \lambda) \circ D$, where $\lambda\colon J \to H$ is a linking and $D\colon \to I \oplus J$ is discrete for I.*

Proof (outline) The linking λ must do four things: increase the locality from J to H since D is to be minimal for I; close some names in J to form the closed free links of G; include any idle edges of G; and identify some names in J by a substitution since D is to be discrete. ∎

We are now ready to prove that

Proposition 3.4 (discrete parameters suffice) *The ground rules generated by a parametric reaction rule are unchanged if parameters are constrained to be discrete.*

Proof Consider a ground rule $(\,(R \oplus \omega) \circ a,\ (R' \oplus \omega') \circ a'\,)$ generated as in Definition 3.2. Then we have $a \colon I \oplus H$ and $a' = \overline{f}_{\vec{\iota}}(a) \colon I' \oplus H'$ where $H' = \overline{f}(H)$. We also know that the wirings $\omega \colon H \to L$ and $\omega' \colon H' \to L$ agree on the names of H'. Further, By Proposition 3.3 we have $a = (\mathsf{id}_I \oplus \lambda) \circ d$ with $d \colon I \oplus J$ discrete for I.

It will be enough to show that our given ground rule can be expressed in the form

$$(\,(R \oplus \zeta) \circ d,\ (R' \oplus \zeta') \circ d'\,)$$

for suitable wirings ζ and ζ', where $d' = \overline{f}_{\vec{\iota}}(d) \colon I' \oplus J'$ and $J' = \overline{f}(J)$. We have

$$(R \oplus \omega) \circ a \ = \ (R \oplus \omega) \circ (\mathsf{id}_I \oplus \lambda) \circ d \ = \ (R \oplus \zeta) \circ d \quad \text{by Proposition 2.4}$$

where $\zeta \stackrel{\text{def}}{=} \omega \circ \lambda$. Next we have

$$\begin{array}{rcll} (R' \oplus \omega') \circ a' & = & (R' \oplus \omega') \circ \overline{f}_{\vec{\iota}}((\mathsf{id}_I \oplus \lambda) \circ d) & \\ & = & (R' \oplus \omega') \circ (\mathsf{id}_{I'} \oplus \overline{f}(\lambda)) \circ \overline{f}_{\vec{\iota}}(d) & \text{by Propositions 2.4, 2.13} \\ & = & (R' \oplus \zeta') \circ d' & \end{array}$$

where $\zeta' \stackrel{\text{def}}{=} \omega' \circ \overline{f}(\lambda)$. But then ζ and ζ' agree on the names of J', so we are done. ∎

Having verified that discrete parameters generate all reactions, we turn attention to the wirings (ω, ω') that act on the names (X, X') of a parameter d and its instance $d' = \overline{f}_{\vec{\iota}}(d)$. Wirings are more general than required to export these names (suitably located); for example they may close some names, or apply a substitution. Since the latter operations can be applied by an external context D, we should expect the same reaction relation to be generated by ground rules that use just placings (π, π') in place of arbitrary wirings. We shall now justify this expectation; but we have to allow that d' exports fewer names than d, since the instantiation may discard parts of d. So we expect $\pi \colon H \to J$ and $\pi' \colon H' \to J'$ where $J = J' \oplus J''$.

Proposition 3.5 (placings suffice) *The relation relation defined for a parametric reaction rule in Definition 3.2 is generated by the ground rules*

$$(\,(R \oplus \pi) \circ d,\ (R' \oplus \pi' \oplus J'') \circ d'\,)$$

where d' is the instance of a discrete parameter d and (π, π') are placings.

Proof Consider a ground rule defined with wirings:

$$(\,(R \oplus \omega) \circ d,\ (R' \oplus \omega') \circ d'\,)\ .$$

We wish to show that its reactions can be generated instead by using placings.

We know that $\omega \colon H \to L$ and $\omega' \colon H' \to L$ have link maps that agree on the names of d'. Let $L = \vec{W}$ with width ℓ. Let $H = \vec{X}$ and $H' = \vec{X}'$ with $X'_i = X_{f(i)}$. It will be enough to show that we can decompose the wirings as follows:

$$\begin{array}{ll} \omega = \lambda \circ \pi & \pi \colon H \to J \\ \omega' = \lambda \circ (\pi' \oplus J'') & \pi \colon H' \to J',\ J = J' \oplus J'' \end{array}$$

where the linking λ is shared, and can therefore be decomposed from the wirings and absorbed in to a surrounding active context D.

To achieve this, we first define J. It must have exactly the names of $H = \vec{X}$, since a placing has an identity link map. For locality, we take $J = \vec{Y}$ where, for each $j \in \ell$,

$$Y_j \stackrel{\text{def}}{=} \{x \in X \mid \omega(x) \in W_j \text{ or } x \text{ closed in } \omega\} .$$

Thus a name closed by ω is located everywhere in J. Now λ, π and π' are fully determined once J is fixed; for λ must have identity place map and the link map of ω, while π and π' must have identity link maps and the place maps of ω and of ω' respectively.

The construction is therefore complete; we leave it to the reader to check that the scoping rule is obeyed in all three cases. ∎

It appears that this is as far as we can simplify the ground rules needed to generate the reaction relation from a parametric rule. It concludes this initial study of bigraphs with multiply located names. Further work is needed to assess the value of such a naming discipline, and this should involve the use of bigraphs for practical purposes such as modelling interactions in structured physical environments. It also affects the way to model fundamental calculi, such as the λ-calculus, in bigraphs. By studying known proofs of confluence (the Church–Rosser theorem) in the bigraphical setting, we hope generalise them to establish partial or total confluence in a wide range of applications. This will be the topic of a sequel to the present paper.

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References

[1] Jensen, O.H. and Milner, R. (2004), Bigraphs and mobile processes (revised). Technical Report 580, University of Cambridge Computer Laboratory. Available from `http://www.cl.cam.ac.uk/users/rm135`.