9: Viterbi Algorithm for HMM Decoding

Machine Learning and Real-world Data

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(slides adapted from Simone Teufel)

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Last session: estimating parameters of an HMM

- The dishonest casino, dice edition.
- Two hidden states: L (loaded dice), F (fair dice).
- You don't know which dice is currently in use. You can only observe the numbers that are thrown.
- You estimated transition and emission probabilities (Problem 1 from last time).
- We are now turning to Problem 4.
- We want the HMM to find out when the fair dice was out, and when the loaded dice was out.
- We need to write a decoder.

Decoding: finding the most likely path

- Definition of decoding: Finding the most likely hidden state sequence X that explains the observation O given the HMM parameters $\mu = (A, B)$.

$$\begin{aligned}\hat{X} &= \operatorname*{argmax}_X P(X, O|\mu) \\ &= \operatorname*{argmax}_X P(O|X, B)P(X|A) \\ &= \operatorname*{argmax}_{X_1 \ldots X_T} \prod_{t=1}^{T} P(O_t|X_t, B)P(X_t|X_{t-1}, A)\end{aligned}$$

- Search space of possible state sequences X is O(N^T); too large for brute force search.

Viterbi is a Dynamic Programming Application

(Reminder from Algorithms course)
We can use Dynamic Programming if two conditions apply:

- Optimal substructure property
  - An optimal state sequence $X_1 \dots X_j \dots X_T$ contains inside it the sequence $X_1 \dots X_j$, which is also optimal
- Overlapping subsolutions property
  - If both X_t and X_u are on the optimal path, with $u > t$, then the calculation of the probability for being in state X_t is part of each of the many calculations for being in state X_u.

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The intuition behind Viterbi

- Here's how we can save ourselves a lot of time.
- Because of the Limited Horizon of the HMM, we don't need to keep a complete record of how we arrived at a certain state.
- For the first-order HMM, we only need to record one previous step.
- Just do the calculation of the probability of reaching each state once for each time step (variable δ).
- Then memoise this probability in a Dynamic Programming table
- This reduces our effort to $O(N^2T)$.
- This is for the first order HMM, which only has a memory of one previous state.

Viterbi: main data structure

- Memoisation is done using a *trellis*.
- A trellis is equivalent to a Dynamic Programming table.
- The trellis is $(N+2) \times (T+2)$ in size, with states j as rows and time steps t as columns.
- Each cell j, t records the Viterbi probability $\delta_j(t)$, the probability of the most likely path that ends in state s_j at time t:

$$\delta_j(t) = \max_{1 \leq i \leq N} [\delta_i(t-1)\, a_{ij}\, b_j(O_t)]$$

- This probability is calculated by maximising over the best ways of going to s_j for each s_i.
- a_{ij}: the transition probability from s_i to s_j
- $b_j(O_t)$: the probability of emitting O_t from destination state s_j

Viterbi algorithm, initialisation

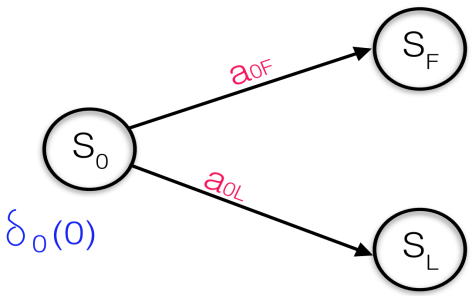
Note: the probability of a state starting the sequence at $t = 0$ is just the probability of it emitting the first symbol.

Viterbi algorithm, initialisation

Viterbi algorithm, initialisation

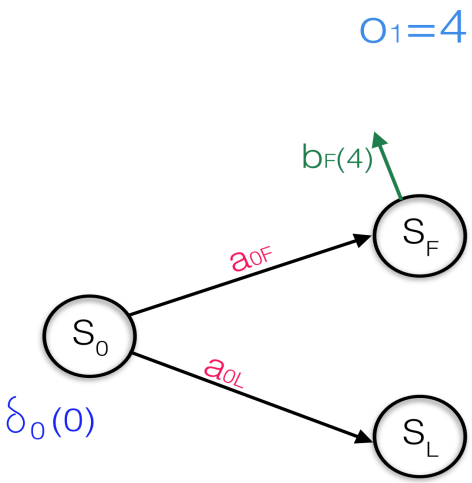
Viterbi algorithm, initialisation

Viterbi algorithm, main step



$X_1 = F$ $X = L$ $X = L$

Viterbi algorithm, main step: observation is 4



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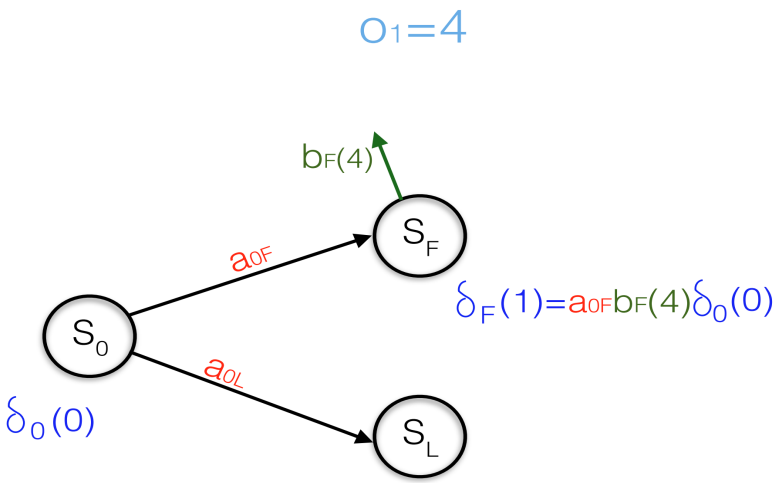


$X_1 = F$ $X = L$ $X = L$

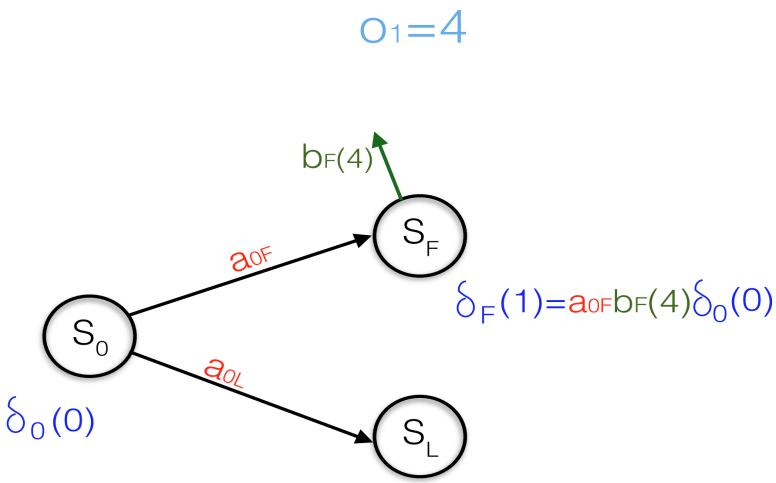
Viterbi algorithm, main step, ψ

- $\psi_j(t)$ is a helper variable that stores the $t-1$ state index i on the highest probability path.

$$\psi_j(t) = \underset{1 \leq i \leq N}{\operatorname{argmax}} [\delta_i(t-1)\, a_{ij}\, b_j(O_t)]$$

- In the backtracing phase, we will use ψ to find the previous cell/state in the best path.

Viterbi algorithm, main step: observation is 4



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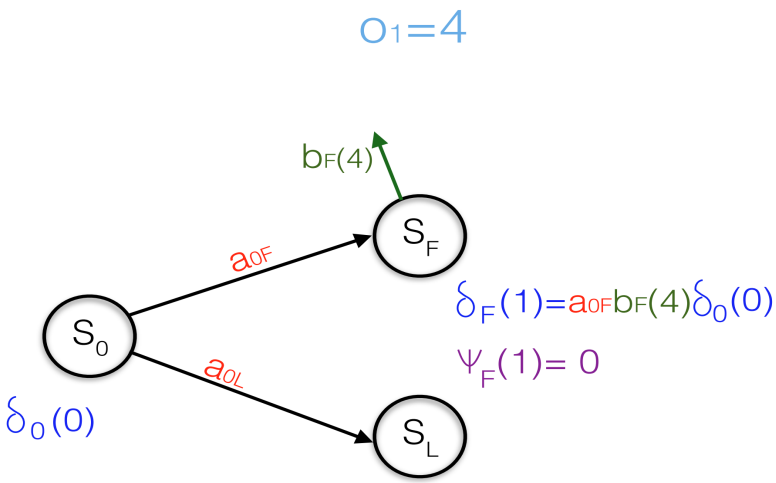

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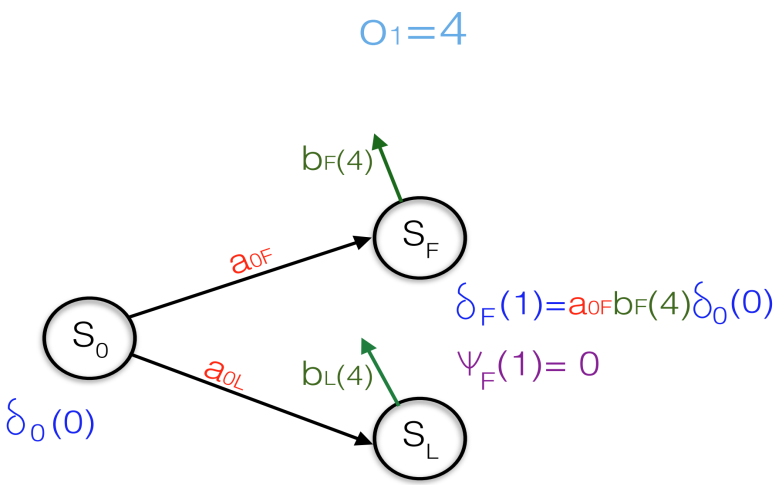

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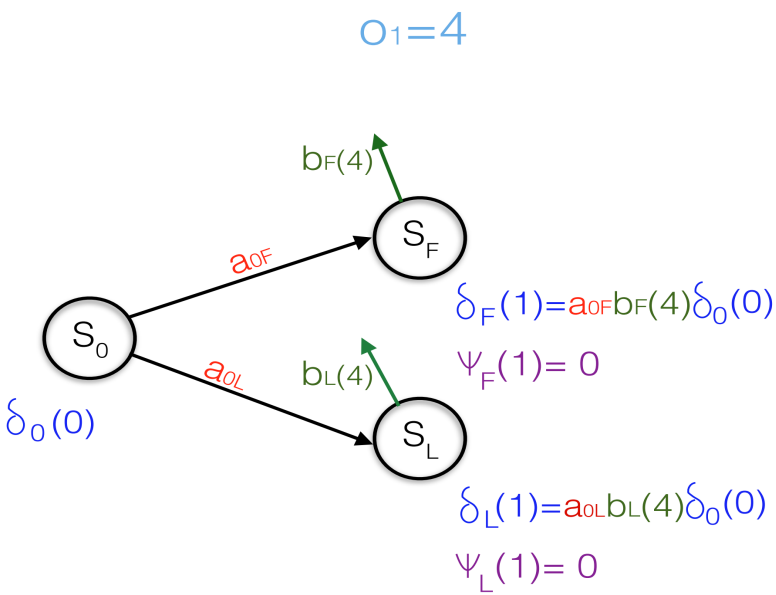
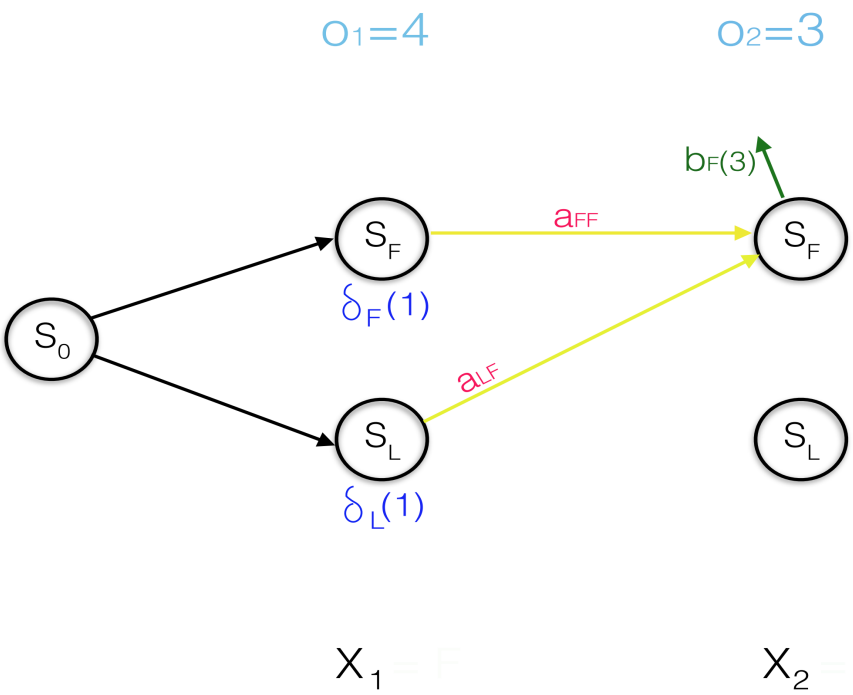


Viterbi algorithm, main step: observation is 3



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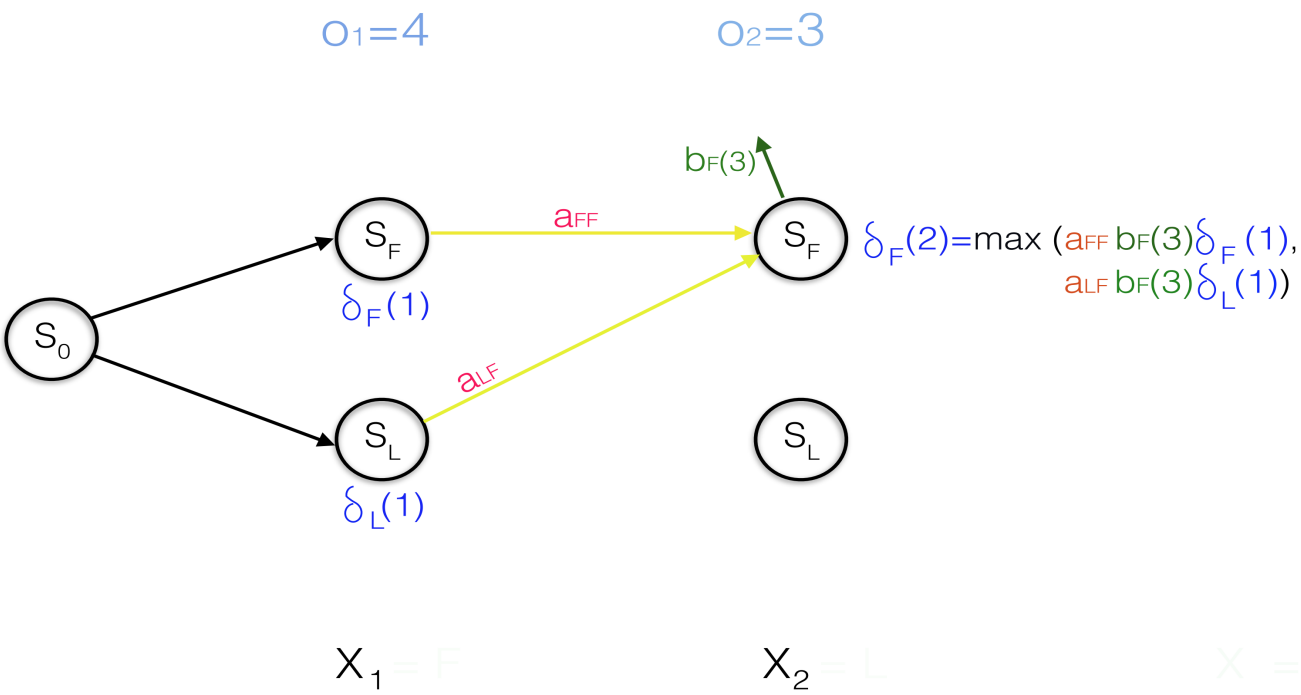

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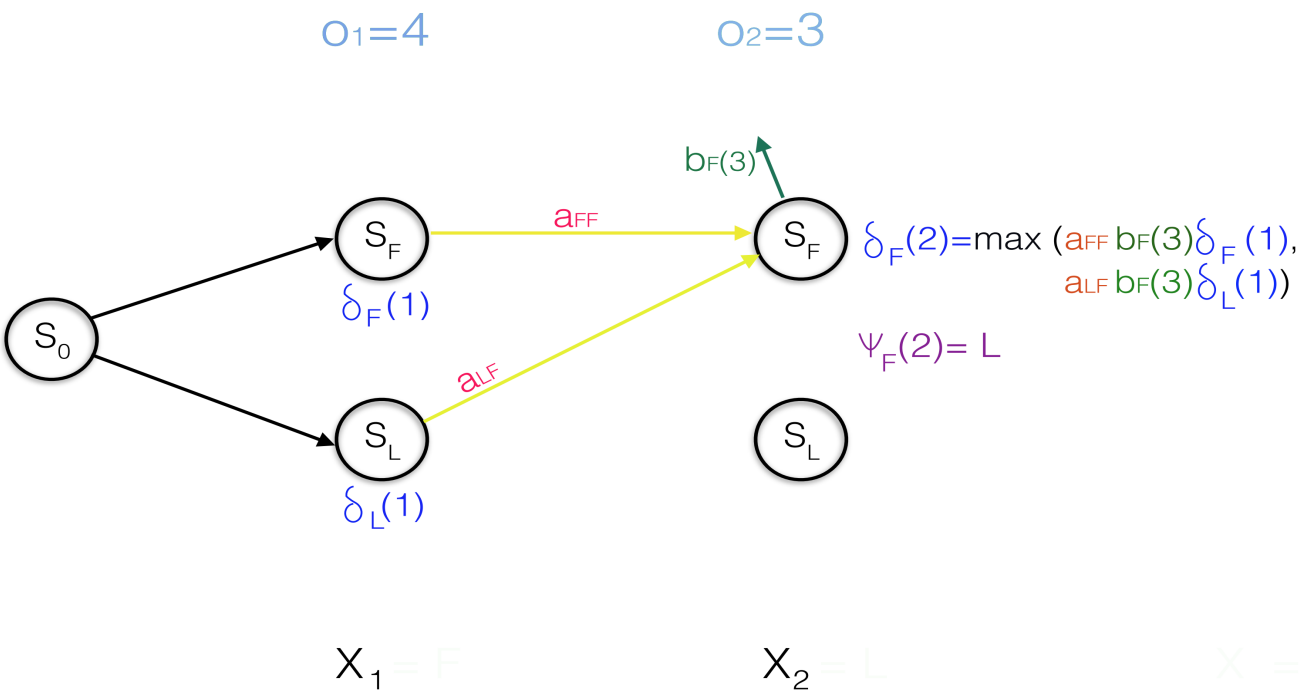
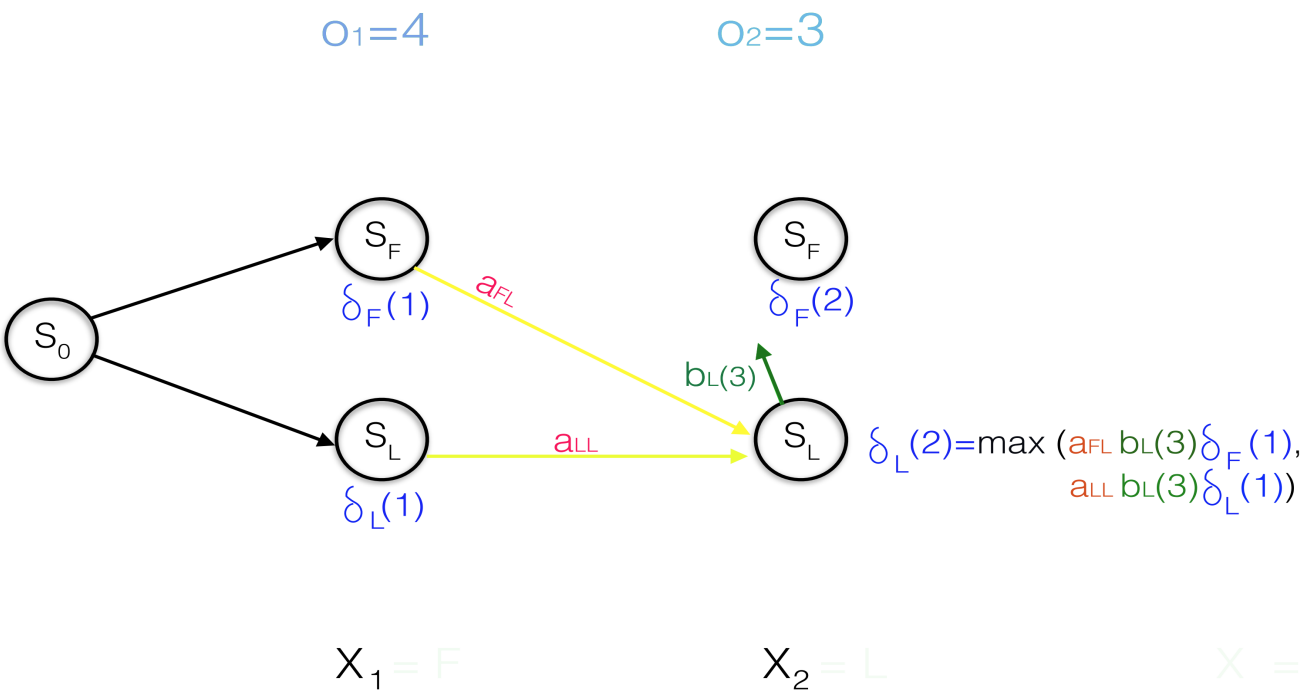


Viterbi algorithm, main step: observation is 3



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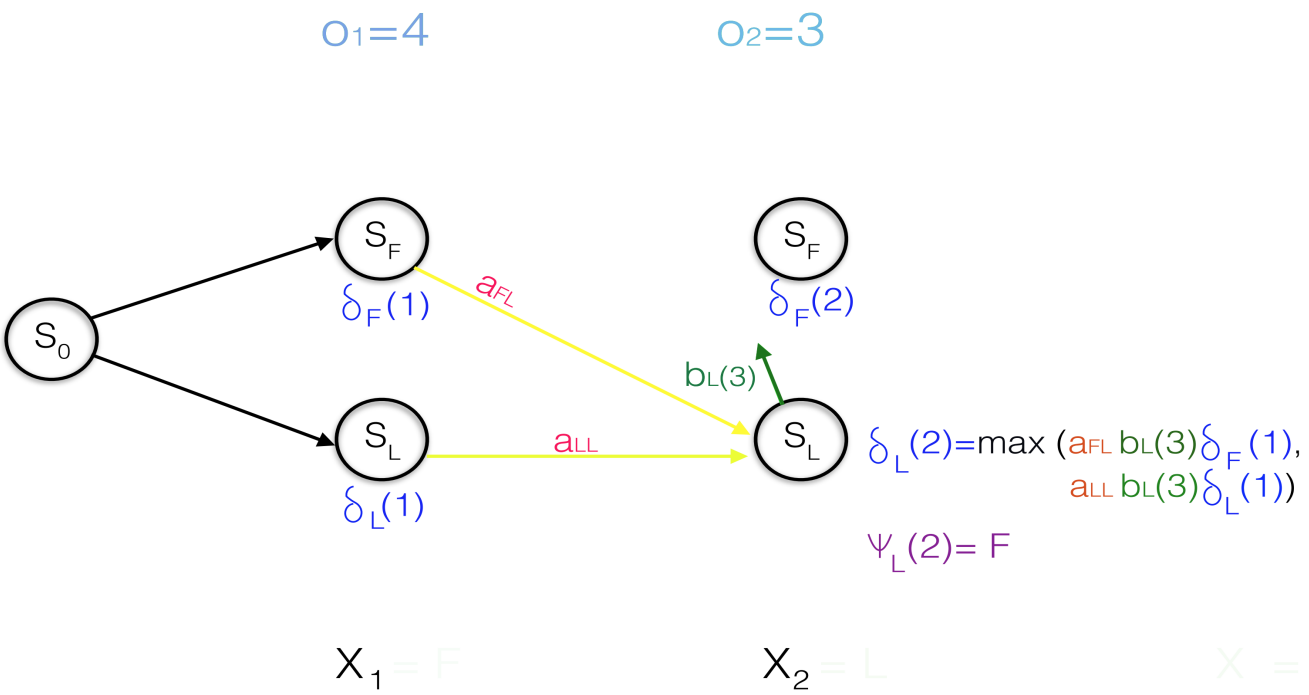
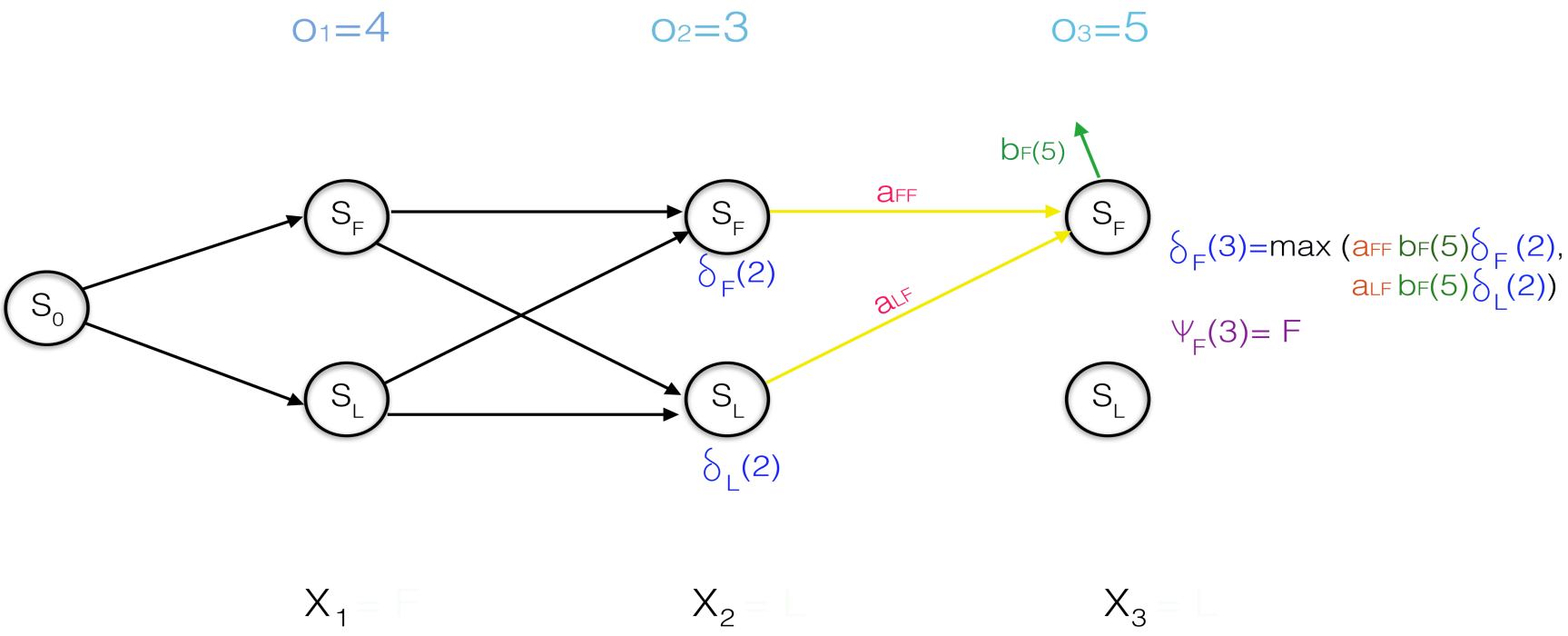


Viterbi algorithm, main step: observation is 5



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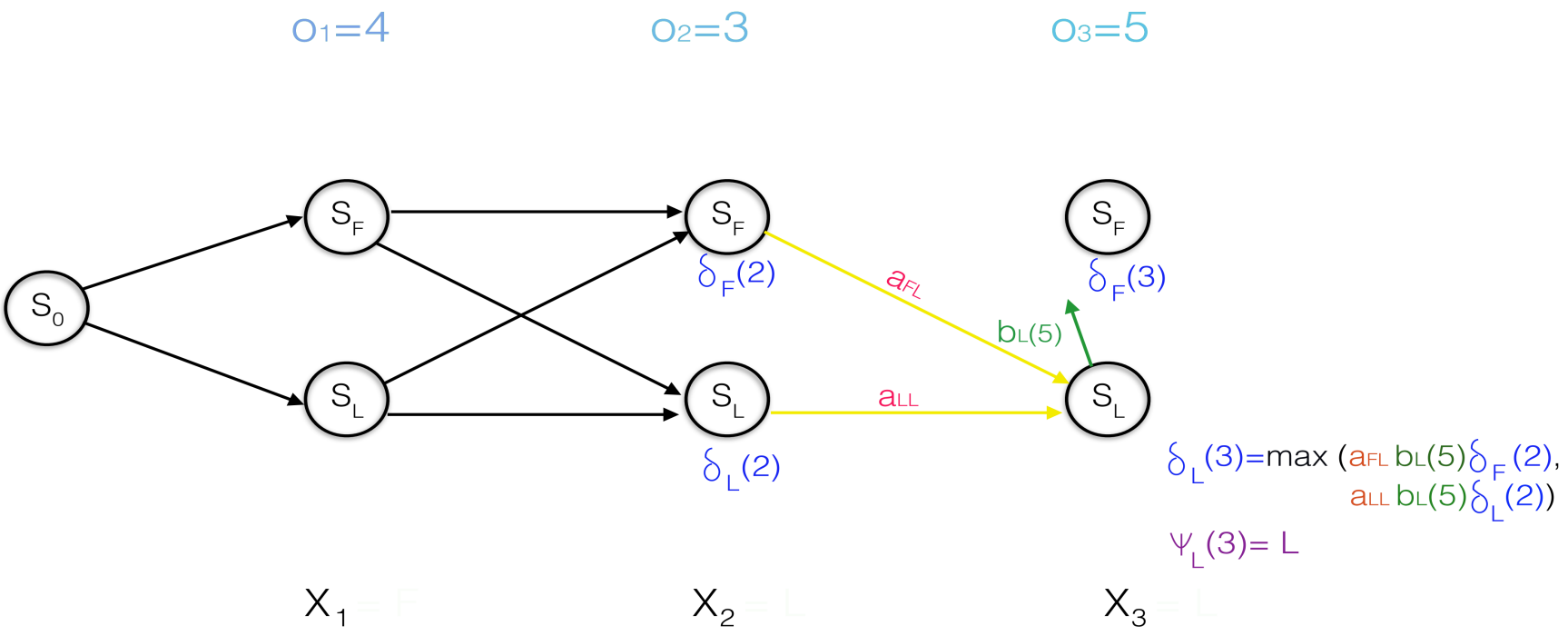
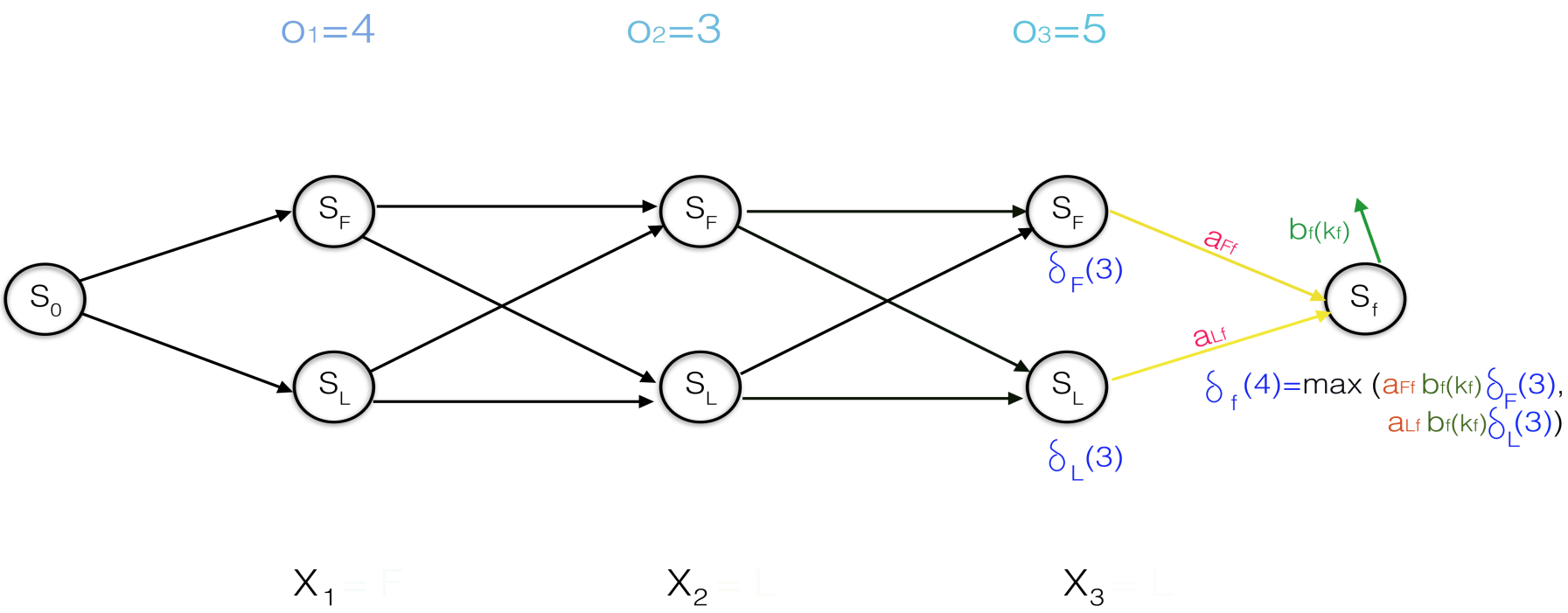
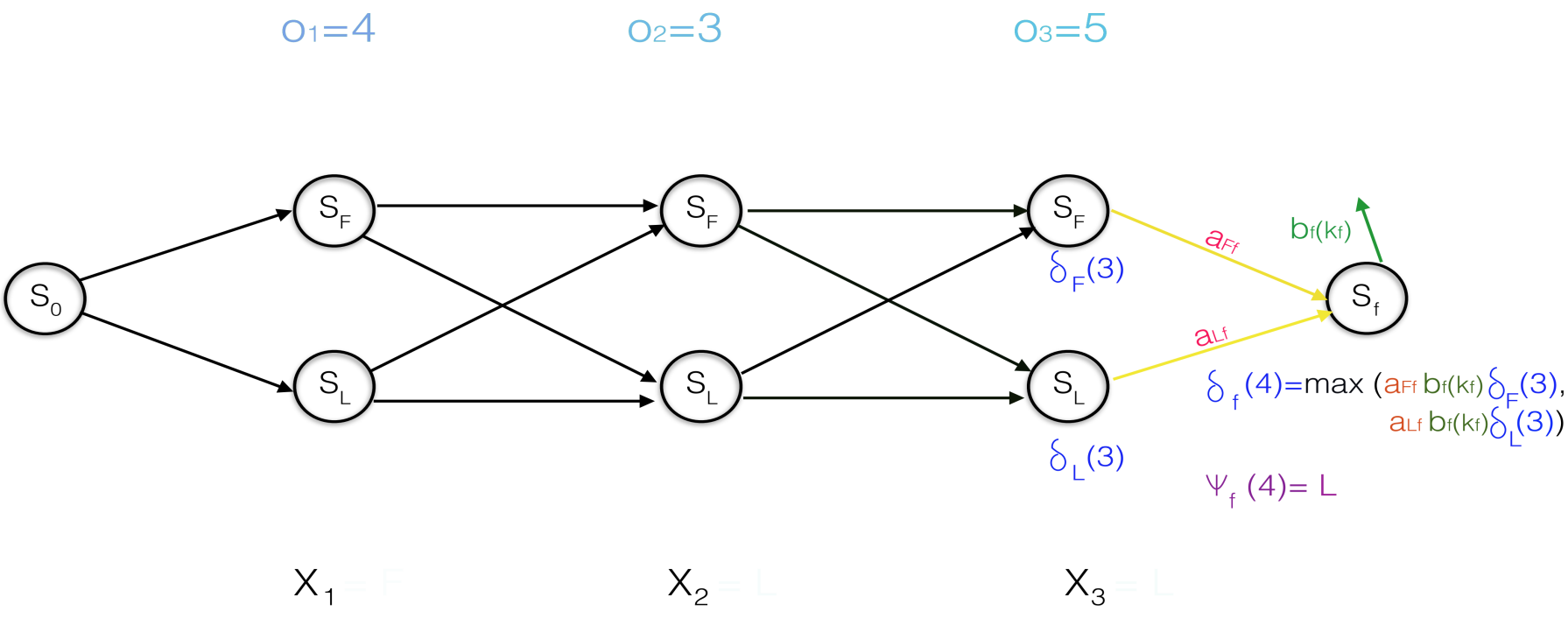


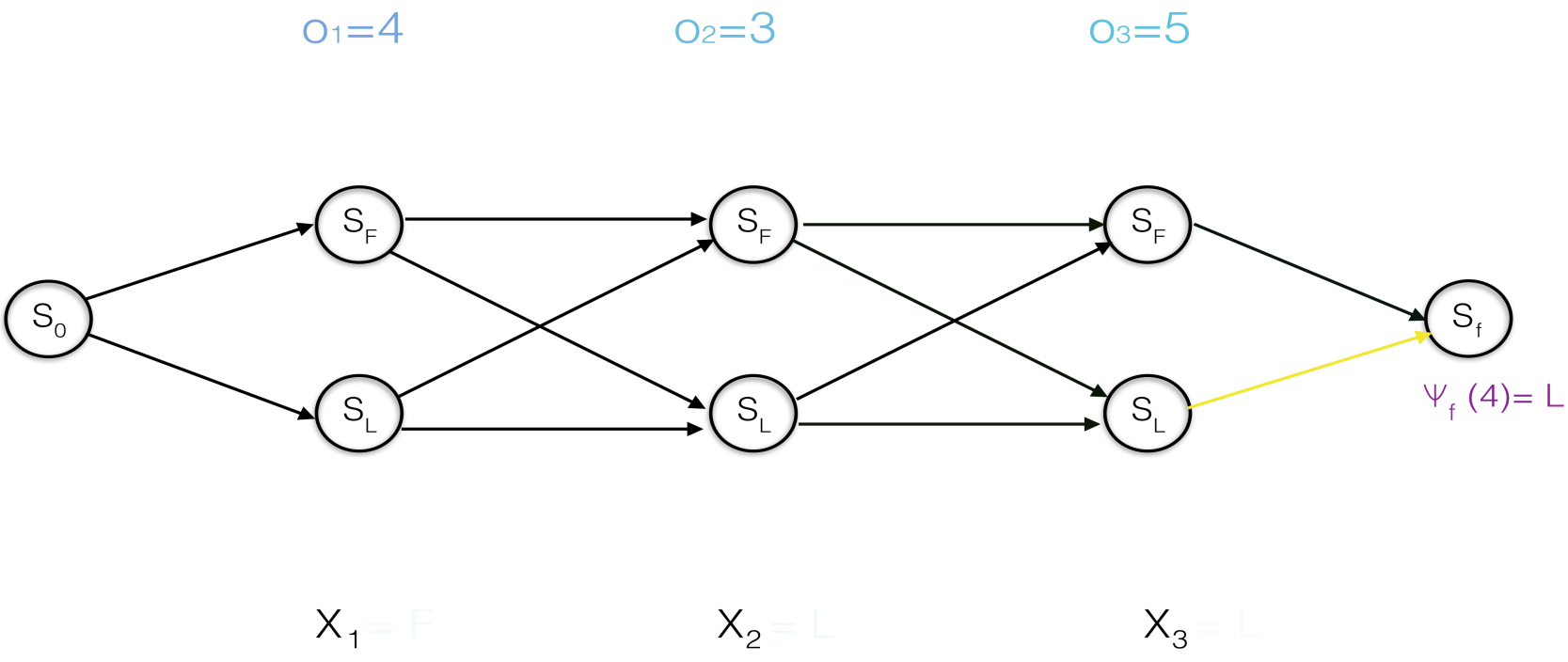
Viterbi algorithm, termination



Viterbi algorithm, termination



Viterbi algorithm, backtracing



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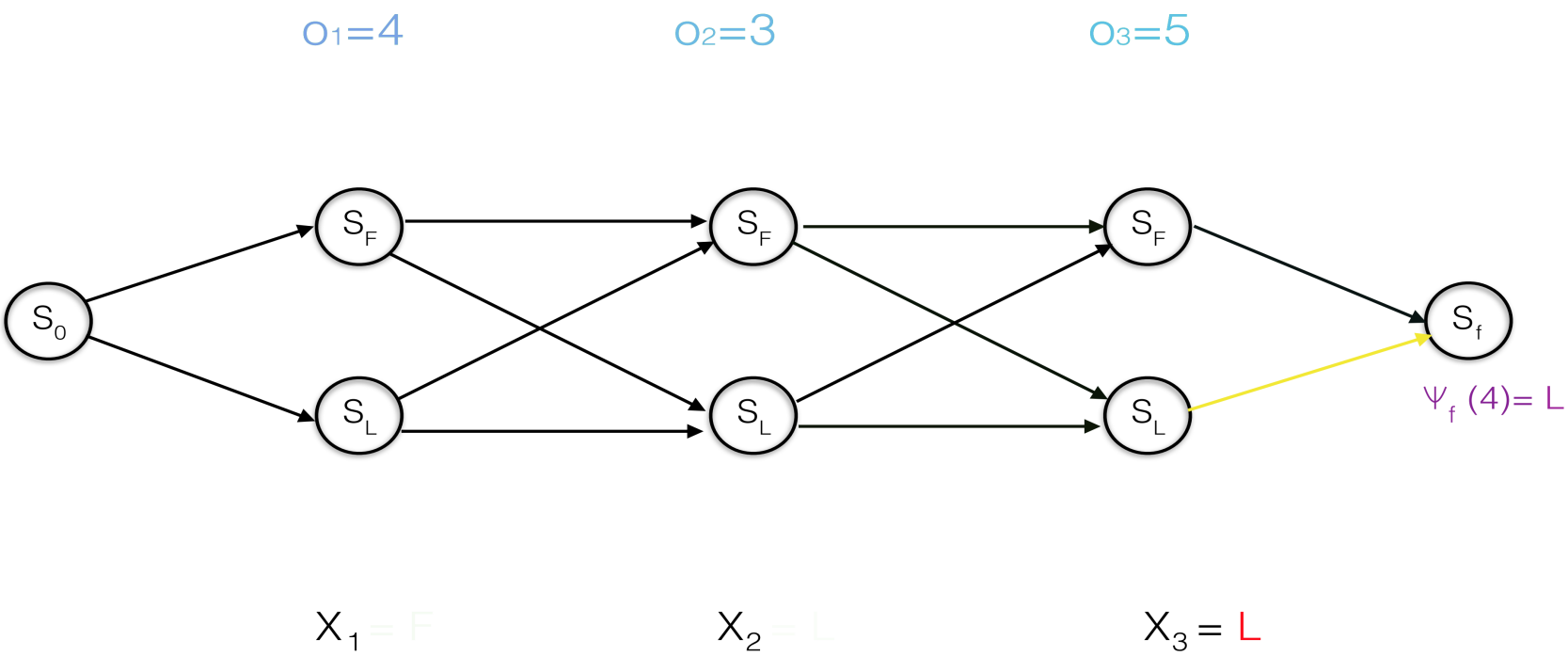

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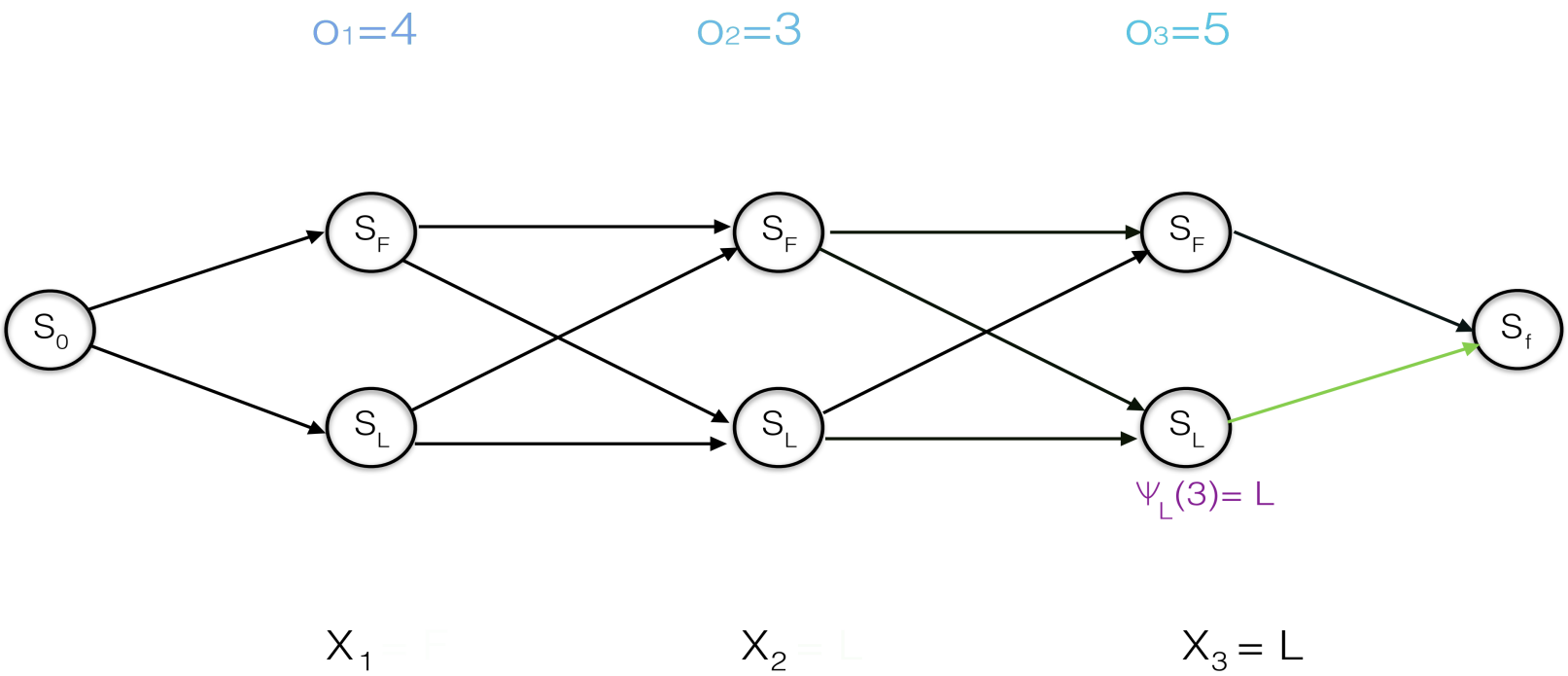

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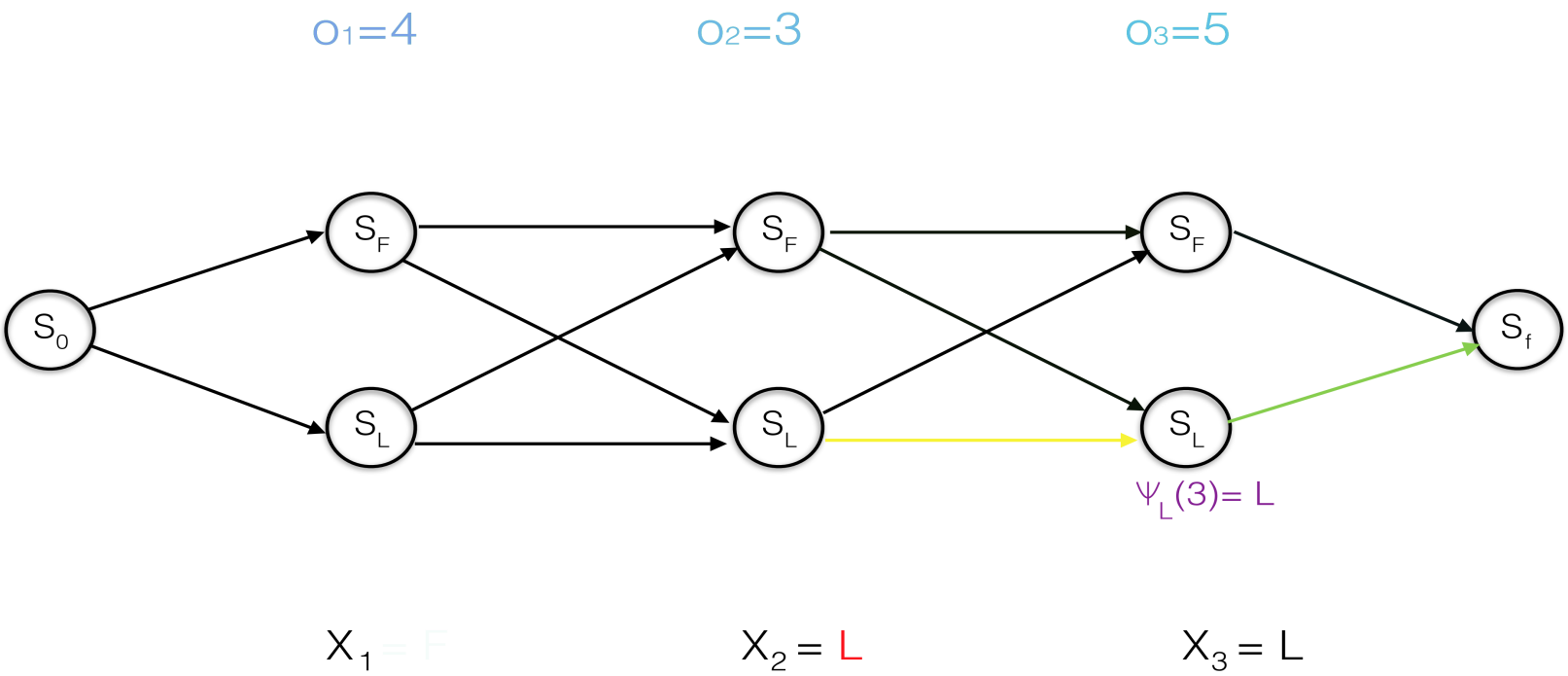

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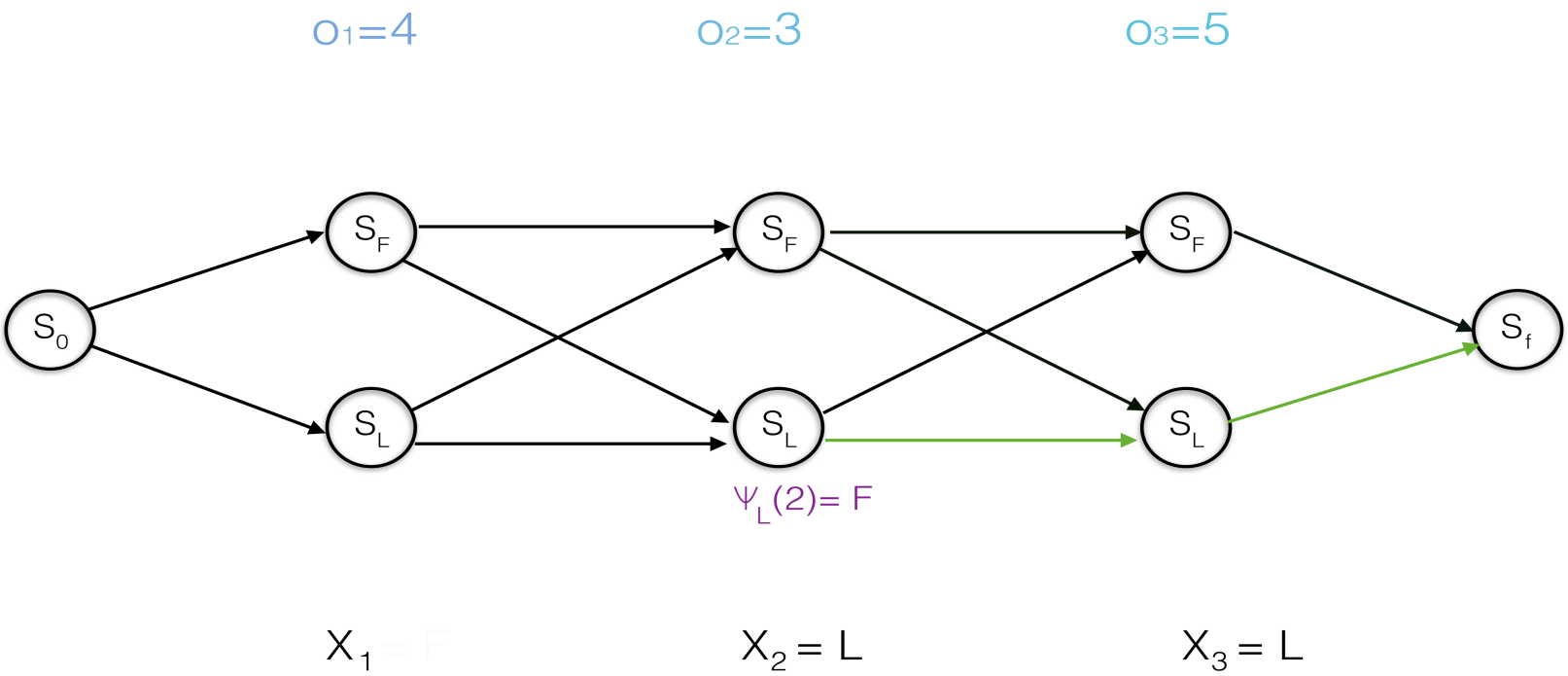

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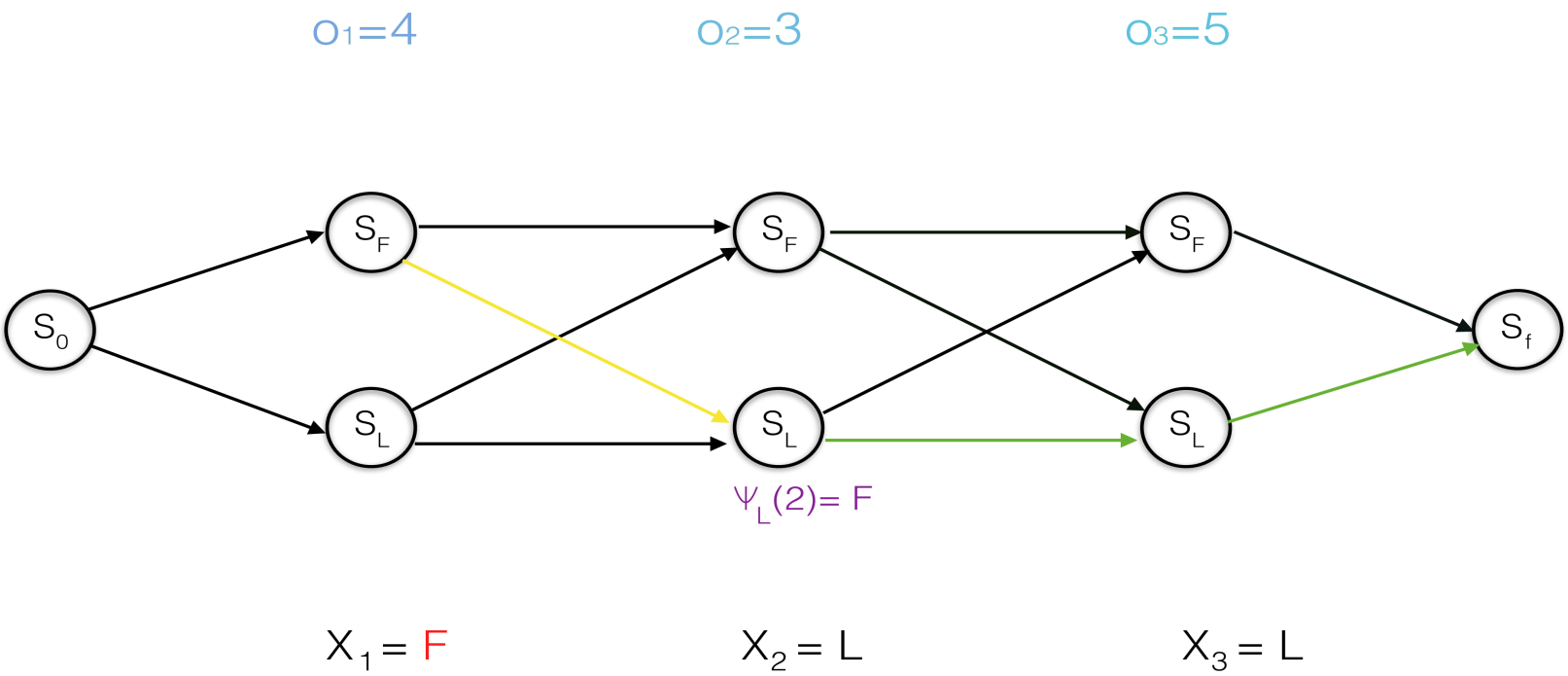

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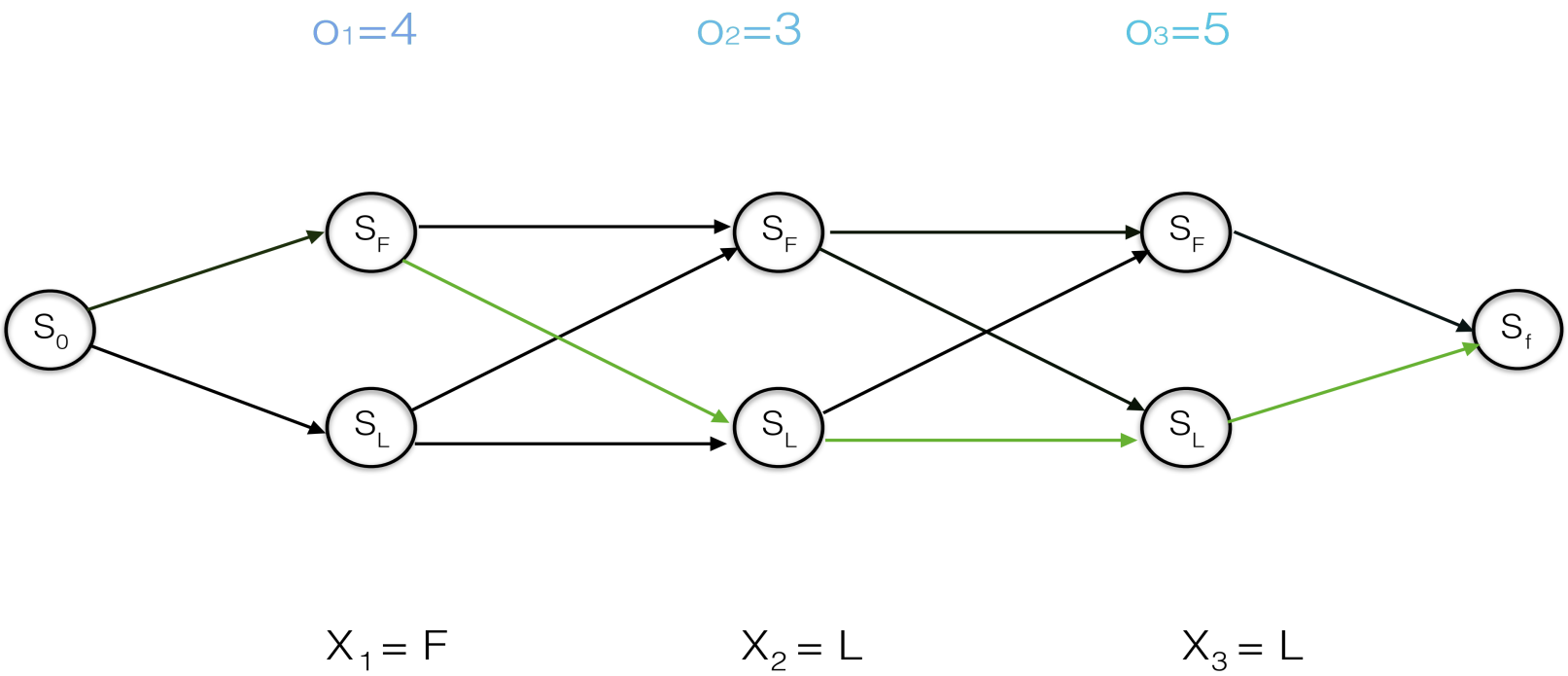

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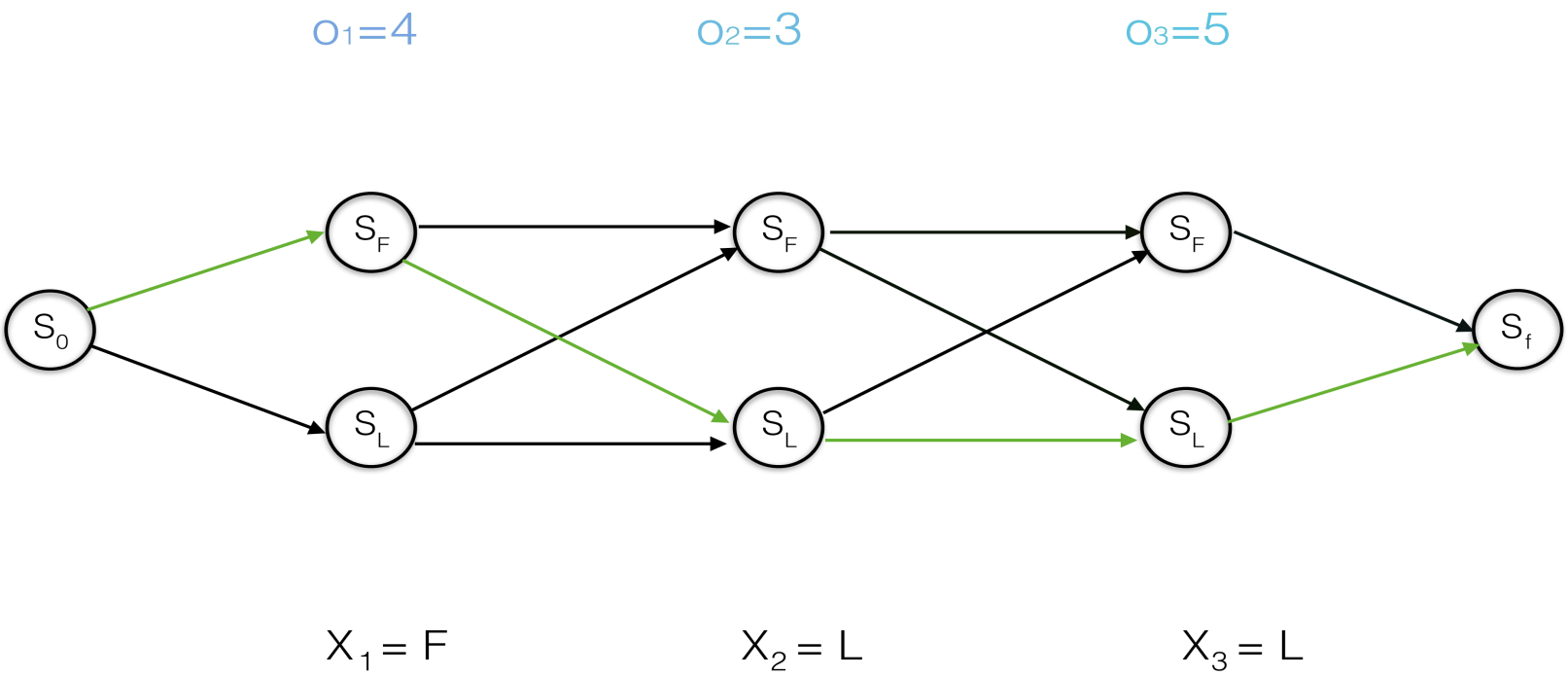

Why is it necessary to keep N states at each time step?

- We have convinced ourselves that it's not necessary to keep more than N ("real") states per time step.
- But could we cut down the table to just a one-dimensional table of T time slots by choosing the probability of the best path overall ending in that time slot, in any of the states?
  - This would be the greedy choice
  - But think about what could happen in a later time slot.
  - You could encounter a zero or very low probability concerning all paths going through your chosen state s_j at time t.
  - Now a state s_k that looked suboptimal in comparison to s_j at time t becomes the best candidate.
  - As we don't know the future, this could happen to any state, so we need to keep the probabilities for each state at each time slot.
- But thankfully, no more.

Precision and Recall

- So far, we have measured system success in accuracy or agreement in Kappa.
- But sometimes it's only one type of instances that we find interesting.
- We don't want a summary measure that averages over interesting and non-interesting instances, as accuracy does.
- In those cases, we use precision, recall and F-measure.
- These metrics are imported from the field of information retrieval, where the difference (in numbers) beween interesting and non-interesting examples is particularly high.
- Accuracy doesn't work well when the types of instances are unbalanced.

True positives, false negatives. . .

		System says: L	System says: F	System says: Total
Truth is:	L	TP	FN	TP+FN
	F	FP	TN	FP+TN
	Total	TP+FP	FN+FP	TP+FP+FN+FP

L is the category we are interested in.

- TP are the true positives.
  - The system correctly declared them as positive.
- FN are the false negatives.
  - The system didn't declare them as as a positive, but should have.
- TN are the true negatives.
  - The system didn't declare them as a positive, and was right.
- FP are the false positives.
  - The system declared them as a positive, but shouldn't have.

Precision and Recall

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Truth is:	L	TP	FN	TP+FN
	F	FP	TN	FP+TN
	Total	TP+FP	FN+TN	TP+FP+FN+TN

- Precision of L: $P_L = \frac{TP}{TP+FP}$
- Recall of L: $R_L = \frac{TP}{TP+FN}$
- F-measure of L: $F_L = \frac{2P_L R_L}{P_L+R_L}$
- Accuracy: $A = \frac{TP+TN}{TP+FP+FN+TN}$

Your task today

Task 8:

- Implement the Viterbi algorithm.
- Run it on the dice dataset and measure precision of L (P_L), recall of L (R_L) and F-measure of L (F_L).

Literature

- Collin's notes: `http://www.cs.columbia.edu/~mcollins/hmms-spring2013.pdf`
- Jurafsky and Martin, 3rd Edition, `https://web.stanford.edu/~jurafsky/slp3/8.pdf`, Chapter 8.4 (but careful, notation!)
- Smith, Noah A. (2004). Hidden Markov Models: All the Glorious Gory Details: `https://www.cs.cmu.edu/~nasmith/papers/smith.tut04a.pdf`