Introduction to Probability

Lecture 10: Estimators (Part I)

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UNIVERSITY OF CAMBRIDGE

Outline

Introduction

Setting: We can take random samples in the form of i.i.d. random variables $X_1, X_2, \ldots, X_n$ from an unknown distribution.

- Taking enough samples allows us to estimate the mean (WLLN, CLT)
- Using indicator variables, we can estimate $\mathbf{P}[X \leq a]$ for any $a \in \mathbb{R}$
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- How can we measure the accuracy of an estimator?
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$$\text{Measurement} = \text{Quantity of Interest} + \text{Measurement Error}$$

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Physical Experiments:

Measurement $=$ Quantity of Interest $+$ Measurement Error

Empirical Distribution Functions

Definition of Empirical Distribution Function (Empirical CDF)

Let $X_1, X_2, \ldots, X_n$ be i.i.d. samples, and F be the corresponding distribution function. For any $a \in \mathbb{R}$, define

$$F_n(a) := \frac{\text{number of } X_i \in (-\infty, a]}{n}.$$

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Remark

The Weak Law of Large Numbers implies that for any $\epsilon > 0$ and $a \in \mathbb{R}$,

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Thus by taking enough samples, we can estimate the entire distribution (including its expectation and variance).

Empirical Distribution Functions (Example 1/2)

Example 1

Consider throwing an unbiased dice 8 times, and let the realisation be:

$$(x_1, x_2, \ldots, x_8) = (4, 1, 4, 3, 1, 6, 4, 1).$$

What is the Empirical Distribution Function $F_8(a)$?

Answer

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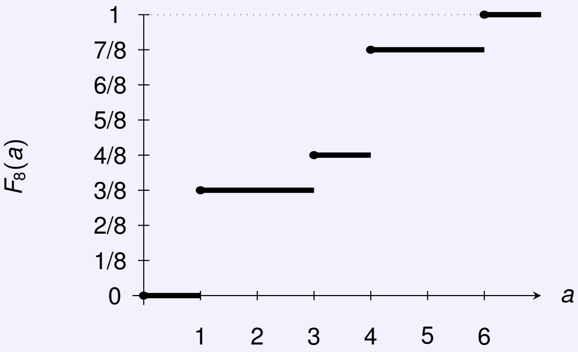

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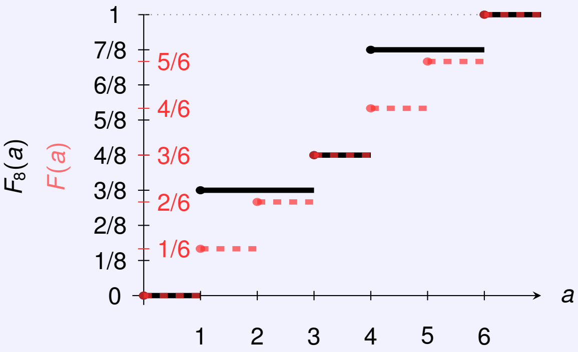
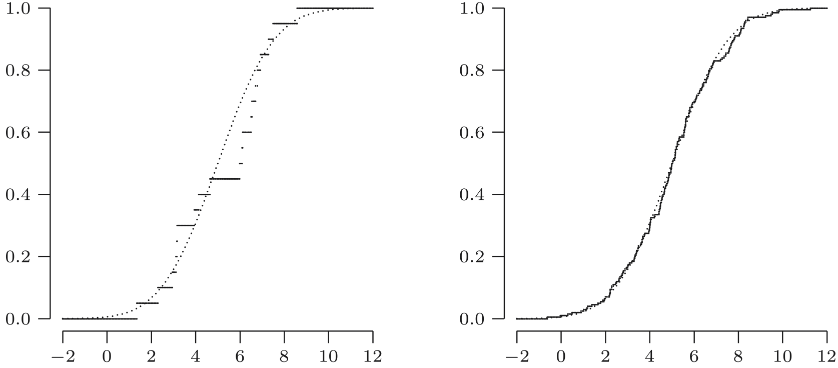


Empirical Distribution Functions (Example 2/2)



Source: Modern Introduction to Statistics

Figure: Empirical Distribution Functions of samples from a Normal Distribution $\mathcal{N}(5, 4)$ ($n = 20$ left, $n = 200$ right)

An Example of an Estimation Problem

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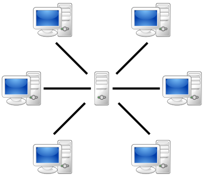
Consider the packages arriving at a network server.



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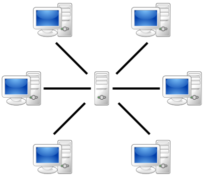

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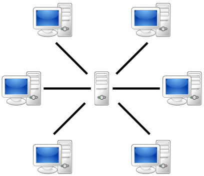

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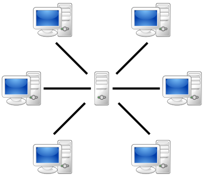

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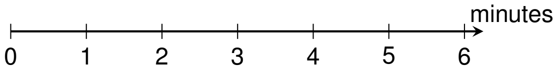

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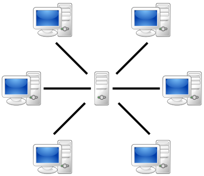

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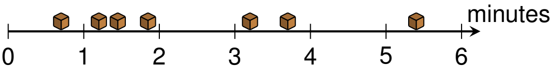

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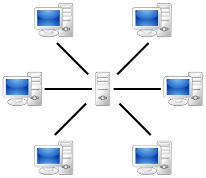

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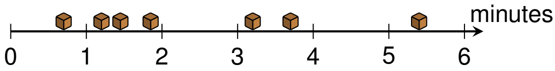

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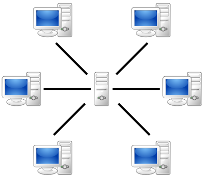

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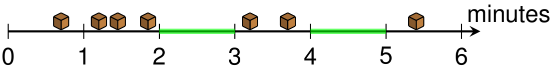

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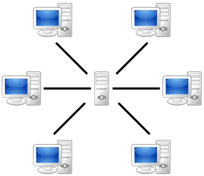

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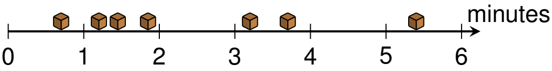

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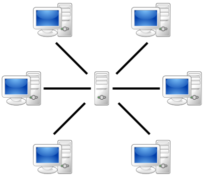

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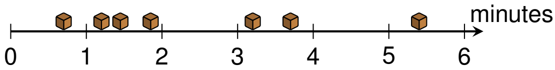

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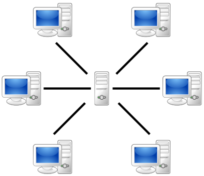

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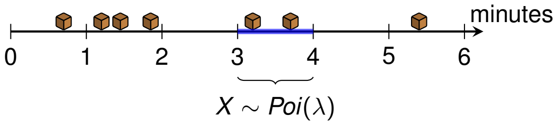

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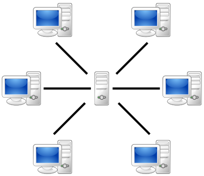

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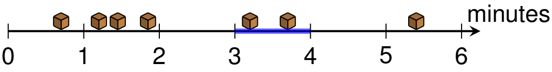

$X \sim Poi(\lambda)$

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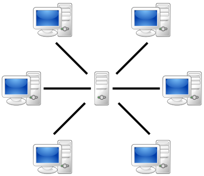

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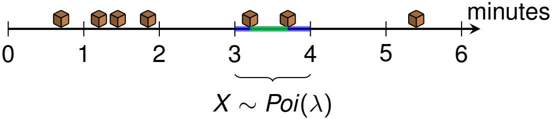

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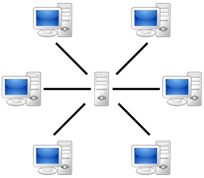

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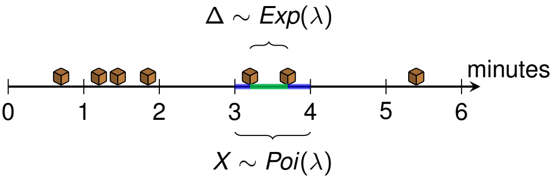

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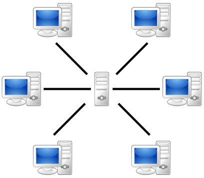

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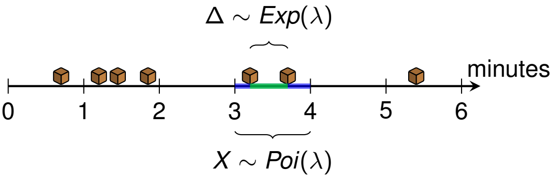
Waiting Time (Lecture 5, Slide 22)



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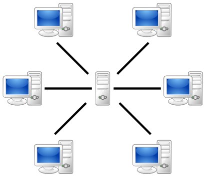
Estimator for λ



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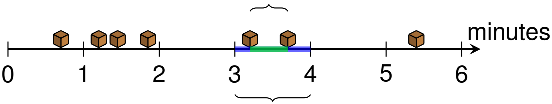

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An Example of an Estimation Problem

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Consider the packages arriving at a network server.

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  2. percentage of minutes during which no packets arrive — Estimator for $e^{-\lambda}$
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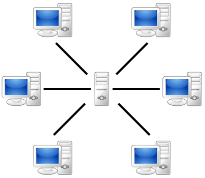

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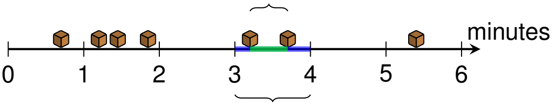

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Definition of Estimator

An estimate is a value t that only depends on the dataset $x_1, x_2, \ldots, x_n$, i.e.,

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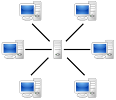
Questions:

- What makes an estimator suitable? $\rightsquigarrow$ unbiased (later: MSE)
- Does an unbiased estimator always exist? How to compute it?
- If there are several unbiased estimators, which one to choose?

Outline

Example: Arrival of Packets (1/3)



- Samples: Given $X_1, X_2, \ldots, X_n$ i.i.d., $X_i \sim Pois(\lambda)$
- Meaning: X_i is the number of packets arriving in minute i

Example 2

Estimate λ by using the sample mean $\overline{X}_n$.

Answer

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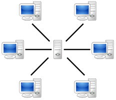

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We have

$$\overline{X}_n := \frac{X_1 + X_2 + \cdots + X_n}{n},$$

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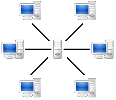

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$$h(X_1, X_2, \ldots, X_n) := \overline{X}_n.$$

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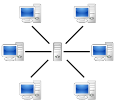
- Samples: Given $X_1, X_2, \ldots, X_n$ i.i.d., $X_i \sim Pois(\lambda)$
- Meaning: X_i is the number of packets arriving in minute i



Example 2

Estimate λ by using the sample mean $\overline{X}_n$.

Answer

We have

$$\overline{X}_n := \frac{X_1 + X_2 + \cdots + X_n}{n},$$

and $\mathbf{E}\left[\overline{X}_n\right] = \mathbf{E}\left[X_1\right] = \lambda$. This suggests the estimator:

$$h(X_1, X_2, \ldots, X_n) := \overline{X}_n.$$

Applying the Weak Law of Large Numbers:

$$\lim_{n\to\infty} \mathbf{P}\left[\left|\overline{X}_n - \lambda\right| > \epsilon\right] = 0 \qquad \text{for any } \epsilon > 0.$$

Example: Arrival of Packets (2/3)

Example 3a

Estimate the probability of zero arrivals, which is $e^{-\lambda}$.

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Then

$$\mathbf{E}[\,Y_i\,] = \mathbf{P}[\,X_i = 0\,] = e^{-\lambda},$$

and thus we can define an estimator by

$$h_1(X_1, X_2, \ldots, X_n) := \frac{Y_1 + Y_2 + \cdots + Y_n}{n} = \overline{Y_n}.$$

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This suggests to estimate $e^{-\lambda}$ by $e^{-\overline{X}_n}$. Hence our estimator is

$$h_2(X_1, X_2, \ldots, X_n) := e^{-\overline{X}_n}.$$

Behaviour of the Estimators

- Suppose we have $n = 30$ and we want to estimate $e^{-\lambda}$
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For most values of λ, both estimators will never return the exact value of $e^{-\lambda}$ on the basis of 30 observations.

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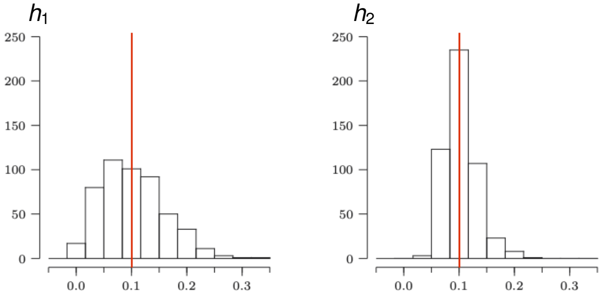

Source: Modern Introduction to Statistics

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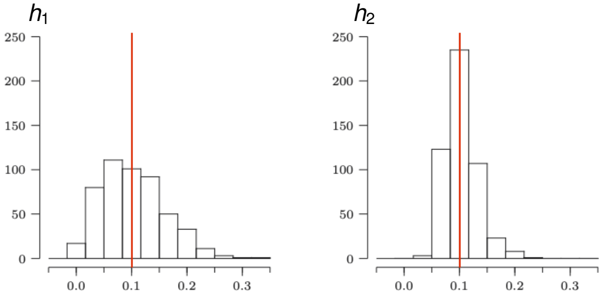

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Both estimators concentrate around the true value 0.1, but the second estimator appears to be more concentrated.

Unbiased Estimators and Bias

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An estimator T is called an unbiased estimator for the parameter θ if

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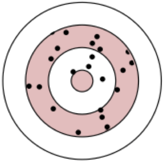

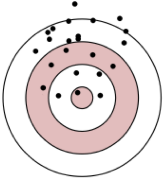

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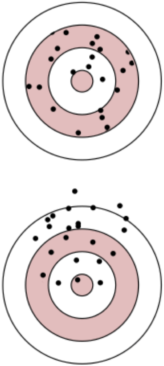

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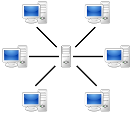
Which of the two estimators h_1, h_2 are unbiased?



Analysis of the Bias of the First Estimator

Example 4a

Is $h_1(X_1, X_2, \ldots, X_n) = \frac{Y_1+Y_2+\cdots+Y_n}{n}$ an unbiased estimator for $e^{-\lambda}$?

Answer

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For any random variable X, and any convex function $g : \mathbb{R} \to \mathbb{R}$, we have

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- This follows by Jensen's inequality, and the inequality is strict since $g : z \mapsto e^{-z}$ is strictly convex and $\overline{X}_n$ is not constant.
- Thus $h_2(X_1, X_2, \ldots, X_n)$ is not unbiased – it has positive bias.

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Asymptotic Bias of the Second Estimator (non-examinable)

Example 4c

$\mathbf{E}\left[\, h_2(X_1,\ldots,X_n) \,\right] \stackrel{n\to\infty}{\longrightarrow} e^{-\lambda}$ (hence it is asymptotically unbiased).

Answer

- Recall $h_2(X_1,\ldots,X_n) = e^{-\overline{X}_n}$. For any $0 \leq k \leq n$,

$$\mathbf{P}\left[\, h_2(X_1,\ldots,X_n) = e^{-k/n} \,\right] = \mathbf{P}\left[\, \sum_{i=1}^{n} X_i = k \,\right] = \mathbf{P}\left[\, Z = k \,\right],$$

where $Z \sim Pois(n\cdot\lambda)$ (since $Pois(\lambda_1) + Pois(\lambda_2) = Pois(\lambda_1+\lambda_2)$)

$$\Rightarrow \qquad \mathbf{P}\left[\, h_2(X_1,\ldots,X_n) = e^{-k/n} \,\right] = \frac{e^{-n\lambda}\cdot(n\lambda)^k}{k!}$$

$$\Rightarrow \qquad \mathbf{E}\left[\, h_2(X_1,\ldots,X_n) \,\right] = \sum_{k=0}^{\infty} e^{-n\lambda}\cdot\frac{(n\lambda^k)}{k!}\cdot e^{-k/n}$$

By LOTUS

$$= e^{-n\lambda}\cdot e^{n\lambda e^{-1/n}} \sum_{k=0}^{\infty} e^{-n\lambda e^{-1/n}}\cdot\frac{(n\lambda e^{-1/n})^k}{k!}$$

$$= e^{-n\lambda\cdot(1-e^{-1/n})}\cdot 1$$

$$\stackrel{n\to\infty}{\approx} e^{-n\lambda\cdot(1-1+1/n+O(1/n^2))} = e^{-\lambda+O(\lambda/n)}.$$

since $e^x = 1 + x + O(x^2)$ for small x

Hence in the limit, the positive bias of h_2 diminishes.

Outline

Introduction

Defining and Analysing Estimators

More Examples

Unbiased Estimator for Expectation and Variance

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$$\overline{X}_n := \frac{X_1 + X_2 + \cdots + X_n}{n}$$
is an unbiased estimator for μ.
- Furthermore, for $n \geq 2$,
$$S_n = S_n(X_1, \ldots, X_n) := \frac{1}{n-1} \cdot \sum_{i=1}^{n} \left(X_i - \overline{X}_n \right)^2$$
is an unbiased estimator for σ^2.

Example 5

We need to prove: $\mathbf{E}[\, S_n \,] = \sigma^2$.

Answer

Multiplying by $n-1$ yields:

$$\begin{aligned}(n-1)\cdot S_n &= \sum_{i=1}^{n}\left(X_i - \overline{X}_n\right)^2 \\ &= \sum_{i=1}^{n}\left(X_i - \mu + \mu - \overline{X}_n\right)^2\end{aligned}$$

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Let us now take expectations:

Example 5

We need to prove: $\mathbf{E}[S_n] = \sigma^2$.

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Multiplying by $n-1$ yields:

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By Lec. 8, Slide 21: $\mathbf{E}\left[(\overline{X}_n-\mu)^2\right] = \mathbf{V}\left[\overline{X}_n\right] = \sigma^2/n$

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An Unbiased Estimator may not always exist

Example 6

Suppose that we have one sample $X \sim Bin(n, p)$, where $0 < p < 1$ is unknown but n is known. Prove there is no unbiased estimator for $1/p$.

Answer

An Unbiased Estimator may not always exist

Example 6

Suppose that we have one sample $X \sim Bin(n, p)$, where $0 < p < 1$ is unknown but n is known. Prove there is no unbiased estimator for $1/p$.

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$$\begin{aligned} \mathbf{E}\left[\, T(X) \,\right] &= \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \cdot T(k) \\ &\leq M \cdot \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = M. \end{aligned}$$

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- The next proof will work even if $p \in [a, b]$ for $0 < a < b \leq 1$.

An Unbiased Estimator may not always exist (cntd. - non-examinable)

Example 6 (cntd.)

Suppose that we have one sample $X \sim Bin(n, p)$, where $0 < p < 1$ is unknown but n is known. Prove there is no unbiased estimator for $1/p$.

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Suppose that we have one sample $X \sim Bin(n, p)$, where $0 < p < 1$ is unknown but n is known. Prove there is no unbiased estimator for $1/p$.

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- Suppose there exists an unbiased estimator with $\mathbf{E}[\,T(X)\,] = 1/p$.

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- Last term is a polynomial of degree $n+1$ with constant term zero

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- Last term is a polynomial of degree $n+1$ with constant term zero $\Rightarrow p \cdot \mathbf{E}[\,T(X)\,] - 1$ is a (non-zero) polynomial of degree $\leq n+1$

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  $\Rightarrow$ $p \cdot \mathbf{E}[T(X)] - 1$ is a (non-zero) polynomial of degree $\leq n + 1$

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  $\Rightarrow$ $\mathbf{E}[T(X)]$ can be equal to $1/p$ for at most $n + 1$ values of p, and thus cannot be an unbiased.