Introduction to Probability

Background Prerequisites: Counting, combinatorics, probability space, axioms

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Outline

Set theory

Counting

Combinatorics

Probability space

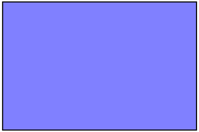
Axioms

Union bound

Problem setting

- Example problem: What is the probability of getting exactly 1 heads in 3 tosses of a fair coin?
- Prerequisites: set theory (language of sets).
- Many basic probability problems are counting problems.

Set theory



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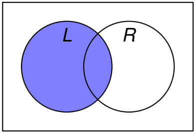

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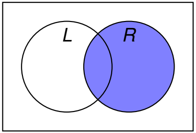

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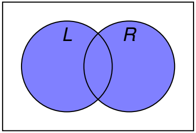
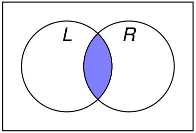


$L \cup R$



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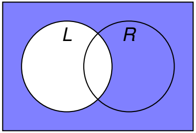
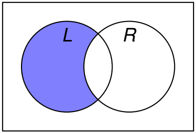


L^c



$L - R$

Outline

What is counting?

- An experiment in probability: experiment $\longrightarrow$ outcome
- Counting: How many possible outcomes can occur from performing this experiment?
- Can be generalised: 2 experiments, two outcomes, what is a joint outcome of 2 experiments?

Example of counting

Example

How many possible outcomes are there when rolling 1 die?

Answer

6 outcomes {1,2,3,4,5,6}

Example of counting

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How many possible outcomes are there when rolling 1 die?

Answer

6 outcomes {1,2,3,4,5,6}

Example

How many possible outcomes are there when rolling 2 dice?

Answer

36 outcomes

$$\begin{aligned}
&\{(1,1),(1,2),\ldots,(1,6),\\
&(2,1),(2,2),\ldots,(2,6),\\
&\vdots\\
&(6,1),(6,2),\ldots,(6,6)\}
\end{aligned}$$

Generalising counting

- r experiments
- experiment 1: n_1 outcomes

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- experiment 2: based on n_1 inputs has n_2 outcomes

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- experiment 3: based on combined outcome of experiment 1 and 2, so $n_1 \cdot n_2$ inputs has n_3 outcomes

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Generalising counting

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  $\vdots$
- total of $n_1 \cdot n_2 \cdots n_r$ possible outcomes of r experiments

Example

Example

University committee consists of 4 UGs, 5 PGs, 7 profs, 2 non-uni people. A subcommittee of 4, consisting of 1 person from each category, is to be chosen. How many different subcommittees are possible?

Answer

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Answer

The choice of a subcommittee is the combined outcome of the 4 separate experiments of choosing a single representative from each of the categories. Thus: $4 \cdot 5 \cdot 7 \cdot 2 = 280$ possible subcommittees.

Sum rule

An experiment has either one of m outcomes or one of n outcomes, where none of the outcomes in both sets are the same. Then there are $m + n$ possible outcomes of the experiment.

Set definition of Sum rule

$|A| = m$ or $|B| = n$ where $A \cap B = \emptyset$
then $\#$ outcomes: $|A| + |B| = m + n$

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Example

I can travel either to Italy to Rome, Naples, Milan, Venice and Florence, or to Spain to Madrid or Barcelona. How many cities can I travel to?

Answer

$|Italy| + |Spain| = 5 + 2 = 7$

Product rule

Experiment has 2 parts. The first part results in one of m outcomes and the second in one of n outcomes regardless of the outcome of the first part. Then there are $m \cdot n$ possible outcomes of the experiment.

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Example

How many possible outcomes are there from rolling two 6-sided dice?

Answer

$|Dice_1| \cdot |Dice_2| = 6 \cdot 6 = 36$

Inclusion-exclusion

The outcome of an experiment can be either from set A or set B where A and B may overlap.

Generalised Sum rule

$|A| = m$ or $|B| = n$ where it may be $A \cap B \neq \emptyset$
then $\#$ outcomes: $|A \cup B| = |A| + |B| - |A \cap B|$

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Example

An 8-bit string is sent over a network. The receiver only accepts strings that either start with 01 or end with 10. How many 8-bit strings will the receiver accept?

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Answer

strings starting with 01 in set A: 01?????? thus	$\lvert A\rvert = 2^6 = 64$
strings ending with 10 in set B: ??????10 thus	$\lvert B\rvert = 2^6 = 64$
overlaping strings $A \cap B$: 01????10 thus	$\lvert A \cap B\rvert = 2^4 = 16$
total:	$\lvert A \cup B\rvert = 64 + 64 - 16 = 112$

General principle of counting

Generalised Product rule

An experiment has r parts such that part i has n_i outcomes for all $i = 1, \ldots, r$. Then the total number of outcomes for the experiment is:

$$\prod_{i=1}^{r} n_i = n_1 \cdot n_2 \cdots n_r$$

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Non-personalised UK licence plates consist of 2 letters, 2 numbers followed by 3 letters. How many possible licence plates can be generated?

Answer

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Answer

Each one of 7 places on the license plate is a separate event, where letters have 26 possibilities and numbers have 10 possibilities.
Total: $26 \cdot 26 \cdot 10 \cdot 10 \cdot 26 \cdot 26 \cdot 26 = 1{,}188{,}137{,}600$

Pigeonhole principle

Pigeonhole principle

If m objects are place into n buckets, then at least one bucket has at least $\lceil \frac{m}{n} \rceil$ objects.

Reminder:
$\lceil X \rceil$: ceiling – smallest integer that is bigger than X
$\lfloor X \rfloor$: floor – largest integer that is smaller than X

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10 pigeons are placed into 9 pigeonholes. How many pigeons are placed in any one pigeonhole at most?

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Answer

At least one pigeonhole must contain $\lceil \frac{m}{n} \rceil = 2$ pigeons.

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Permutations

Permutation is a counting task of sorting n objects.

Permutation rule (distinct)

A permutation is an ordered arrangement of n distinct objects. Then the number of ways in which these n objects can be permuted (put into unique orderings) is:

$$n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1 = n!$$

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Example

Consider the acronym CAM. How many different ordered arrangements of the letters C, A and M are possible?

Answer

$\{(A, C, M), (A, M, C), (C, A, M), (C, M, A), (M, A, C), (M, C, A)\}$, thus 6 possible permutations, i.e., $3! = 3 \cdot 2 \cdot 1$.

Indistinct permutations

Permutation of indistinct objects

There are n objects and n_1 are the same (indistinguishable), n_2 are the same, ..., n_r are the same. Then the number of distinct permutations of these n objects is:

$$\frac{n!}{n_1! \cdot n_2! \cdots n_r!}$$

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Example

How many distinct bit strings can be formed from two 0's and three 1's?

Answer

$$\frac{5!}{2! \cdot 3!} = \frac{120}{2 \cdot 6} = 10.$$

Combinations

Combinations for one group

A combination in an unordered selection of r objects from a set of n objects. If all objects are distinct, then the number of ways of making the selection is:

$$\frac{n!}{r!(n-r)!} = \binom{n}{r}$$

Reminder: note that $\binom{n}{r}$ is a a binomial coefficient, read as "n choose r".

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$(n-r)!$ ways to permute nonselected objects

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Example

How many ways are there to select 3 unordered objects from a set of 7 objects?

Answer

$$\frac{n!}{r!(n-r)!} = \binom{7}{3} = \frac{7!}{3!4!} = 35$$

Example of counting combinations

Example

How many ways are there to select 3 books from a set of 6 books, if there are two books that should not both be chosen together? For example, you are choosing 3 out of 6 probability books, but don't want to choose both the 8th and 9th edition of the Ross textbook.

Answer

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Case 1: Select 8th Ed and 2 other non-9th Ed – $\binom{4}{2}$ ways to do so.

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Case 1: Select 8th Ed and 2 other non-9th Ed – $\binom{4}{2}$ ways to do so.

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Case 3: Select 3 from books that are not 8th nor 9th Ed – $\binom{4}{3}$ ways to do so.

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Case 3: Select 3 from books that are not 8th nor 9th Ed – $\binom{4}{3}$ ways to do so.

Total: using Sum Rule of counting, we get $\binom{4}{2} + \binom{4}{2} + \binom{4}{3} = 6 + 6 + 4 = 16$.

Multinomial combinations

Combinations for multiple groups of objects

If there are n distinct objects, then the number of ways of selecting r distinct groups of respective sizes $n_1, n_2, \ldots, n_r$ such that $\sum_{i=1}^{r} n_i = n$ is:

$$\frac{n!}{n_1!n_2!\cdots n_r!} = \binom{n}{n_1, n_2, \ldots, n_r}$$

where $\binom{n}{n_1, n_2, \ldots, n_r}$ is known as multinomial coefficient.

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Example

There are 13 children on the playground who need to be split into 3 groups of sizes 6, 4 and 3. How many different divisions are possible?

Answer

$$\binom{13}{6, 4, 3} = \frac{13!}{6!4!3!} = 60060$$

Multinomial example

Example

In order to organise a basketball tournament, 20 children at a playground divide themselves in 4 teams of 5 players. How many different divisions are possible?

Answer

Multinomial example

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In order to organise a basketball tournament, 20 children at a playground divide themselves in 4 teams of 5 players. How many different divisions are possible?

Answer

The answer is **NOT**

$$\binom{20}{5,5,5,5}$$

because the order of the four teams is irrelevant. It would be correct if being in team A were considered different from being in team D. But here we are only interested in the possible divisions, so since there are 4! permutations between team “labels", the answer is

$$\frac{\binom{20}{5,5,5,5}}{4!} = \binom{20}{5,5,5,5,4}$$

Summary of combinatorics

Counting tasks on n objects (without replacement)			
Permutations (sort objects)		Combinations (choose r objects)	
Distinct	Indistinct	Distinct 1 group	Distinct k groups
$n!$	$\frac{n!}{n_1! \cdot n_2! \cdots n_r!}$	$\binom{n}{r} = \frac{n!}{r!(n-r)!}$	$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1! n_2! \cdots n_k!}$

Useful identity: $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$ where $1 \leq r \leq n$

Binomial theorem: $(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$

Outline

Random experiments

- Randomness is described by conducting experiments (or trials) with uncertain outcomes.
- Sample space S: a set of all possible outcomes of an experiment.
- Event E: some subset of S, i.e., $E \subseteq S$.
- Probability $\mathbf{P}$ is a number between 0 and 1 to which we ascribe a meaning: our belief that an event E occurs: $\mathbf{P}[E] \in [0, 1]$.

Sample spaces

Sample space

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Examples

Give sample spaces for the following:

1. Gender of a newborn child
2. Flipping of 2 coins
3. Rolling 2 dice
4. YouTube hours in a day

Answer

1. $S = \{G, B\}$
2. $S = \{(H,H), (H,T), (T,H), (T,T)\}$
3. $S = \{(i,j) : i,j \in \{1,2,3,4,5,6\}\}$
4. $S = \{x : x \in \mathbb{R}, 0 \leq x \leq 24\}$

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Examples

Give event spaces for the following:

1. A newborn child is a girl.
2. There is 1 or more heads on 2 coin flips.
3. At least one of the numbers is a 6 in a rolling of 2 dice.
4. Wasted day where 5 or more hours have been spent on YT.

Answer

1. $E = \{G\}$
2. $E = \{(H, H), (H, T), (T, H)\}$
3. $E = \{(6,1),(6,2),(6,3),(6,4),(6,5),(6,6),(1,6),(2,6),(3,6),(4,6),(5,6)\}$
4. $E = \{x : x \in \mathbb{R}, 5 \leq x \leq 24\}$

Set operations on events

Given event space S and events E and F:

Union: $E \cup F$ is the event containing all outcomes of E **or** F.
$E = \{(H, H), (H, T)\}$ and $F = \{(H, T), (T, T)\}$ then
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Intersection: $E \cap F$ (also denoted EF) is the event containing all outcomes of E **and** F.
$E = \{(H, H), (H, T)\}$ and $F = \{(H, T), (T, T)\}$ then
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Complement: E^c is the event containing all outcomes in S that are **not** in E.
Note, thus we have $E \cup E^c = S$ and $E \cap E^c = \emptyset$.
$S = \{(H, H), (H, T), (T, H), (T, T)\}$ and $E = \{(H, H), (H, T)\}$
then $E^c = \{(T, H), (T, T)\}$

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The usual commutative, associative and distributive laws hold.
De Morgan's laws: $(\bigcup_{i=1}^n E_i)^c = \bigcap_{i=1}^n E_i^c$ and $(\bigcap_{i=1}^n E_i)^c = \bigcup_{i=1}^n E_i^c$

Outline

Probability definition

Frequentist definition of probability

$$\mathbf{P}[E] = \lim_{n \to \infty} \frac{n(E)}{n}$$

where $n = \#$ of total trials and $n(E) = \#$ trials where E occurs.

Interpretation of probability:

- Probability of desired event E is the ratio of the # of trials that result in an outcome in E to the number of trials performed (in the limit as your number of trials approaches infinity).
- $\mathbf{P}[E]$ is a measure of the chance of E occurring.
- Often probability is a measure of the individual's degree of belief of E occurring (Bayesian definition).
- Interpretation is a mess, a philosophical argument.
- Choice of interpretation doesn't matter, as long as the axioms of probability hold.

Probability axioms

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Axiom 2: Probability of the sample space S is $\mathbf{P}[S] = 1$

Axiom 3: If E and F are mutually exclusive ($E \cap F = \emptyset$), then
$\mathbf{P}[E] + \mathbf{P}[F] = \mathbf{P}[E \cup F]$.
In general, for all mutually exclusive events $E_1, E_2, \ldots$

$$\mathbf{P}\left[\bigcup_{i=1}^{\infty} E_i\right] = \sum_{i=1}^{\infty} \mathbf{P}[E_i]$$

Probability identities

Proposition 1: $\mathbf{P}\left[E^c \right] = 1 - \mathbf{P}[E] = \mathbf{P}[S] - \mathbf{P}[E]$

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Proposition 3: $\mathbf{P}[E \cup F] = \mathbf{P}[E] + \mathbf{P}[F] - \mathbf{P}[EF]$

Probability identities

Proposition 1: $\mathbf{P}\left[\, E^c \,\right] = 1 - \mathbf{P}[\, E \,] = \mathbf{P}[\, S \,] - \mathbf{P}[\, E \,]$

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Proposition 4 (general inclusion-exclusion principle):

$$\mathbf{P}\left[\bigcup_{i=1}^{n} E_i\right] = \sum_{r=1}^{n} (-1)^{r+1} \sum_{i_1 < i_2 < \cdots < i_r}^{n} \mathbf{P}[\, E_{i1} \cap \cdots \cap E_{i_r} \,]$$

(Proofs in book).

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Probability with equally likely outcomes

For sample space S in which all outcomes are equally likely, we have $\mathbf{P}[\text{each outcome}] = \frac{1}{|S|}$ and for any event $E \subseteq S$,

$$\mathbf{P}[E] = \frac{\#\text{ outcomes in } E}{\#\text{ outcomes in } S} = \frac{|E|}{|S|}$$

Examples

Example

You order 2 dishes online with probability of 0.6 of liking the first dish, 0.4 of liking the second dish, and 0.3 of liking both dishes. What is the probability you will like neither dish?

Answer

E_i: event "you like dish i".

$$\mathbf{P}[\text{you will like neither dish}] = \mathbf{P}[(E_1 \cup E_2)^c] = 1 - \mathbf{P}[E_1 \cup E_2] = 1 - (\mathbf{P}[E_1] + \mathbf{P}[E_2] - \mathbf{P}[E_1 \cap E_2]) = 1 - (0.6 + 0.4 - 0.3) = 0.3$$

Examples

Example

3 people are randomly selected from a group of 11 people which is made of 5 women and 6 men. What is the probability that 2 women and 1 man are selected?

Answer

$S = \binom{11}{3}$ are all subsets of size 3 from 11 people.
Random selection means each subset is equally likely.
$\binom{5}{2}\binom{6}{1}$ are all subsets with 2 women and 1 man.

$$\mathbf{P}\left[\,2 \text{ women,} 1 \text{ man}\,\right] = \frac{\binom{5}{2}\binom{6}{1}}{\binom{11}{3}} = \frac{4}{11}$$

Birthday paradox

Example

If n people are in a room, what is the probability that 2 have the same birthday? (Assume that there are 365 days and probability of being born on a given day is $\frac{1}{365}$).

Answer

Simpler to calculate probability that "no two people in the room have the same birthday" ($= \mathbf{P}[E_n^c]$) where $E_n =$"two people have birthday on the same day", and then use $\mathbf{P}[E_n] = 1 - \mathbf{P}[E_n^c]$.

$$|S| = 365^n$$

$$|E_n^c| = 365 \cdot 364 \cdots (365 - n + 1) \text{ (\# of ways to have no two people with the same bday)}$$

$$\mathbf{P}\left[E_n^c\right] = \frac{365 \cdot 364 \cdots (365 - n + 1)}{365^n}$$

$$\mathbf{P}[E_n] = 1 - \frac{365 \cdot 364 \cdots (365 - n + 1)}{365^n} \text{ (\# of ways two people have the same bday)}$$

$$\text{if } n = 23 \text{ then } \mathbf{P}[E_{23}] = 50.7\%$$

$$\text{if } n = 70 \text{ then } \mathbf{P}[E_{70}] = 99.9\%$$

Outline

Set theory

Counting

Combinatorics

Probability space

Axioms

Union bound

Boole's inequality

Union bound AKA Boole's inequality

For any events $E_1, E_2, \ldots, E_n$ we have

$$\mathbf{P}\left[\bigcup_{i=1}^{n} E_i\right] \leq \sum_{i=1}^{n} \mathbf{P}\left[\, E_i \,\right]$$

For E_1 and E_2 it is easy to see:

$$\mathbf{P}\left[\, E_1 \cup E_2 \,\right] = \mathbf{P}\left[\, E_1 \,\right] + \mathbf{P}\left[\, E_2 \,\right] - \mathbf{P}\left[\, E_1 \cap E_2 \,\right] \leq \mathbf{P}\left[\, E_1 \,\right] + \mathbf{P}\left[\, E_2 \,\right].$$

Useful in applications that need to show that the probability of union for some events is less than some value.

E.g., in random graphs that are used to analyse social networks, wireless networks, the internet: given nodes and edges with associated probabilities, what is the probability that there exists an isolated node in the graph that is not connected to any other nodes in the graph.

Summary of probability problems

- Find the sample space S.
- Define events of interest E.
- Determine outcome probabilities.
- Compute event probabilities.