University of Cambridge
2022/23 Part II / Part III / MPhil ACS
Category Theory
Exercise Sheet 3
by Andrew Pitts

1. Show that for any objects X and Y in a cartesian closed category $\mathbf{C}$, there are functions

$$f \in \mathbf{C}(X,Y) \mapsto \ulcorner f \urcorner \in \mathbf{C}(1, Y^X)$$
$$g \in \mathbf{C}(1, Y^X) \mapsto \overline{g} \in \mathbf{C}(X,Y)$$

that give a bijection between the set $\mathbf{C}(X,Y)$ of $\mathbf{C}$-morphisms from X to Y and the set $\mathbf{C}(1, Y^X)$ of $\mathbf{C}$-morphisms from the terminal object 1 to the exponential Y^X. [Hint: use the isomorphism (7) from Exercise Sheet 2, question 2.]

2. Show that for any objects X and Y in a cartesian closed category $\mathbf{C}$, the morphism $\mathtt{app} : Y^X \times X \to Y$ satisfies $\mathtt{cur}(\mathtt{app}) = \mathtt{id}_{Y^X}$. [Hint: recall from equation (4) on Exercise Sheet 2 that $\mathtt{id}_{Y^X} \times \mathtt{id}_X = \mathtt{id}_{Y^X \times X}$.]

3. Suppose $f : Y \times X \to Z$ and $g : W \to Y$ are morphisms in a cartesian closed category $\mathbf{C}$. Prove that

$$\mathtt{cur}(f \circ (g \times \mathtt{id}_X)) = (\mathtt{cur}\, f) \circ g \in \mathbf{C}(W, Z^X) \tag{1}$$

[Hint: use Exercise Sheet 2, question 1c.]

4. Let $\mathbf{C}$ be a cartesian closed category. For each $\mathbf{C}$-object X and $\mathbf{C}$-morphism $f : Y \to Z$, define

$$f^X \triangleq \mathtt{cur}(Y^X \times X \xrightarrow{\mathtt{app}} Y \xrightarrow{f} Z) \in \mathbf{C}(Y^X, Z^X) \tag{2}$$

   (a) Prove that $(\mathtt{id}_Y)^X = \mathtt{id}_{Y^X}$.

   (b) Given $f \in \mathbf{C}(Y \times X, Z)$ and $g \in \mathbf{C}(Z, W)$, prove that

$$\mathtt{cur}(g \circ f) = g^X \circ \mathtt{cur}\, f \in \mathbf{C}(Y, W^X) \tag{3}$$

   (c) Deduce that if $u \in \mathbf{C}(Y,Z)$ and $v \in \mathbf{C}(Z,W)$, then $(v \circ u)^X = v^X \circ u^X \in \mathbf{C}(Y^X, W^X)$.

   [Hint: for part (4a) use question 2; for part (4b) use Exercise Sheet 2, question 1c.]

5. Let $\mathbf{C}$ be a cartesian closed category. For each $\mathbf{C}$-object X and $\mathbf{C}$-morphism $f : Y \to Z$, define

$$X^f \triangleq \mathtt{cur}(X^Z \times Y \xrightarrow{\mathtt{id} \times f} X^Z \times Z \xrightarrow{\mathtt{app}} X) \in \mathbf{C}(X^Z, X^Y) \tag{4}$$

   (a) Prove that $X^{\mathtt{id}_Y} = \mathtt{id}_{X^Y}$.

   (b) Given $g \in \mathbf{C}(W, X)$ and $f \in \mathbf{C}(Y \times X, Z)$, prove that

$$\mathtt{cur}(f \circ (\mathtt{id}_Y \times g)) = Z^g \circ \mathtt{cur}\, f \in \mathbf{C}(Y, Z^W) \tag{5}$$

   (c) Deduce that if $u \in \mathbf{C}(Y,Z)$ and $v \in \mathbf{C}(Z,W)$, then $X^{(v \circ u)} = X^u \circ X^v \in \mathbf{C}(X^W, X^Y)$.

[Hint: for part (5a) use question 2; for part (5b) use Exercise Sheet 2, question 1c.]

6. Let $\mathbf{C}$ be a cartesian closed category in which every pair of objects X and Y possesses a binary coproduct $X \xrightarrow{\mathtt{inl}_{X,Y}} X + Y \xleftarrow{\mathtt{inr}_{X,Y}} Y$. For all objects $X, Y, Z \in \mathbf{C}$ construct an isomorphism $(Y + Z) \times X \cong (Y \times X) + (Z \times X)$. [Hint: you may find it helpful to use some of the properties from question 4.]

7. Using the natural deduction rules for Intuitionistic Propositional Logic (given in Lecture 6), give proofs of the following judgements. In each case write down a corresponding typing judgement of the Simply Typed Lambda Calculus.

   (a) $\diamond, \psi \vdash (\varphi \Rightarrow \psi) \Rightarrow \psi$

   (b) $\diamond, \varphi \vdash (\varphi \Rightarrow \psi) \Rightarrow \psi$

   (c) $\diamond, ((\varphi \Rightarrow \psi) \Rightarrow \psi) \Rightarrow \psi \vdash \varphi \Rightarrow \psi$

8. (a) Given simple types A, B, C, give terms s and t of the Simply Typed Lambda Calculus that satisfy the following typing and $\beta\eta$-equality judgements:

$$\diamond, x : (A \times B) \to C \vdash s : A \to (B \to C) \tag{6}$$

$$\diamond, y : A \to (B \to C) \vdash t : (A \times B) \to C \tag{7}$$

$$\diamond, x : (A \times B) \to C \vdash t[s/y] =_{\beta\eta} x : (A \times B) \to C \tag{8}$$

$$\diamond, y : A \to (B \to C) \vdash s[t/x] =_{\beta\eta} y : A \to (B \to C) \tag{9}$$

   (b) Explain why question (8a) implies that for any three objects X,Y and Z in a cartesian closed category $\mathbf{C}$, there are morphisms

$$f : Z^{(X \times Y)} \to (Z^Y)^X \tag{10}$$

$$g : (Z^Y)^X \to Z^{(X \times Y)} \tag{11}$$

   that give an isomorphism $Z^{(X \times Y)} \cong (Z^Y)^X$ in $\mathbf{C}$.

9. Make up and solve a question like question 8 ending with an isomorphism $X^1 \cong X$ for any object X in a cartesian closed category $\mathbf{C}$ (with terminal object 1).

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Question 1 Recalling the isomorphism $1 \times X \cong X$ from question 2 on Exercise Sheet 2, define

$$\ulcorner f \urcorner = \mathtt{cur}(1 \times X \xrightarrow[\cong]{\pi_2} X \xrightarrow{f} Y)$$
$$\overline{g} = X \xrightarrow[\cong]{\langle\langle\rangle, \mathtt{id}_X\rangle} 1 \times X \xrightarrow{e \times \mathtt{id}_X} Y^X \times X \xrightarrow{\mathtt{app}} Y$$

Thus

$$\begin{aligned}
\ulcorner \overline{g} \urcorner &= \mathtt{cur}(\mathtt{app} \circ (g \times \mathtt{id}_X) \circ \langle\langle\rangle, \mathtt{id}_X\rangle \circ \pi_2) \\
&= \mathtt{cur}(\mathtt{app} \circ (g \times \mathtt{id}_X)) \quad \text{since } \pi_2 : 1 \times X \to X \text{ is an iso with inverse } \langle\langle\rangle, \mathtt{id}_X\rangle \\
&= g \quad \text{by the uniqueness part of the universal property of exponentials}
\end{aligned}$$

and

$$\begin{aligned}
\overline{\ulcorner f \urcorner} &= \mathtt{app} \circ (\mathtt{cur}(f \circ \pi_2) \times \mathtt{id}_X) \circ \langle\langle\rangle, \mathtt{id}_X\rangle \\
&= f \circ \pi_2 \circ \langle\langle\rangle, \mathtt{id}_X\rangle \quad \text{by definition of } \mathtt{cur}(f \circ \pi_2) \\
&= f \quad \text{since } \pi_2 : 1 \times X \to X \text{ is an iso with inverse } \langle\langle\rangle, \mathtt{id}_X\rangle
\end{aligned}$$

Question 2 By definition, $\mathtt{cur}(\mathtt{app})$ is the unique morphism $f \in \mathbf{C}(Y^X, Y^X)$ satisfying $\mathtt{app} \circ (f \times \mathtt{id}_X) = \mathtt{app}$. But from Exercise Sheet 2 question 1c, we have $\mathtt{id}_{Y^X} \times \mathtt{id}_X = \mathtt{id}_{Y^X \times X}$ and hence $\mathtt{app} \circ (f \times \mathtt{id}_X) = \mathtt{app}$ also holds when $f = \mathtt{id}_{Y^X}$. Therefore $\mathtt{id}_{Y^X} = \mathtt{cur}(\mathtt{app})$.

Question 3 Note that

$$\begin{aligned}
\mathtt{app} \circ (((\mathtt{cur}\, f) \circ g) \times \mathtt{id}_X) &= \mathtt{app} \circ (\mathtt{cur}\, f \times \mathtt{id}_X) \circ (g \times \mathtt{id}_X) && \text{by Ex. Sh. 2, question 1c} \\
&= f \circ (g \times \mathtt{id}_X) && \text{by definition of } \mathtt{cur}\, f
\end{aligned}$$

and therefore $(\mathtt{cur}\, f) \circ g = \mathtt{cur}(f \circ (g \times \mathtt{id}_X))$, by the uniqueness part of the universal property of exponentials.

Question 4

(a) $(\mathtt{id}_Y)^X \triangleq \mathtt{cur}(\mathtt{id}_Y \circ \mathtt{app}) = \mathtt{cur}(\mathtt{app}) = \mathtt{id}_{Y^X}$, by question 2.

(b) $$\begin{aligned}
\mathtt{app} \circ ((g^X \circ \mathtt{cur}\, f) \times \mathtt{id}_X) &= \mathtt{app} \circ (g^X \times \mathtt{id}_X) \circ (\mathtt{cur}\, f \times \mathtt{id}_X) && \text{by Ex. Sh. 2, question 1c} \\
&= g \circ \mathtt{app} \circ (\mathtt{cur}\, f \times \mathtt{id}_X) && \text{by definition of } g^X \\
&= g \circ f && \text{by definition of } \mathtt{cur}\, f
\end{aligned}$$

and therefore $g^X \circ \mathtt{cur}\, f = \mathtt{cur}(g \circ f)$, by the uniqueness part of the universal property of exponentials.

(c) $$\begin{aligned} g^X \circ f^X &= g^X \circ \mathtt{cur}(f \circ \mathrm{app}) && \text{by definition of } f^X \\ &= \mathtt{cur}(g \circ f \circ \mathrm{app}) && \text{by part (b)} \\ &\triangleq (g \circ f)^X \end{aligned}$$

Question 5

(a) $X^{\mathtt{id}_Y} \triangleq \mathtt{cur}(\mathtt{app} \circ (\mathtt{id}_{X^Y} \times \mathtt{id}_Y)) = \mathtt{cur}(\mathtt{app} \circ \mathtt{id}_{X^Y \times Y}) = \mathtt{cur}(\mathtt{app}) = \mathtt{id}_{X^Y}$, by question 2.

(b) $$\begin{aligned} \mathtt{app} \circ ((Z^g \circ \mathtt{cur}\, f) \times \mathtt{id}_W) &= \mathtt{app} \circ (Z^g \times \mathtt{id}_W) \circ (\mathtt{cur}\, f \times \mathtt{id}_W) && \text{by Ex.Sh. 2, question 1c} \\ &= \mathtt{app} \circ (\mathtt{id}_Y \times g) \circ (\mathtt{cur}\, f \times \mathtt{id}_W) && \text{by definition of } Z^g \\ &= \mathtt{app} \circ (\mathtt{cur}\, f \times \mathtt{id}_X) \circ (\mathtt{id}_Y \times g) && \text{by Ex.Sh. 2, question 1c} \\ &= f \circ (\mathtt{id}_Y \times g) && \text{by definition of } \mathtt{cur}\, f \end{aligned}$$

and therefore $Z^g \circ \mathtt{cur}\, f = \mathtt{cur}(f \circ (\mathtt{id}_Y \times g))$, by the uniqueness part of the universal property of exponentials.

(c) $$\begin{aligned} X^u \circ X^v &= X^u \circ \mathtt{cur}(\mathtt{app} \circ (\mathtt{id} \times v)) && \text{by definition of } X^v \\ &= \mathtt{cur}(\mathtt{app} \circ (\mathtt{id} \times v) \circ (\mathtt{id} \times u)) && \text{by part (b)} \\ &= \mathtt{cur}(\mathtt{app} \circ (\mathtt{id} \times (v \circ u))) && \text{by Ex.Sh. 2, question 1c} \\ &\triangleq X^{(v \circ u)} \end{aligned}$$

Question 6 The universal property of the coproduct $X + Y$ says that for all $f \in \mathbf{C}(X, Z)$ and $g \in \mathbf{C}(Y, Z)$ there is a unique morphism $[f, g] \in \mathbf{C}(X + Y, Z)$ with $[f, g] \circ \mathtt{inl}_{X,Y} = f$ and $[f, g] \circ \mathtt{inr}_{X,Y} = g$. Given objects $X, Y, Z \in \mathbf{C}$, from

$$\begin{aligned} \mathtt{cur}(\mathtt{inl}_{Y \times X, Z \times X}) &: Y \to ((Y \times X) + (Z \times X))^X \\ \mathtt{cur}(\mathtt{inr}_{Y \times X, Z \times X}) &: Z \to ((Y \times X) + (Z \times X))^X \end{aligned}$$

we get

$$[\mathtt{cur}(\mathtt{inl}_{Y \times X, Z \times X}), \mathtt{cur}(\mathtt{inr}_{Y \times X, Z \times X})] : Y + Z \to ((Y \times X) + (Z \times X))^X$$

and hence

$$i \triangleq \mathtt{app} \circ ([\mathtt{cur}(\mathtt{inl}_{Y \times X, Z \times X}), \mathtt{cur}(\mathtt{inr}_{Y \times X, Z \times X})] \times \mathtt{id}_X) \in \mathbf{C}((Y + Z) \times X, (Y \times X) + (Z \times X))$$

In the other direction, define

$$j \triangleq [\mathtt{inl}_{Y,Z} \times \mathtt{id}_X, \mathtt{inr}_{Y,Z} \times \mathtt{id}_X] \in \mathbf{C}((Y \times X) + (Z \times X), (Y + Z) \times X)$$

To see that $i \circ j = \mathtt{id}$, note that

$$\begin{aligned} i \circ j \circ \mathtt{inl} &= i \circ (\mathtt{inl} \times \mathtt{id}) && \text{by definition of } j \\ &= \mathtt{app} \circ ([\mathtt{cur}\, \mathtt{inl}, \mathtt{cur}\, \mathtt{inr}] \times \mathtt{id}) \circ (\mathtt{inl} \times \mathtt{id}) && \text{by definition of } i \\ &= \mathtt{app} \circ (([\mathtt{cur}\, \mathtt{inl}, \mathtt{cur}\, \mathtt{inr}] \circ \mathtt{inl}) \times \mathtt{id}) && \text{by Ex.Sh. 2, question 1c} \\ &= \mathtt{app} \circ (\mathtt{cur}\, \mathtt{inl} \times \mathtt{id}) && \text{by definition of } [\_, \_] \\ &= \mathtt{inl} && \text{by definition of } \mathtt{cur}\, \_ \\ &= \mathtt{id} \circ \mathtt{inl} \end{aligned}$$

and similarly, $i \circ j \circ \texttt{inr} = \texttt{id} \circ \texttt{inr}$; therefore by the uniqueness part of the universal property for coproducts we have $i \circ j = \texttt{id}$. To see that $j \circ i = \texttt{id}$, note that

$$\begin{array}{rll}
\texttt{cur}(j \circ i) & = j^X \circ \texttt{cur}\, i & \text{by (3)} \\
& = j^X \circ [\texttt{cur}\,\texttt{inl}, \texttt{cur}\,\texttt{inr}] & \text{by definition of } i \\
& = [j^X \circ \texttt{cur}\,\texttt{inl}, j^X \circ \texttt{cur}\,\texttt{inr}] & \text{by the dual of property (1) for products from Ex.Sh. 2} \\
& = [\texttt{cur}(j \circ \texttt{inl}), \texttt{cur}(j \circ \texttt{inr})] & \text{by (3)} \\
& = [\texttt{cur}(\texttt{inl} \times \texttt{id}), \texttt{cur}(\texttt{inr} \times \texttt{id})] & \text{by definition of } j \\
& = [(\texttt{cur}\,\texttt{id}) \circ \texttt{inl}, (\texttt{cur}\,\texttt{id}) \circ \texttt{inr}] & \text{by (1)} \\
& = (\texttt{cur}\,\texttt{id}) \circ [\texttt{inl}, \texttt{inr}] & \text{by the dual of property (1) for products from Ex.Sh. 2} \\
& = (\texttt{cur}\,\texttt{id}) \circ \texttt{id} & \text{by uniqueness part of univ. property of coproducts} \\
& = \texttt{cur}\,\texttt{id} &
\end{array}$$

and hence $j \circ i = \texttt{app}(\texttt{cur}(j \circ i) \times \texttt{id}) = \texttt{app}(\texttt{cur}\,\texttt{id} \times \texttt{id}) = \texttt{id}$.

Question 7

(a) IPL proof tree

$$\dfrac{\dfrac{\dfrac{}{\diamond, \psi \vdash \psi}\,(\textsc{ax})}{\diamond, \psi, \varphi \Rightarrow \psi \vdash \psi}\,(\textsc{wk})}{\diamond, \psi \vdash (\varphi \Rightarrow \psi) \Rightarrow \psi}\,(\Rightarrow\textsc{i})$$

STLC typing judgement $\diamond, y : \psi \vdash \lambda f : \varphi \Rightarrow \psi.\, y : (\varphi \Rightarrow \psi) \Rightarrow \psi$

(b) IPL proof tree

$$\dfrac{\dfrac{\dfrac{}{\diamond, \varphi, \varphi \Rightarrow \psi \vdash \varphi \Rightarrow \psi}\,(\textsc{ax}) \qquad \dfrac{\dfrac{}{\diamond, \varphi \vdash \varphi}\,(\textsc{ax})}{\diamond, \varphi, \varphi \Rightarrow \psi \vdash \varphi}\,(\textsc{wk})}{\diamond, \varphi, \varphi \Rightarrow \psi \vdash \psi}\,(\Rightarrow\textsc{e})}{\diamond, \varphi \vdash (\varphi \Rightarrow \psi) \Rightarrow \psi}\,(\Rightarrow\textsc{i})$$

STLC typing judgement $\diamond, y : \psi \vdash \lambda f : \varphi \Rightarrow \psi.\, f\, x : (\varphi \Rightarrow \psi) \Rightarrow \psi$

(c) IPL proof tree, where $\theta \triangleq ((\varphi \Rightarrow \psi) \Rightarrow \psi) \Rightarrow \psi$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\diamond, \theta \vdash \theta}\,(\textsc{ax})}{\diamond, \theta, \varphi \vdash \theta}\,(\textsc{wk}) \qquad \dfrac{\dfrac{\dfrac{}{\diamond, \theta, \varphi, \varphi \Rightarrow \psi \vdash \varphi \Rightarrow \psi}\,(\textsc{ax}) \qquad \dfrac{\dfrac{}{\diamond, \theta, \varphi \vdash \varphi}\,(\textsc{ax})}{\diamond, \theta, \varphi, \varphi \Rightarrow \psi \vdash \varphi}\,(\textsc{wk})}{\diamond, \theta, \varphi, \varphi \Rightarrow \psi \vdash \psi}\,(\Rightarrow\textsc{e})}{\diamond, \theta, \varphi \vdash (\varphi \Rightarrow \psi) \Rightarrow \psi}\,(\Rightarrow\textsc{i})}{\diamond, \theta, \varphi \vdash \psi}\,(\Rightarrow\textsc{e})}{\diamond, \theta \vdash \varphi \Rightarrow \psi}\,(\Rightarrow\textsc{i})$$

STLC typing judgement $\diamond, f : \theta \vdash \lambda x : \varphi.\, f(\lambda g : \varphi \Rightarrow \psi.\, g\, x) : \varphi \Rightarrow \psi$

Question 8

(a) $s \triangleq \lambda a : A.\, \lambda b : B.\, x\,(a, b)$

$t \triangleq \lambda c : A \times B.\, y\,(\texttt{fst}\,c)\,(\texttt{snd}\,c)$

Proof of (6), where $\Gamma \triangleq \diamond, x : (A \times B) \to C, a : A, b : B$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\cdots}\,(\textsc{var})}{\cdots}\,(\textsc{var'})}{\Gamma \vdash x : A \times B \to C}\,(\textsc{var'}) \qquad \dfrac{\dfrac{\dfrac{}{\cdots}\,(\textsc{var})}{\Gamma \vdash a : A}\,(\textsc{var'}) \quad \dfrac{}{\Gamma \vdash b : B}\,(\textsc{var})}{\Gamma \vdash (a, b) : A \times B}\,(\textsc{pair})}{\Gamma \vdash x(a, b) : C}\,(\textsc{app})}{\diamond, x : (A \times B) \to C \vdash s : A \to (B \to C)}\,(\lambda^2)
$$

Proof of (7), where $\Gamma' \triangleq \diamond, y : A \to (B \to C), c : A \times B$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\cdots}\,(\textsc{var})}{\Gamma' \vdash y : A \to (B \to C)}\,(\textsc{var'}) \qquad \dfrac{\dfrac{}{\Gamma' \vdash c : A \times B}\,(\textsc{var})}{\Gamma' \vdash \texttt{fst}\,c : A}\,(\textsc{fst})}{\Gamma' \vdash y\,(\texttt{fst}\,c) : B \to C}\,(\textsc{app}) \qquad \dfrac{\dfrac{}{\Gamma' \vdash c : A \times B}\,(\textsc{var})}{\Gamma' \vdash \texttt{snd}\,c : B}\,(\textsc{snd})}{\Gamma' \vdash y\,(\texttt{fst}\,c)\,(\texttt{snd}\,c) : C}\,(\textsc{app})}{\diamond, y : A \to (B \to C) \vdash t : (A \times B) \to C}\,(\lambda)
$$

Proof of (8) (not laid out as a tree):

$$
\begin{array}{lll}
t[s/y] & \triangleq \lambda c : A \times B.\, (\lambda a : A. \lambda b : B.\, x\,(a, b))\,(\texttt{fst}\,c)\,(\texttt{snd}\,c) & \\
& =_{\beta\eta} \lambda c : A \times B.\, x\,(\texttt{fst}\,c, \texttt{snd}\,c) & \beta\text{-conversion, twice} \\
& =_{\beta\eta} \lambda c : A \times B.\, x\,c & \eta\text{-conv. at type } A \times B \\
& =_{\beta\eta} x & \eta\text{-conv. at type } (A \times B) \to C
\end{array}
$$

Proof of (9) (not laid out as a tree):

$$
\begin{array}{lll}
s[t/x] & \triangleq \lambda a : A. \lambda b : B.\, (\lambda c : A \times B.\, y\,(\texttt{fst}\,c)\,(\texttt{snd}\,c))\,(a, b) & \\
& =_{\beta\eta} \lambda a : A. \lambda b : B.\, y\,(\texttt{fst}(a, b))\,(\texttt{snd}(a, b)) & \beta\text{-conversion,} \\
& =_{\beta\eta} \lambda a : A. \lambda b : B.\, y\,a\,b & \beta\text{-conversion, twice} \\
& =_{\beta\eta} \lambda a : A.\, y\,a & \eta\text{-conv. at type } B \to C \\
& =_{\beta\eta} y & \eta\text{-conv. at type } A \to (B \to C)
\end{array}
$$

(b) In part (8a), if we take A, B, C to be ground types that are interpreted in $\mathbf{C}$ by the objects X, Y, Z, then the interpretations of (6) and (7) give morphisms

$$
f \triangleq \left(Z^{X \times Y} \cong 1 \times Z^{X \times Y} \xrightarrow{M[\![\diamond, x:(A \times B) \to C \vdash s : A \to (B \to C)]\!]} (Z^Y)^X \right)
$$

$$
g \triangleq \left((Z^Y)^X \cong 1 \times (Z^Y)^X \xrightarrow{M[\![\diamond, y : A \to (B \to C) \vdash t : (A \times B) \to C]\!]} Z^{X \times Y} \right)
$$

with the required domains and codomains. Furthermore, by the semantics of substitution and the Soundness Theorem for STLC, (8) implies

$$\begin{aligned} g \circ f &= \left(Z^{X \times Y} \cong 1 \times Z^{X \times Y} \xrightarrow{M[\![\diamond, x:(A \times B) \to C \vdash t[s/y]:(A \times B) \to C]\!]} Z^{X \times Y} \right) \\ &= \left(Z^{X \times Y} \cong 1 \times Z^{X \times Y} \xrightarrow{M[\![\diamond, x:(A \times B) \to C \vdash x:(A \times B) \to C]\!]} Z^{X \times Y} \right) \\ &= \left(Z^{X \times Y} \cong 1 \times Z^{X \times Y} \xrightarrow{\pi_2} Z^{X \times Y} \right) \\ &= \mathtt{id}_{Z^{(X \times Y)}} \end{aligned}$$

and similarly (9) implies $f \circ g = \mathtt{id}_{(Z^Y)^X}$.

For the record, f and g can be described using the structure of a cartesian closed category as follows:

$$\begin{aligned} f &\triangleq \mathtt{cur}\left(\mathtt{cur}\left((Z^{(X \times Y)} \times X) \times Y \xrightarrow{\langle \pi_1 \circ \pi_1, \langle \pi_2 \circ \pi_1, \pi_2 \rangle\rangle} Z^{(X \times Y)} \times (X \times Y) \xrightarrow{\mathsf{app}} Z\right)\right) \\ g &\triangleq \mathtt{cur}\left((Z^Y)^X \times (X \times Y) \xrightarrow{\langle\langle \pi_1, \pi_1 \circ \pi_2 \rangle, \pi_2 \circ \pi_2 \rangle} ((Z^Y)^X \times X) \times Y \xrightarrow{\mathsf{app} \times \mathtt{id}_Y} Z^Y \times Y \xrightarrow{\mathsf{app}} Z\right) \end{aligned}$$

However, it is quite tedious to use these descriptions to verify that f and g are mutually inverse.

Question 9 The STLC terms you need to use are

$$\begin{aligned} \diamond, x : \mathtt{unit} \to A &\vdash x\,() : A \\ \diamond, y : A &\vdash \lambda z : \mathtt{unit}.\, y : \mathtt{unit} \to A \end{aligned}$$