Primitive recursive functions

Basic functions 0, suc, and U^i_n.

Substitution Given primitive recursive functions $g, h_1, \ldots, h_m$,

$$f(x_1, \ldots, x_n) = g(h_1(x_1, \ldots, x_n), \ldots, h_m(x_1, \ldots, x_n))$$

is primitive recursive.

Primitive recursion Given primitive recursive functions g and h, the function

$$\begin{aligned} f(0, x_1, \ldots, x_n) &= g(x_1, \ldots, x_n) \\ f(\mathsf{suc}(y), x_1, \ldots, x_n) &= h(f(y, x_1, \ldots, x_n), y, x_1, \ldots x_n) \end{aligned}$$

is primitive recursive.

Interpretation continued

Primitive recursion The function f such that

$$\begin{aligned} f(0, x_1, \ldots, x_n) &= g(x_1, \ldots, x_n) \\ f(\mathsf{suc}(y), x_1, \ldots, x_n) &= h(f(y, x_1, \ldots, x_n), y, x_1, \ldots x_n) \end{aligned}$$

is given by

$$\begin{aligned} F \equiv Y\ & (\lambda f y x_1 \ldots x_n.\, \textbf{if}\, (\textbf{iszero}\, y)(G x_1 \ldots x_n) \\ & (H(f(\textbf{pre}\,(y)))x_1 \ldots x_n)(\textbf{pre}\,(y))x_1 \ldots x_n) \end{aligned}$$

Interpreting arithmetic functions

Definition

If f is an n-place function over the natural numbers, then f is **λ-definable** if there exists some λ-term F such that

$$F\underline{k}_1 \ldots \underline{k}_n = \underline{f(k_1, \ldots, k_n)}$$

for all $k_1, \ldots, k_n \in \mathbb{N}$.

Interpretation as λ-terms

Basic functions

0 is given by $\underline{0} \equiv \lambda f x.x$;

suc is given by **suc** $\equiv \lambda n f x.n f(f x)$;

U^i_n is given by $\lambda x_1 \ldots x_n.x_i$.

Substitution The function

$$f(x_1, \ldots, x_n) = g(h_1(x_1, \ldots, x_n), \ldots, h_m(x_1, \ldots, x_n))$$

is given by

$$F \equiv \lambda x_1 \ldots x_n.G(H_1 x_1 \ldots x_n) \ldots (H_m x_1 \ldots x_n)$$

Total recursive functions

Definition

The primitive recursive functions are total.

Given an n-place total recursive function g such that for every $x_2, \ldots, x_n$ there exists y such that $[g(y, x_2, \ldots, x_n) = x_1$, then the function

$$f(x_1, \ldots, x_n) = \text{least } y \text{ st. } [g(y, x_2, \ldots, x_n) = x_1]$$

is total.

Fact

A function f is λ-definable if and only if it is a total recursive function.

Partial recursive functions

Definition

The primitive recursive functions are partial recursive functions.

The partial recursive functions are obtained by adding the **minimisation operator**: given partial recursive function g,

$$f(x_1, \ldots, x_n) = \begin{cases} \text{least } y \text{ st. } [g(y, x_2, \ldots, x_n) = x_1] & \text{if } y \text{ exists} \\ \text{undefined} & \text{otherwise} \end{cases}$$

is a partial recursive function.

Total recursive functions as λ-terms

λ-term

The total recursive function

$$f(x_1, \ldots, x_n) = \text{ the least } y \text{ s.t. } [g(y, x_2, \ldots, x_n) = x_1]$$

is interpreted by

$$F \equiv \lambda x_1 \ldots x_n.$$
$$Y(\lambda h y.\mathbf{if}(\mathbf{equals} x_1 (G y x_2 \ldots x_n)) y (h(\mathbf{suc}(y)))\underline{0}$$

Equals

$$equals \equiv \lambda m\, n.\mathbf{iszero}\ (\mathbf{add}\ (\mathbf{sub}\ m\ n)(\mathbf{sub}\ n\ m))$$

Second fixed-point theorem

Theorem If F is an arbitrary λ-term then there exists a term X such that

$$F\ulcorner X\urcorner = X.$$

Proof Let

$$\begin{aligned} W &\equiv \lambda x.F\,(\mathbf{AP}\,x\,(\mathbf{Num}\,x)) \\ X &\equiv W\ulcorner W\urcorner \end{aligned}$$

It is easy to show that $X \twoheadrightarrow F(\ulcorner X\urcorner)$.

Gödel numbering for λ-terms

Definition Define $\#M$ by

$$\begin{aligned} \#(x_i) &= 2^i \\ \#(\lambda x_i.M) &= 3^i 5^{\#M} \\ \#(MN) &= 7^{\#M} 11^{\#N} \end{aligned}$$

Notation Let $\ulcorner M\urcorner \equiv \underline{\#(M)}$.

Facts There exist λ-terms **AP** and **Num** such that

$$\begin{aligned} \mathbf{AP}\ulcorner M\urcorner\ulcorner N\urcorner &= \ulcorner MN\urcorner \\ \mathbf{Num}\,\underline{n} &= \ulcorner \underline{n}\urcorner \end{aligned}$$

Undecidability result

Theorem There is no λ-term **halts** such that

$$\mathbf{halts}\ulcorner M\urcorner = \begin{cases} \mathbf{true} & M \text{ has a normal form} \\ \mathbf{false} & M \text{ has no normal form} \end{cases}$$

Proof Suppose that **halts** exists. Let

$$D \equiv \lambda x.\mathbf{if}\,(\mathbf{halts}\ x)\Omega\underline{0}, \quad \Omega \equiv (\lambda x.xx)(\lambda x.xx).$$

By the second fixed-point theorem, there exists X such that

$$X = D\ulcorner X\urcorner = \mathbf{if}\,(\mathbf{halts}\ \ulcorner X\urcorner)\Omega\underline{0}$$

This leads to a contradiction.