Lists

Constructors

$$\begin{aligned} \textbf{nil} &\equiv \textbf{pair true true} \\ \textbf{cons} &\equiv \lambda xy.\ \textbf{pair false}\ (\textbf{pair}\ x\ y) \end{aligned}$$

Destructors

$$\begin{aligned} \textbf{null} &\equiv \textbf{fst} \\ \textbf{hd} &\equiv \lambda z.\ \textbf{fst}\ (\textbf{snd}\ z) \\ \textbf{tl} &\equiv \lambda z.\ \textbf{snd}\ (\textbf{snd}\ z) \end{aligned}$$

Append

$$\textbf{append} = \lambda xy.\ \textbf{if}\ (\textbf{null}\ x)\ y\ (\textbf{cons}\ (\textbf{hd}\ x)(\textbf{append}\ (\textbf{tl}\ x)\ y))$$

Fixed-point combinators

Definition

A **combinator** is a λ-term containing no free variables.

Definition

A **fixed-point combinator** is a combinator Y such that

$$YM = M(YM)$$

for **arbitrary** λ-term M.

λ-term for append

$$\textbf{append}\ Z\ W \ = \textbf{if}\ (\textbf{null}\ Z)\ W\ (\textbf{cons}\ (\textbf{hd}\ Z)(\textbf{append}\ (\textbf{tl}\ Z)\ W))$$

Solution

$$\textbf{append} \equiv Y(\underbrace{\lambda gzw.\ \textbf{if}\ (\textbf{null}\ z)\ w\ (\textbf{cons}\ (\textbf{hd}\ z)(g\ (\textbf{tl}\ z)\ w))}_{\mathbf{P}})$$

Justification

$$\begin{aligned} \textbf{append}\ Z\ W &\equiv Y(P)\,Z\,W \\ &= P(Y(P))\,Z\,W \\ &\equiv \textbf{if}\ (\textbf{null}\ Z)\ W\ (\textbf{cons}\ (\textbf{hd}\ Z)\ \underbrace{Y(P)}_{\mathbf{append}}\ (\textbf{tl}\ Z)\ W) \end{aligned}$$

Solving recursive equations

Equation $M = PM$

Solution $M \equiv Y(P)$

Equation $Mx_1 \ldots x_n = PM, \quad x_i \notin \mathbf{FV}(M)$

Solution $M \equiv Y(\lambda g x_1 \ldots x_n . Pg)$

Equations $M = PMN$ **and** $N = QMN$

Solution $L \equiv Y(\lambda g.\mathbf{pair}\,(P(\mathbf{fst}\,g)(\mathbf{snd}\,g))(Q(\mathbf{fst}\,g)(\mathbf{snd}\,g)))$

$M \equiv \mathbf{fst}\,L, \quad N \equiv \mathbf{snd}\,L$

Other examples

Factorial function

$$\mathbf{fact}\ N = \mathbf{if}\ (\mathbf{iszero}\ N)\ \underline{1}\ (\mathbf{mult}\ N(\mathbf{fact}\ (\mathbf{pre}\ N)))$$

Infinite list of zeros

$$\mathbf{zeroes} = \mathbf{cons}\ \underline{0}\ \mathbf{zeroes}$$

Examples of fixed-point combinators

Curry's $Y \equiv \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$.

Fact $YM = M(YM)$ for arbitrary M.

Turing's $\Theta \equiv AA$ where $A \equiv \lambda xy.y(xxy)$.

Fact $\Theta F \twoheadrightarrow F(\Theta F)$ for arbitrary F.