Formal Definition

The **reduction relation** on terms, written $M \to N$, is defined inductively by the following axioms and rules (identifying α-equivalent terms):

$$(\lambda x.M)N \to M[N/x]$$

$$\lambda x.(Mx) \to M, \qquad x \notin \mathsf{FV}(M)$$

$$\frac{M \to M'}{\lambda x.M \to \lambda x.M'}$$

$$\frac{M \to M'}{MN \to M'N} \qquad \frac{N \to N'}{MN \to MN'}$$

Normal Forms

Definitions

A **β- redex** is a term of the form $(\lambda x.M)N$.

An **η-redex** is a term of the form $\lambda x.Mx$ for $x \notin \mathsf{FV}(M)$.

A term M is **in normal form** if it contains no β- or η- redexes.

A term M **has a normal form** if $M \twoheadrightarrow N$ and N is in normal form.

Proposition

A term is in normal form if and only if it looks like

$$\lambda x_1 \ldots . \lambda x_m . y M_1 \ldots M_k, \text{ for } m, k \geq 0$$

where $M_1, \ldots M_k$ have the same shape.

Induction on the structure of λ-terms

To show property $P(M)$ for all terms M, it suffices to prove

the **base case:** $P(x)$

two induction steps:

- assume property $P(M)$ for arbitrary M and prove $P(\lambda x.M)$;
- assume properties $P(M)$ and $P(N)$ for arbitrary M and N, and prove $P(MN)$.

The **equality relation** between two terms, written $M = N$, is defined inductively by the following axioms and rules (identifying α-equivalent terms):

axioms

$$(\lambda x.M)N = M[N/x]$$

$$\lambda x.Mx = M, \quad x \notin \mathsf{FV}(M)$$

equivalence rules

$$M = M \qquad \frac{M = N}{N = M} \qquad \frac{L = M \quad M = N}{L = N}$$

congruence rules

$$\frac{M = M'}{\lambda x.M = \lambda x.M'} \qquad \frac{M = M'}{MN = M'N} \qquad \frac{M = M'}{LM = LM'}$$

The Church-Rosser Property

If $M = N$ then there exists a term L such that $M \twoheadrightarrow L$ and $N \twoheadrightarrow L$.

Consequences

If a term has a normal form, it is unique (up to α-equivalence).

The λ-calculus is consistent (non-trivial).

Equality

Definition

We write $M = N$ if M can be transformed into N by performing finitely many (possibly zero) reductions and expansions.

Proposition

If $M \twoheadrightarrow L$ and $N \twoheadrightarrow L$ then $M = N$

The Strip Lemma

$$\begin{tikzcd} & M \arrow[dl] \arrow[dr, two heads] & \\ M_1 \arrow[dr, two heads] & & M_2 \arrow[dl, two heads] \\ & L & \end{tikzcd}$$

The Diamond Property

Let $M \twoheadrightarrow M_1$ and $M \twoheadrightarrow M_2$, then there exists a term L such that $M_1 \twoheadrightarrow L$ and $M_2 \twoheadrightarrow L$:

$$\begin{tikzcd} & M \arrow[dl, two heads] \arrow[dr, two heads] & \\ M_1 \arrow[dr, two heads] & & M_2 \arrow[dl, two heads] \\ & L & \end{tikzcd}$$