Definition The **set of all bound variables** in M, denoted by $\mathsf{BV}(M)$, is defined inductively by

$$
\begin{aligned}
\mathsf{BV}(x) &= \emptyset \\
\mathsf{BV}(\lambda x.M) &= \mathsf{BV}(M) \cup \{x\} \\
\mathsf{BV}(MN) &= \mathsf{BV}(M) \cup \mathsf{BV}(N)
\end{aligned}
$$

Definition The **set of all free variables** in M, denoted by $\mathsf{FV}(M)$, is defined inductively by

$$
\begin{aligned}
\mathsf{FV}(x) &= \{x\} \\
\mathsf{FV}(\lambda x.M) &= \mathsf{FV}(M) - \{x\} \\
\mathsf{FV}(MN) &= \mathsf{FV}(M) \cup \mathsf{FV}(N)
\end{aligned}
$$

Definition

The term $M[N/y]$ is **safe** provided the bound variables of M are disjoint from the free variables of N:

$$\mathsf{BV}(M) \cap \mathsf{FV}(N) = \emptyset.$$

Definition of the λ-terms

The terms of the λ-calclus, known as **λ-terms**, are constructed recursively from a given set of variables V. They may take one of the following forms:

x variable;

$\underbrace{(\lambda x.M)}_{x \text{ bound}}$ abstraction, where M is a term;

(MN) application, where M and N are terms.

Notation $M \equiv N$ means that M is identical to N.

Substitution

Definition

The term $M[L/y]$, the result of substituting L for all free occurrences of y in M, is defined inductively by

$$
\begin{aligned}
x[L/y] &\equiv \begin{cases} L & \text{if } x \equiv y \\ x & \text{otherwise} \end{cases} \\
(\lambda x.M)[L/y] &\equiv \begin{cases} (\lambda x.M) & \text{if } x \equiv y \\ (\lambda x.M[L/y]) & \text{otherwise} \end{cases} \\
(MN)[L/y] &\equiv (M[L/y]N[L/y])
\end{aligned}
$$

β-conversion

$$(\lambda x.M)N \to_\beta M[N/x]$$

Remark

Formally, this is only defined when $\mathsf{BV}(M) \cap \mathsf{FV}(N) = \emptyset$. Since we identify terms which are α-equivalent, we usually omit this side-condition. Notice that we always have

$$(\lambda x.M)N \equiv_\alpha (\lambda z.M[z/x])N \to_\beta M[z/x][N/z]$$

for $z \notin \mathsf{BV}(M) \cup \mathsf{FV}(M)$.

η-conversion

$$(\lambda x.(Mx)) \to_\eta M, \quad \text{if } x \notin \mathsf{FV}(M).$$

Remark

This rule embodies a principal of **extensionality**: two functions are equal if they always return equal results given equal arguments (as long as we identify terms which are α-equal).

α-conversion

$$(\lambda x.M) \to_\alpha (\lambda y.M[y/x]), \quad y \notin \mathsf{FV}(M) \cup \mathsf{BV}(M)$$

Notation

We sometimes write $M \equiv_\alpha N$ when M and N are **α-equivalent**: that is, when N can be obtained from M by a finite (perhaps empty) series of α-conversions.

Remark

We shall usually ignore the distinction between terms which are α-equivalent.

Examples

1. $(\lambda x.x)z \to_\beta z$;
2. $(\lambda x.y)z \to_\beta y$;
3. $(\lambda x.f(x)(x))z \to_\beta f(z)(z)$;
4. $(\lambda z.(zy))(\lambda x.x) \to_\beta (\lambda x.x)y \to_\beta y$.
5. $\Omega \equiv (\lambda x.xx)(\lambda x.xx) \to_\beta (\lambda x.xx)(\lambda x.xx)$

Formal Definition

The **reduction relation** on terms, written $M \rightarrow N$, is defined inductively by the following axioms and rules (identifying α-equivalent terms):

$$(\lambda x.M)N \rightarrow M[N/x]$$

$$\lambda x.(Mx) \rightarrow M, \qquad x \notin \mathsf{FV}(M)$$

$$\frac{M \rightarrow M'}{\lambda x.M \rightarrow \lambda x.M'}$$

$$\frac{M \rightarrow M'}{MN \rightarrow M'N} \qquad \frac{N \rightarrow N'}{MN \rightarrow MN'}$$

Reduction

Informal description

We say that M reduces to N, and write $M \rightarrow N$, if N is obtained from M by applying a β-conversion or an η-conversion to a subterm of M.

Notation

We write $M \twoheadrightarrow N$ if $M \rightarrow M_1 \rightarrow \ldots \rightarrow M_k \equiv N$ for $k \geq 0$.

Normal Forms

Definitions

A **β- redex** is a term of the form $(\lambda x.M)N$.

An **η-redex** is a term of the form $\lambda x.Mx$ for $x \notin \mathsf{FV}(M)$.

A term M is **in normal form** if it contains no β- or η- redexes.

A term M **has a normal form** if $M \twoheadrightarrow N$ and N is in normal form.