# Complexity Theory

## Lecture 4

Anuj Dawar

http://www.cl.cam.ac.uk/teaching/2223/Complexity

# Composites

Consider the decision problem (or *language*) Composite defined by:

$$\{x \mid x \text{ is not prime}\}$$

This is the complement of the language Prime.

Is Composite $\in$ P?

Clearly, the answer is yes if, and only if, Prime $\in$ P.

# Hamiltonian Graphs

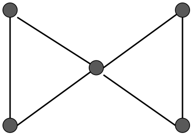
Given a graph $G = (V, E)$, a *Hamiltonian cycle* in $G$ is a path in the graph, starting and ending at the same node, such that every node in $V$ appears on the cycle *exactly once*.

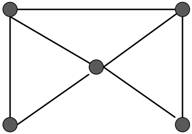
A graph is called *Hamiltonian* if it contains a Hamiltonian cycle.

The language HAM is the set of encodings of Hamiltonian graphs.

Is HAM $\in$ P?

# Examples





The first of these graphs is not Hamiltonian, but the second one is.

# Graph Isomorphism

Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, is there a *bijection*

$$\iota : V_1 \to V_2$$

such that for every $u, v \in V_1$,

$$(u, v) \in E_1 \quad \text{if, and only if,} \quad (\iota(u), \iota(v)) \in E_2.$$

Is $\mathrm{Graph\ Isomorphism} \in \mathsf{P}$?

# Polynomial Verification

The problems Composite, SAT, HAM and Graph Isomorphism have something in common.

In each case, there is a *search space* of possible solutions.

> *the numbers less than $x$; truth assignments to the variables of $\phi$; lists of the vertices of $G$; a bijection between $V_1$ and $V_2$.*

The size of the search is *exponential* in the length of the input.

Given a potential solution in the search space, it is *easy* to check whether or not it is a solution.

# Verifiers

A verifier $V$ for a language $L$ is an algorithm such that

$$L = \{x \mid (x, c) \text{ is accepted by } V \text{ for some } c\}$$

If $V$ runs in time polynomial in the length of $x$, then we say that

*$L$ is polynomially verifiable.*

Many natural examples arise, whenever we have to construct a solution to some design constraints or specifications.

# Nondeterminism

If, in the definition of a Turing machine, we relax the condition on $\delta$ being a function and instead allow an arbitrary relation, we obtain a *nondeterministic Turing machine*.

$$\delta \subseteq (Q \times \Sigma) \times ((Q \cup \{\mathrm{acc}, \mathrm{rej}\}) \times \Sigma \times \{R, L, S\}).$$

The yields relation $\to_M$ is also no longer functional.

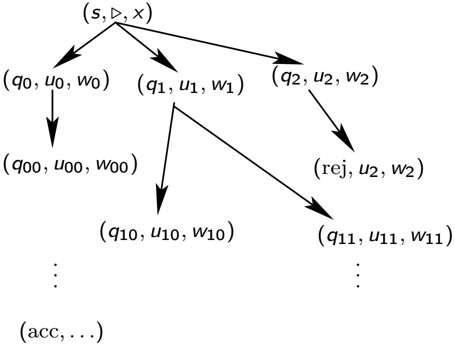
We still define the language accepted by $M$ by:

$$\{x \mid (s, \triangleright, x) \to_M^\star (\mathrm{acc}, w, u) \text{ for some } w \text{ and } u\}$$

though, for some $x$, there may be computations leading to accepting as well as rejecting states.

# Computation Trees

With a nondeterministic machine, each configuration gives rise to a tree of successive configurations.



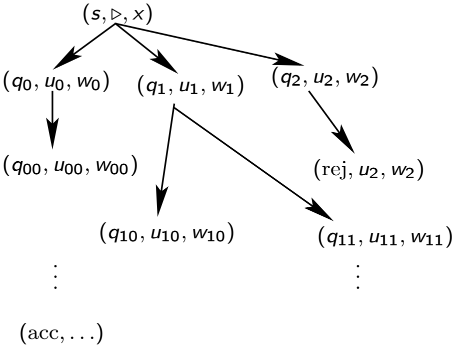
# Nondeterministic Complexity Classes

We have already defined TIME($f$) and SPACE($f$).

NTIME($f$) is defined as the class of those languages $L$ which are accepted by a *nondeterministic* Turing machine $M$, such that for every $x \in L$, there is an accepting computation of $M$ on $x$ of length $O(f(n))$, where $n$ is the length of $x$.

$$\mathsf{NP} = \bigcup_{k=1}^{\infty} \mathsf{NTIME}(n^k)$$

# Nondeterminism




For a language in NTIME($f$), the height of the tree can be bounded by $f(n)$ when the input is of length $n$.

# NP

*A language $L$ is polynomially verifiable if, and only if, it is in* NP.

To prove this, suppose $L$ is a language, which has a verifier $V$, which runs in time $p(n)$.

The following describes a *nondeterministic algorithm* that accepts $L$

1. input $x$ of length $n$
2. nondeterministically guess $c$ of length $\leq p(n)$
3. run $V$ on $(x, c)$

# NP

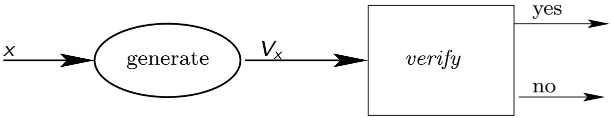
In the other direction, suppose $M$ is a nondeterministic machine that accepts a language $L$ in time $n^k$.

We define the *deterministic algorithm* $V$ which on input $(x, c)$ simulates $M$ on input $x$.
At the $i^{\text{th}}$ nondeterministic choice point, $V$ looks at the $i^{\text{th}}$ character in $c$ to decide which branch to follow.
If $M$ accepts then $V$ accepts, otherwise it rejects.

$V$ is a polynomial verifier for $L$.

# Generate and Test

We can think of nondeterministic algorithms in the generate-and test paradigm:




Where the *generate* component is nondeterministic and the *verify* component is deterministic.

# Reductions

Given two languages $L_1 \subseteq \Sigma_1^\star$, and $L_2 \subseteq \Sigma_2^\star$,

A *reduction* of $L_1$ to $L_2$ is a *computable* function

$$f : \Sigma_1^\star \to \Sigma_2^\star$$

such that for every string $x \in \Sigma_1^\star$,

$$f(x) \in L_2 \text{ if, and only if, } x \in L_1$$