Artificial Intelligence

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Artificial Intelligence

Introduction: aims, history, rational action, and agents

Reading: AIMA chapters 1, 2, 28 and 29.

Introduction: what are our aims?

Artificial Intelligence (AI) is currently at the top of its *periodic hype-cycle*.

Much of this has been driven by *philosophers* and *people with something to sell*.

Introduction: what are our aims?

What is the purpose of Artificial Intelligence (AI)? If you're a *philosopher* or a *psychologist* then perhaps it's:

- To *understand intelligence*.
- To understand *ourselves*.

Philosophers have worked on this for at least 2000 years. They've also wondered about:

- *Can* we do AI? *Should* we do AI? What are the *ethical implications*?
- Is AI *impossible*? (Note: I didn't write *possible* here, for a good reason...)

Despite 2000 years of work by philosophers, there's essentially *nothing* in the way of results.

Introduction: what are our aims?

Luckily, we were sensible enough not to pursue degrees in philosophy—we're scientists/engineers, so while we might have *some* interest in such pursuits, our perspective is different:

- Brains are *amazing*. They are small (about 3 pounds), *incredibly* densely packed (orders of magnitude over silicon, and in three dimensions), energy efficient (about 20 watts) but apparently slow (not quite so clear-cut).
- Brains are incredibly good at some tasks—we want to understand a specific form of *computation.*
- It would be nice to be able to *construct* intelligent systems, for all manner of reasons.
- It is also nice to *make and sell cool stuff*.

Historically speaking, this view *seems to be the more successful...*

AI has been entering our lives for decades, almost without us being aware of it.

But be careful: brains are *much more complex than you think.*

Introduction: now is a fantastic time to investigate AI

In many ways this is a young field, having only really got under way in 1956 with the *Dartmouth Conference*:

`en.wikipedia.org/wiki/Dartmouth_workshop`

- This means we can actually *do* things. It's as if we were physicists before anyone thought about atoms, or gravity, or....
- Also, we know what we're trying to do is *possible.* (Unless we think humans don't exist. *NOW STEP AWAY FROM THE PHILOSOPHY* before *SOMEONE GETS HURT!!!!*)

Perhaps I'm being too hard on them; there was some good groundwork: *Socrates* wanted an algorithm for *"piety"*, leading to *Syllogisms.* Ramon Lull's *concept wheels* and other attempts at mechanical calculators. Rene Descartes' *Dualism* and the idea of mind as a *physical system.* Wilhelm Leibnitz's opposing position of *Materialism.* (The intermediate position: mind is *physical* but *unknowable.*) The origin of *knowledge*: Francis Bacon's *Empiricism*, John Locke: *"Nothing is in the understanding, which was not first in the senses"*. David Hume: we obtain rules by repeated exposure: *Induction.* Further developed by Bertrand Russell and in the *Confirmation Theory* of Carnap and Hempel.

More recently: the connection between *knowledge* and *action*? How are actions *justified*? If to achieve the end you need to achieve something intermediate, consider how to achieve that, and so on. This approach was implemented in Newell and Simon's 1957 *General Problem Solver (GPS).*

What has been achieved?

Artificial Intelligence (AI) is currently at the top of its *periodic hype-cycle.*

As a result, it's important to maintain a sense of perspective. And remember, AI has been deployed for *decades* with relatively little fanfare.

Notable successes:

- Perception: vision, speech processing, inference of emotion from video, scene labelling, touch sensing, artificial noses...
- Logical reasoning: Prolog, expert systems, CYC, Bayesian reasoning, Watson...
- Playing games: chess, backgammon, go, robot football...
- Diagnosis of illness in various contexts. Drug design...
- Theorem proving: Robbin's conjecture, formalization of the Kepler conjecture, Boolean Pythagorean Triples problem (longest proof in history)...
- Planning...
- And many more... (most of which don't include the word *'DEEP'*!)

What has been achieved?

Artificial Intelligence (AI) is currently at the top of its *periodic hype-cycle.*

As a result, it's important to maintain some sense of perspective.

- There are equally many areas in which we currently *can't do things very well.*
- When AI has a success, the ideas in question tend to *stop being called AI*.

Do you consider the fact that *your phone can do speech recognition* to be a form of AI?

The nature of the pursuit

What is AI? This is not necessarily a straightforward question.

It depends on who you ask...

We can find many definitions and a rough categorisation can be made depending on whether we are interested in:

- The way in which a system *acts* or the way in which it *thinks*.
- Whether we want it to do this in a *human* way or a *rational* way.

Here, the word *rational* has a special meaning: it means *doing the correct thing in given circumstances*.

What is AI, version one: acting like a human

Alan Turing proposed what is now known as the *Turing Test*.

- A human judge is allowed to interact with an AI program via a terminal.
- This is the *only* method of interaction.
- If the judge can't decide whether the interaction is produced by a machine or another human then the program passes the test.

In the *unrestricted* Turing test the AI program may also have a camera attached, so that objects can be shown to it, and so on.

The Turing test is informative, and (very!) hard to pass. (See the *Loebner Prize*...)

- It requires many abilities that seem necessary for AI, such as learning. *BUT*: a human child would probably not pass the test.
- Sometimes an AI system needs human-like acting abilities—for example *expert systems* often have to produce explanations—but *not always*.
- No, LLMs *don't pass it*...
- ...and even when they (or something else) does, *the test is problematic*.

What is AI, version two: thinking like a human

There is always the possibility that a machine *acting* like a human does not actually *think*. The *cognitive modelling* approach to AI has tried to:

- Deduce *how humans think*—for example by *introspection* or *psychological experiments*.
- Copy the process by mimicking it within a program.

An early example of this approach is the *General Problem Solver* produced by Newell and Simon in 1957. They were concerned with whether or not the program reasoned in the same manner that a human did.

$$\text{Computer Science} + \text{Psychology} = \textit{Cognitive Science}$$

What is AI, version three: thinking rationally and the "laws of thought"

The idea that intelligence reduces to *rational thinking* is a very old one, going at least as far back as Aristotle as we've already seen.

The general field of *logic* made major progress in the 19th and 20th centuries, allowing it to be applied to AI.

- We can *represent* and *reason* about many different things.
- The *logicist* approach to AI.

This is a very appealing idea, but there are obstacles. It is hard to:

- Represent *commonsense knowledge*.
- Deal with *uncertainty*.
- Reason without being tripped up by *computational complexity*.
- Sometimes it's necessary to act when there's *no* logical course of action.
- Sometimes inference is *unnecessary* (reflex actions).

These will be recurring themes in this course, and in *Machine Learning and Bayesian Inference* next year.

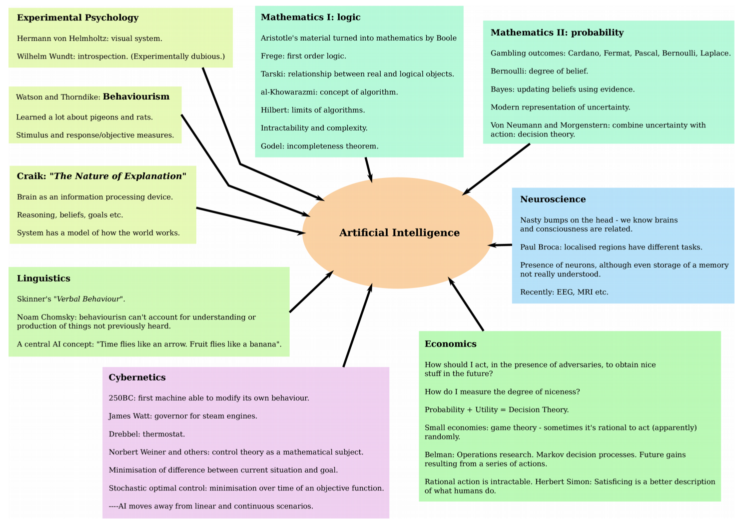
What is AI, version four: acting rationally

Basing AI on the idea of *acting rationally* means attempting to design systems that act to *achieve their goals* given their *beliefs*.

- Thinking about this in engineering terms, it seems *almost inevitably* to lead us towards the usual subfields of AI. What might be needed?
- The concepts of *action*, *goal* and *belief* can be defined precisely making the field suitable for scientific study.
- This is important: if we try to model AI systems on humans, we can't even propose *any* sensible definition of *what a belief or goal is.*
- In addition, humans are a system that is still changing and adapted to a very specific environment.
- All of the things needed to pass a Turing test seem necessary for rational acting, so this seems preferable to the *acting like a human* approach.
- The logicist approach can clearly form *part* of what's required to act rationally, so this seems preferable to the *thinking rationally* approach alone.

As a result, we will focus on the idea of designing systems that *act rationally*.

Other fields that have contributed to AI



What's in this course?

This course introduces some of the fundamental areas that make up AI:

- An outline of the background to the subject.
- An introduction to the idea of an *agent*.
- Solving problems in an intelligent way by *search*.
- Solving problems represented as *constraint satisfaction* problems.
- Playing *games*.
- *Knowledge representation, and reasoning*.
- *Planning*.
- *Learning* using *neural networks*.

Strictly speaking, this course covers what is often referred to as *"Good Old-Fashioned AI"*. (Although "Old-Fashioned" is a misleading term.)

The nature of the subject changed when the importance of *uncertainty* was fully appreciated. *Machine Learning and Bayesian Inference* covers this more recent material.

What's *not* in this course?

- The classical AI programming languages *Prolog* and *Lisp*.
- A great deal of all the areas on the last slide!
- Perception: *vision*, *hearing* and *speech processing*, *touch* (force sensing, knowing where your limbs are, knowing when something is bad), *taste*, *smell*.
- Natural language processing.
- Acting on and in the world: *robotics* (effectors, locomotion, manipulation), *control engineering*, *mechanical engineering*, *navigation*.
- Areas such as *genetic algorithms/programming*, *swarm intelligence*, *artificial immune systems* and *fuzzy logic*, for reasons that I will expand upon during the lectures.
- *Uncertainty* and much further probabilistic material. (You'll have to wait until next year.)

Introductory reading that *isn't nonsense*

- Francis Crick, *"The recent excitement about neural networks"*, Nature (1989) is still entirely relevant:

  www.nature.com/articles/337129a0/

- The *Loebner Prize in Artificial Intelligence*:

  aisb.org.uk/aisb − events/

  provides a good illustration of how hard it is to pass the Turing test.
- Marvin Minsky, *"Why people think computers can't"*, AI Magazine (1982) is an excellent response to nay-saying philosophers.

  ojs.aaai.org//index.php/aimagazine/article/view/376

- Go: www.nature.com/articles/nature16961 (And note the inclusion of "tree search" in the title...)
- The Cyc project: www.cyc.com

Lots of this material does not include the word "deep"...

Introductory reading that *isn't nonsense*

- *AI in the UK: ready, willing and able?*
  House of Lords, Select Committee on Artificial Intelligence

  publications.parliament.uk/pa/ld201719/ldselect/ldai/100/100.pdf

- *Machine learning: the power and promise of computers that learn by example*
  The Royal Society

  royalsociety.org/topics − policy/projects/machine − learning/

- *Building machines that learn and think like people*
  Brenden M. Lake *et al*, Behavioral and Brain Sciences, Cambridge University Press, 2017.
  Link

Text book

The course is based on the relevant parts of:

Artificial Intelligence: A Modern Approach (AIMA), Fourth Edition (2022). Stuart J. Russell and Peter Norvig, Pearson Series in Artificial Intelligence.

(Earlier Editions may suffice.) An alternative source is:

Artificial Intelligence: Foundations of Computational Agents (AIFCA), Second Edition (2017). David L. Poole and Alan K. Mackworth, CUP.

For more depth on specific areas see:

Dechter, R. (2003). *Constraint processing*. Morgan Kaufmann.

Cawsey, A. (1998). *The essence of artificial intelligence*. Prentice Hall.

Ghallab, M., Nau, D. and Traverso, P. (2004). *Automated planning: theory and practice*. Morgan Kaufmann.

Bishop, C.M. (2006). *Pattern recognition and machine learning*. Springer.

Brachman, R. J. and Levesque, H. J. (2004). *Knowledge Representation and Reasoning*. Morgan Kaufmann.

AAAAAAAAARRRRGH!!!!!

AAAAAAAARRRGH!!!
SHUT UP AND TELL ME ABOUT DEEP STUFF!!!!!!!

No. You're in the wrong lecture series...

AIMA: total length = *1166 pages*.

AIME: chapter on *Deep Learning* = *39 pages*.

Proportion of the *most respected AI text in history* devoted to Deep Learning:

3.34%

Draw your own conclusions...

Anyway, I *will* be covering *backpropagation*. If you understand that the rest is just *architecture and scale*...

(And often an *appalling lack of reproducability*...)

Prerequisites

The prerequisites for the course are: first order logic, some algorithms and data structures, discrete and continuous mathematics, and basic computational complexity.

DIRE WARNING:

No doubt you want to know something about *machine learning*, given the recent peek in interest.

In the lectures on *machine learning* I will be talking about *neural networks.*

I will introduce the *backpropagation algorithm*, which is the foundation for both *classical neural networks* and the more fashionable *deep learning* methods.

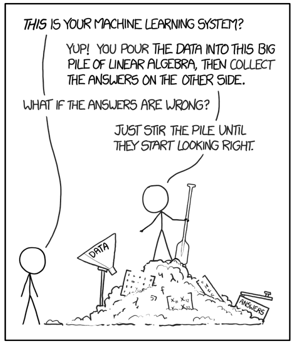
This means you will need to be able to *differentiate* and also handle *vectors and matrices.*

If you've forgotten how to do this *you WILL get lost—I guarantee it!!!*

Neural networks: the terrible truth...

There are two ways to approach machine learning:

- The wrong way: load PyTorch, call a library function, and *hope for the best*.
- The right way: *understand what you're doing*.



Credit: `xkcd.com/1838/`.
License: `creativecommons.org/licenses/by-nc/2.5/legalcode`.

Prerequisites

Self test:

1. Let
$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} a_i x_i^2$$
where the a_i are constants. Can you compute $\partial f / \partial x_j$ where $1 \leq j \leq n$?
2. Let $f(x_1, \ldots, x_n)$ be a function. Now assume $x_i = g_i(y_1, \ldots, y_m)$ for each x_i and some collection of functions g_i. Assuming all requirements for differentiability and so on are met, can you write down an expression for $\partial f / \partial y_j$ where $1 \leq j \leq m$?

If the answer to either of these questions is "no" then it's time for some revision. (You have about three weeks notice, so I'll assume you know it!)

And finally…

There are some important points to be made regarding *computational complexity*.

First, you might well hear the term *AI-complete* being used a lot. What does it mean?

AI-complete: only solvable if you can solve AI in its entirety.

For example: high-quality automatic translation from one language to another.

To produce a genuinely good translation of *Moby Dick* from English to Cantonese is likely to be AI-complete.

And finally…

More practically, you will often hear me make the claim that *everything that's at all interesting in AI is at least NP-complete.*

There are two ways to interpret this:

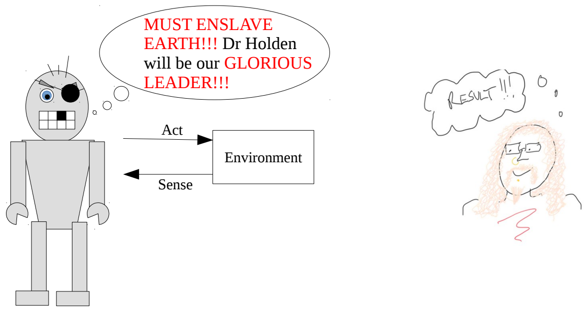
1. The wrong way: "It's all a waste of time.[1]" OK, so it's a partly understandable interpretation. *BUT* the fact that Boolean satisfiability is intractable *does not* mean we can't solve large instances in practice…
2. The right way: "It's an opportunity to design nice approximation algorithms." In reality, the algorithms that are *good in practice* are ones that try to *often* find a *good* but not necessarily *optimal* solution, in a *reasonable* amount of time and memory.

[1] In essence, a comment on a course assessment a couple of years back to the effect of: "Why do you teach us this stuff if it's all futile?"

Agents

There are many different definitions for the term *agent* within AI.

Allow me to introduce EVIL ROBOT.



We will use the following simple definition: *an agent is any device that can sense and act upon its environment*.

Agents

This definition can be very widely applied: to humans, robots, pieces of software, and so on.

We are taking quite an *applied* perspective. We want to *make things* rather than *copy humans*. So:

1. How can we judge an agent's performance?
2. How can an agent's *environment* affect its design?
3. Are there sensible ways in which to think about the *structure* of an agent?

Recall that we are interested in devices that *act rationally*, where 'rational' means doing the *correct thing* under *given circumstances*.

Measuring performance

Item 1: How can we judge an agent's performance?

- Any measure of performance is likely to be *problem-specific*.
  - Even a simple email filter is an agent—it can sense and act. Here the performance measure is straightforward.
  - For a self-driving car, it is more complicated!
- We're usually interested in *expected, long-term performance*.
  - *Expected* performance because usually agents are not *omniscient*—they don't *infallibly* know the outcome of their actions.
    (It is *rational* for you to enter this lecture theatre even if the roof falls in today. An agent capable of detecting and protecting itself from a falling roof might be more *successful* than you, but *not* more *rational*.
  - *Long-term performance* because it tends to lead to better approximations to what we'd consider rational behaviour.

Environments

Item 2: How can an agent's *environment* affect its design?

Some common attributes of an environment have a considerable influence on agent design.

- *Accessible/inaccessible:* do percepts tell you *everything* you need to know about the world?
- *Deterministic/non-deterministic:* does the future depend *predictably* on the present and your actions?
- *Episodic/non-episodic* is the agent run in independent episodes.
- *Static/dynamic:* can the world change while the agent is deciding what to do?
- *Discrete/continuous:* an environment is discrete if the sets of allowable percepts and actions are finite.
- *For multiple agents:* whether the situation is *competitive* or *cooperative*, and whether *communication* is required.

Programming agents

Item 3: Are there sensible ways in which to think about the *structure* of an agent?

A basic agent can be thought of as working according to a straightforward underlying process. To achieve some *goal*:

- *Gather perceptions.*
- Update *working memory* to take account of them.
- On the basis of what's in the working memory, *choose an action* to perform.
- *Update* the working memory to take account of this action.
- *Do* the chosen action.

Obviously, this hides a great deal of complexity:

- A percept might arrive *while an action is being chosen.*
- The world may change *while an action is being chosen.*
- Actions may affect the world in *unexpected ways.*
- We might have *multiple goals*, which *interact* with each other.
- And so on...

Keeping track of the environment, and having a goal

It seems reasonable that an agent should maintain:

- A *description of the current state of its environment*.
- Knowledge of how the environment *changes independently of the agent*.
- Knowledge of how the agent's *actions affect its environment*.

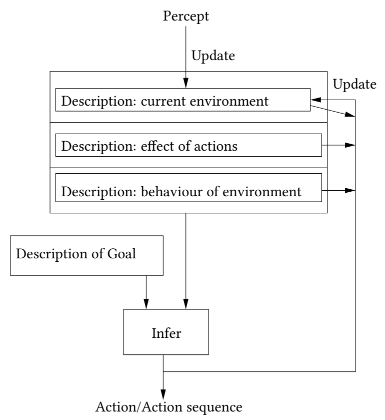
This requires us to do *knowledge representation* and *reasoning*.

It also seems reasonable that an agent should choose a rational course of action depending on its *goal*.

- If an agent has knowledge of how its actions affect the environment, then it has a basis for choosing actions to achieve goals.
- To obtain a *sequence* of actions we need to be able to *search* and to *plan*.

Goal-based agents

We now have a basic design that looks something like this:



Utility-based agents

Introducing goals is still not the end of the story.

- There may be *many* sequences of actions that lead to a given goal, and *some may be preferable to others*.
- We might need to trade-off *conflicting goals*, for example speed and safety.
- An agent may have several goals, but not be certain of achieving any of them. Can it trade-off the likelihood of reaching a goal against the desirability of getting there?

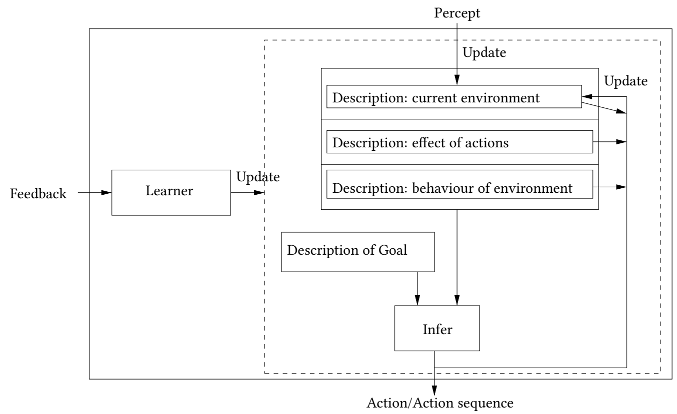
A *utility function* maps a state to a number representing the desirability of that state.

Maximising expected utility over time forms a fundamental model for the design of agents.

Unfortunately, there is insufficient time in this course to properly explore agents based on utility.

Learning agents

It seems reasonable that an agent should *learn from experience*:



What might this entail?

Learning agents

Learning mainly requires two additions:

1. The learner needs some form of *feedback* on the agent's performance. This can come in several different forms.
2. The learner needs a means of *generating new behaviour* in order to find out about the world.

The second point leads to an important trade-off:

1. Should the agent spend time *exploiting* what it's learned so far, if it's achieving a level of success, or...
2. ...should the agent try new things, *exploring* the environment on the basis that it might learn something *really useful* even if it performs *worse in the short term*?

Artificial Intelligence

Problem solving by search

Reading: AIMA chapters 3 and 4.

We begin with what is perhaps the simplest collection of AI techniques: those allowing an *agent* existing within an *environment* to *search* for a *sequence of actions* that *achieves a goal*.

Search algorithms apply to a particularly simple class of problems—we need to identify:

- *An initial state* s_0 from a set S of possible states.

  This models the agent's situation before anything else happens.
- *A set of actions*, denoted A.

  These are modelled by specifying what state will result on performing any available action in any state.

  We can model this using a function $\texttt{action} : A \times S \rightarrow S$: if the agent is in state s and performs action a then its new state is $\texttt{action}(a, s)$.
- *A goal test*: we can tell whether or not the state we're in corresponds to a goal.

  We can model this using a function $\texttt{goal} : S \rightarrow \{\texttt{true}, \texttt{false}\}$.

We also need the idea of *path cost*.

We need another function $\texttt{cost} : A \times S \rightarrow \mathbb{R}$. This denotes the *cost* of *performing an action* a in *state* s.

If the agent starts in state s_0 and takes a sequence of actions $a_0, a_1, \ldots, a_n$ then it moves through a sequence of states

$$s_0 \xrightarrow{\texttt{cost}(a_0,s_0)} s_1 \xrightarrow{\texttt{cost}(a_1,s_1)} s_2 \xrightarrow{\texttt{cost}(a_2,s_2)} \cdots \xrightarrow{\texttt{cost}(a_n,s_n)} s_{n+1}$$

with $s_{i+1} = \texttt{action}(a_i, s_i)$. We then define the *path cost* of this path as

$$p(s_{n+1}) = \sum_{i=0}^{n} \texttt{cost}(a_i, s_i).$$

We generally want a path to a *goal* that has *minimim path cost*.

Note that you have *already seen* problems like this...

You have *already seen* problems like this...

- *Foundations of Computer Science*: talks about searching in *trees*.

  It covers *depth-first*, *breadth-first* and *iterative deepening* search.
- *Algorithms*: talks about searching in *graphs*.

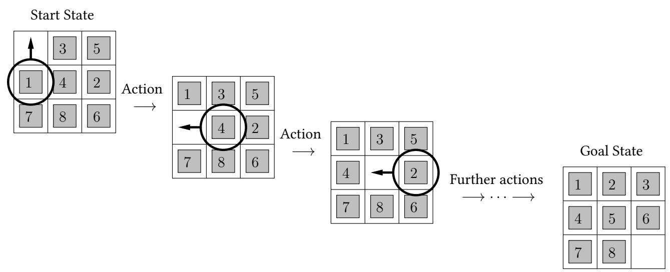
  It also covers *depth-first* and *breadth-first* search, from a more formal perspective.

This is all important stuff, but there's a problem: *none of these methods works in practice for typical AI problems!*

Essentially, the problem is that they are too naïve in the way that they *choose a state to explore* at each step.

I'm going to assume that you know this material and move on...

A simple example: *the 8-puzzle*.



From the *pre-PC dark ages*. Christmas was grim...

Problem solving by search

Here we have:

- *Start state:* a randomly-selected configuration of the numbers 1 to 8 arranged on a 3×3 square grid, with one square empty.
- *Goal state:* the numbers in ascending order with the bottom right square empty.
- *Actions:* `left`, `right`, `up`, `down`. We can move any square adjacent to the empty square into the empty square. (It's not always possible to choose from all four actions.)
- *Path cost:* 1 per move.

The 8-puzzle is very simple. However general sliding block puzzles are a good test case. The general problem is NP-complete. The 5×5 version has about 10^{25} states, and a random instance is in fact quite a challenge.

Problem solving by search

Problems of this kind are very simple, but a surprisingly large number of applications have appeared:

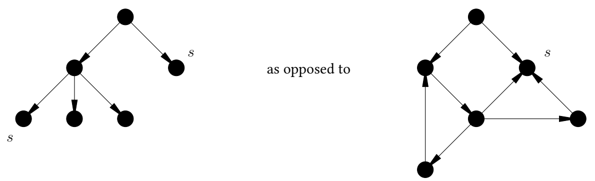
- Route-finding/tour-finding.
- Layout of VLSI systems.
- Navigation systems for robots.
- Sequencing for automatic assembly.
- Searching the internet.
- Design of proteins.

and many others...

Problems of this kind continue to form an active research area.

Search trees versus search graphs

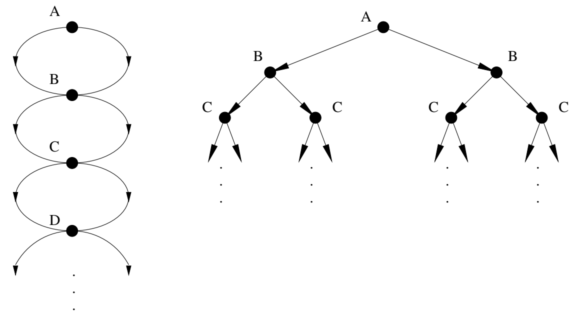
We need to make an important distinction between *search trees* and *search graphs*.



- In a *tree* only *one path* can lead to a given *node*, but a *state* s can appear in multiple nodes.
- In a *graph* a *state* can appear in only one *node*, but may be reached via *multiple paths*.
- In a *graph* we may encounter *cycles*.
- In a *graph* we may encounter *redundant paths*, where multiple paths lead to the same state.

Search trees versus search graphs

Graphs can lead to *problems*:



The *sliding blocks puzzle* for example suffers this way.

So: we start by assuming the search is taking place on a *tree*.

The basic tree-search algorithm

We need to define one more function: `expand` takes any *state* s. It applies all *actions* that can be applied in s and returns the *set of the resulting states*:

$$\texttt{expand}(s) = \{s' | s' = \texttt{action}(a, s) \text{ where } a \text{ is an action possible in } s\}.$$

The algorithm for searching in a tree then looks like this:

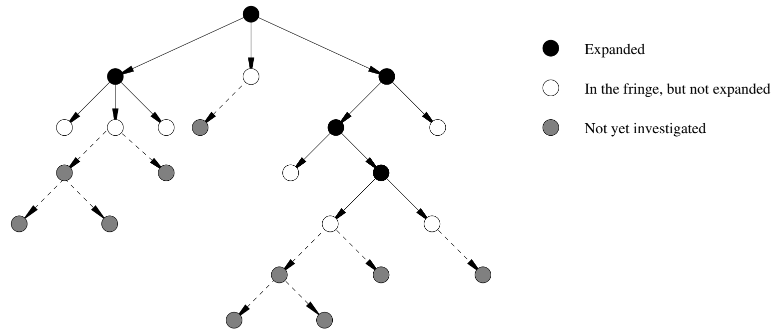
```
fringe = [s_0];
while true do
    if fringe.empty() then
        return NONE;
    s = fringe.remove();
    if goal(s) then
        return (SOME s);
    fringe.addAll(expand(s));
```

The *search strategy* is set by using a *priority queue* to implement the fringe.

The definition of *priority* then sets the way in which the tree is searched.

The basic tree-search algorithm

The process looks like this:



At each iteration, one node from the fringe is expanded. In general, if the *branching factor* is b then the *layer* at *depth* d can have b^d states.

The *entire tree* to depth d can have $\sum_{i=0}^{d} b^d = \frac{b^{d+1}-1}{b-1}$ states.

Graph search

To search in *graphs* we need a way to make sure no state gets visited *more than once*.

We need to add a *closed list*, and add a state to it when the state is *first seen*:

```
closed = [];
fringe = [s_0];
while true do
    if fringe.empty() then
        return NONE;
    s = fringe.remove();
    if goal(s) then
        return (SOME s);
    closed.add(s);
    for s' ∈ expand(s) do
        if (!closed.contains(s') && !fringe.contains(s')) then
            fringe.add (s');
```

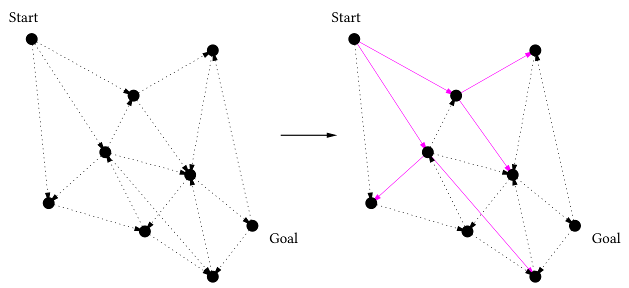

Graph search

There are several points to note regarding graph search:

1. The *closed list* contains all the expanded states.
2. The closed list can be implemented using a *hash table*. So the time taken to *add* or *check membership* can be managable.
3. Both worst case time and space are now *proportional to the size of the state space*. (Which is BIG!!!!)
4. *Memory:* depth first and iterative deepening search are no longer linear space as we need to store the closed list.
5. *Optimality:* when a repeat is found we are *discarding the new possibility even if it is better than the first one*. We may need to check which solution is better and if necessary modify path costs and depths for descendants of the repeated state.

Graph search

Graph search *builds a tree on the graph:*



The graph becomes *separated*: any path from the start to an unexplored node *has to pass through a fringe node.*

The performance of search techniques

How might we judge the performance of a search technique?

We are interested in:

- Whether a solution is found.
- Whether the solution found is a good one in terms of path cost.
- The cost of the search in terms of time and memory.

So

$$\text{the total cost} = \text{path cost} + \text{search cost}$$

If a problem is highly complex it may be worth settling for a *sub-optimal solution* obtained in a *short time*.

And we are interested in:

Completeness: does the strategy *guarantee* a solution is found?

Optimality: does the strategy guarantee that the *best* solution is found?

Once we start to consider these, things get a lot more interesting...

Basic search algorithms

We can immediately define some familiar tree search algorithms:

- New nodes are added to the *head of the queue*. This is *depth-first search.*
- New nodes are added to the *tail of the queue*. This is *breadth-first search.* (You can do the goal test earlier—why is this?)

We will not dwell on these, as they are both *completely hopeless* in practice.

Why is breadth-first search hopeless?

- The procedure is *complete*: it is guaranteed to find a solution if one exists. (Provided $b < \infty$.)
- The procedure is *optimal* if the path cost is a non-decreasing function of node-depth. (For example, when all actions have the same cost.)
- The procedure has *exponential complexity for both memory and time.*

In practice it is the *memory* requirement that is problematic.

Basic search algorithms

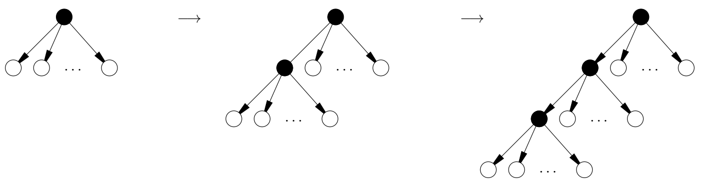
For *depth-first* search:

- With *graph search*: complete if the number of states is *finite*, but not otherwise.
- Not complete for *tree search* because of loops.

Neither is *optimal*.

Basic search methods

With depth-first search: for a given branching factor b and depth d the memory requirement is $O(bd)$.



This is only for *tree search* because we need to store *nodes on the current path* and *the other unexpanded nodes*.

The time complexity for *tree search* is still $O(b^d)$ (if you know you only have to go to depth d). For *graph search* it is the size of the state space.

The search is *no longer optimal*, and may not be *complete*.

Iterative-deepening combines the two, but *we can do better*.

Uniform-cost search

How might we change tree search to try to get to an *optimal solution* while limiting the *time and memory* needed?

The key point: so far we only distinguish *goal states* from *non-goal states*!

None of the searches you've seen so far tries to prioritize the exploration of good states!!!

What is a *good state*?

- Well, at any point in the search we can work out the *path cost* $p(s)$ of whatever state s we've got to.
- How about using the $p(s)$ as the priority for the priority queue?

This is called *Uniform-Cost Search* when implemented as a *graph search*.

Uniform-cost search

It needs a slight modification:

```
closed = [s_0];
fringe = [s_0];
while true do
    if fringe.empty() then
        return NONE;
    s = fringe.remove();
    if goal(s) then
        return (SOME s);
    for s' ∈ expand(s) do
        if (!closed.contains(s') || p(s') < path cost for s' already in closed then
            closed.add (s');
            fringe.add (s');
```

This modification must also be used when implementing the $A^\star$ *search* method, which we will see in a moment.

Uniform-cost search

This is *optimal* because when we select a node it must have the shortest path to that node.

It is complete, provided it is impossible to get stuck within an infinite path:

- Require all costs to have a minimal value of $\epsilon > 0$.
- Require the branching factor to be finte, so $b < \infty$.

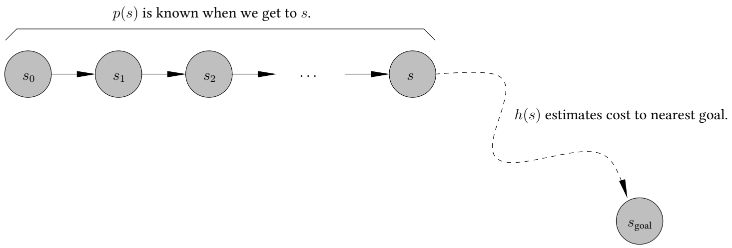
In practice it doesn't work very well: we need *something more subtle*.

But it does suggest the idea of an *evaluation function*: a function that attempts to measure the *desirability of each state*.

Heuristics

Why is *path cost* not a good evaluation function? It is not *directed* in any sense *toward the goal*.

A *heuristic function*, usually denoted $h(s)$, is one that *estimates* the cost of the best path from any state s to a goal. If s is a goal then $h(s) = 0$.



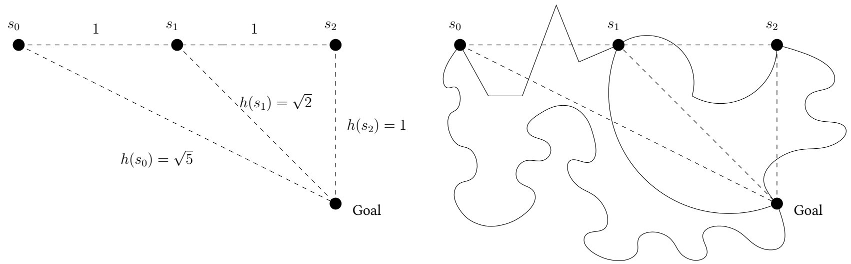
This is a *problem-dependent* measure. We are required either to *design it* using our *knowledge of the problem*, or by some other means.

The last point is critical: *AI is a long way from being independent of human ingenuity*.

Example: route-finding

Example: for route finding a reasonable heuristic function is

$$h(s) = \text{straight line distance from } s \text{ to the nearest goal}$$



Accuracy here obviously depends on what the roads are really like.

Can we use $h(s)$ in choosing a state to explore? If it's *really good* it can work well, but *we can still do better*!

$A^\star$ search

$A^\star$ *search* is the classical *AI-oriented search algorithm*.

$A^\star$ *search* combines the good points of:

- Using $p(s)$ to know how far we've come.
- Using $h(s)$ to estimate how far we have to go.

It does this in a very simple manner: it uses path cost $p(s)$ and also the heuristic function $h(s)$ by forming

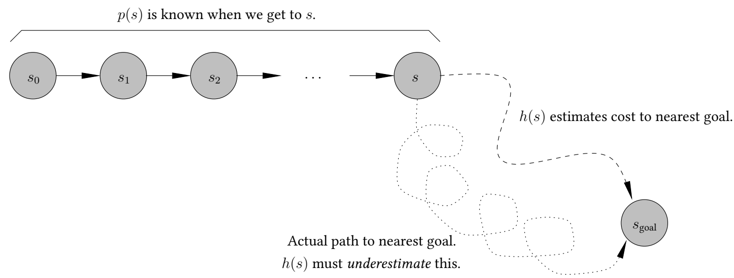
$$f(s) = p(s) + h(s).$$

So: $f(s)$ is the *estimated cost* of a path *through* s.

By using this as a priority for exploring states we get a search algorithm that is *optimal* and *complete* under simple conditions, and can be *vastly superior* to the more naïve approaches.

$A^\star$ search

Definition: an *admissible heuristic* $h(s)$ is one that *never overestimates* the cost of the best path from s to a goal.



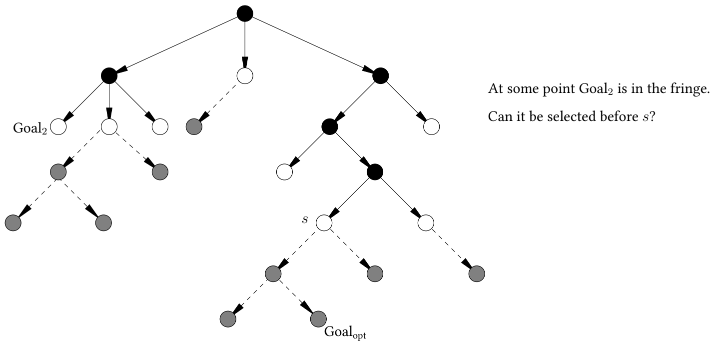
So if $h'(s)$ denotes the *actual* distance from s to the goal we have

$$\forall s . h(s) \leq h'(s).$$

If $h(s)$ is *admissible* then *tree-search* $A^\star$ *is optimal*.

$A^\star$ tree-search is optimal for admissible $h(s)$

To see that *tree-search $A^\star$ is optimal* we reason as follows. Let Goal_{opt} be an optimal goal state with $f(\text{Goal}_{\text{opt}}) = p(\text{Goal}_{\text{opt}}) = f_{\text{opt}}$ (because $h(\text{Goal}_{\text{opt}}) = 0$).



Let Goal_2 be a suboptimal goal state with $f(\text{Goal}_2) = p(\text{Goal}_2) = f_2 > f_{\text{opt}}$. We need to demonstrate that *the search can never select* Goal_2.

$A^\star$ tree-search is optimal for admissible $h(s)$

Let s be a state in the fringe on an optimal path to Goal_{opt}. So

$$f_{\text{opt}} \geq p(s) + h(s) = f(s)$$

because h is admissible.

Now say Goal_2 is chosen for expansion *before* s. This means that

$$f(s) \geq f_2$$

so we've established that

$$f_{\text{opt}} \geq f_2 = p(\text{Goal}_2).$$

But this means that Goal_{opt} is not optimal: a contradiction.

And that's all that's needed for trees. *But for searching on graphs we need a little more...*

$A^\star$ graph search

Unfortunately for *graph search* the situation is trickier...

- Graph search can *discard an optimal* route if that route is not the first one generated.
- We could keep *only the least expensive path.* This means updating, which is extra work, not to mention messy, but sufficient to insure optimality.
- Alternatively, we can impose a further condition on $h(s)$ which *forces the best path to a repeated state to be generated first*.

The required condition is called *monotonicity*. As

$$\text{monotonicity} \longrightarrow \text{admissibility}$$

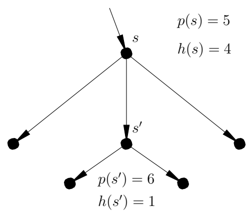
this is an important property.

Monotonicity

Assume h is admissible. Remember that $f(s) = p(s) + h(s)$ so if s' follows s

$$p(s') \geq p(s)$$

and we expect that $h(s') \leq h(s)$ although this does not have to be the case.



Here $f(s) = 9$ and $f(s') = 7$ so $f(s') < f(s)$.

Monotonicity

Monotonicity:

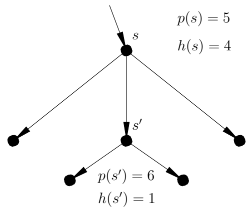
- If it is always the case that $f(s') \geq f(s)$ then $h(s)$ is called *monotonic* or *consistent*.
- $h(s)$ is monotonic if and only if it obeys the *triangle inequality*.

$$h(s) \leq \texttt{cost}(a, s) + h(s')$$

where a is the action moving us from s to s'.

Monotonicity

Why does this make sense?



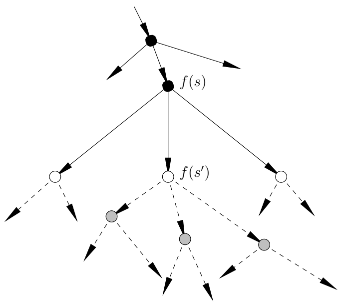
The fact that $f(s) = 9$ tells us the cost of a path through s is *at least* 9 (because $h(s)$ is admissible).

But s' is *on a path through* s. So to say that $f(s') = 7$ makes no sense.

$A^\star$ graph search is optimal for monotonic heuristics

The crucial fact from which optimality follows is that if $h(s)$ is monotonic then the values of $f(s)$ along any path are non-decreasing.

We therefore have the following situation:



You can't deal with s' until everything with $f(s'') < f(s')$ has been dealt with.

Consequently everything with $f(s'') < f_{\text{opt}}$ gets explored. Then one or more things with f_{opt} get found (not necessarily all goals).

$A^\star$ search is complete

$A^\star$ search is *complete* provided:

1. The graph has *finite branching factor*.
2. There is a *finite, positive constant* ϵ such that *each action* has *cost at least* ϵ.

Why is this? The search expands nodes according to increasing $f(s)$. So: the only way it can fail to find a goal is if there are *infinitely many nodes with* $f(s) < f(\text{Goal})$.

There are two ways this can happen:

1. There is a node with an *infinite number of descendants*.
2. There is a path with an *infinite number of nodes* but a *finite path cost*.

Complexity

We won't be *proving* the following, but they are *good things to know*:

- $A^\star$ search has a further desirable property: it is *optimally efficient*.
- This means that no other optimal algorithm that works by constructing paths from the root can *guarantee to examine fewer nodes*.
- *BUT*: despite its good properties we're not done yet...
- ...$A^\star$ search unfortunately still has *exponential time complexity in most cases*.
- As $A^\star$ search also stores all the nodes it generates: once again it is generally *memory that becomes a problem before time*.

IDA* - iterative deepening $A^\star$ search

How might we *improve* the way in which $A^\star$ search uses *memory*?

- Iterative deepening search used depth-first search with a *limit on depth* that is *gradually increased*.
- *IDA** does the same thing *with a limit on* f *cost*.

IDA* - iterative deepening $A^\star$ search

The function contour searches from a specified state s *as far as a specified limit* fLimit *on* f.

It returns either a path from s to a goal, or the *next biggest* value to try for the limit on f.

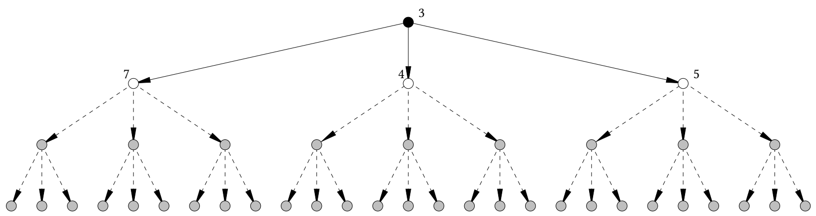
```
function contour(s, fLimit, path)
    nextF = ∞;
    if f(s) > fLimit then
        return ([], f(s));
    if goal(s) then
        return (s :: path, fLimit)
    for s' ∈ expand(s) do
        (newPath, newF) = contour(s', fLimit, s :: path);
        if newPath != [] then
            return (newPath, fLimit);
        nextF = min(nextF, newF);
    return ([], nextF);
```


IDA* - iterative deepening $A^\star$ search

```
function iterativeDeepeningAStar()
    fLimit = f(s0);
    while true do
        (path, fLimit) = contour(s0, fLimit, []);
        if path != [] then
            return path;
        if fLimit == ∞ then
            return [];
```


IDA* - iterative deepening $A^\star$ search

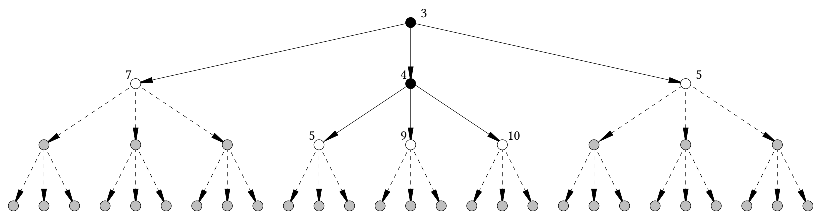
This is a little tricky to unravel, so here is an example:



Initially, the algorithm looks ahead and finds the *smallest* f cost that is *greater than* its current f cost limit. The new limit is 4.

IDA* - iterative deepening $A^\star$ search

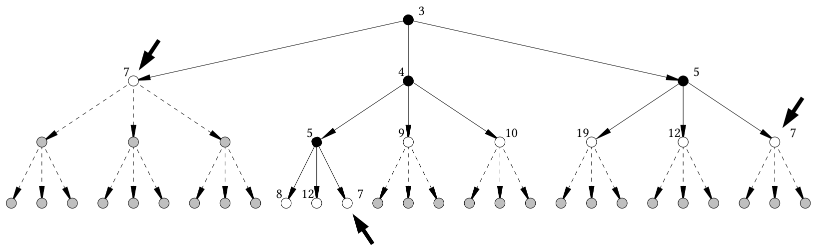
It now does the same again:



Anything with f cost *at most* equal to the current limit gets explored, and the algorithm keeps track of the *smallest* f cost that is *greater than* its current limit. The new limit is 5.

IDA* - iterative deepening $A^\star$ search

And again:



The new limit is 7, so at the next iteration the three arrowed nodes will be explored.

IDA* - iterative deepening $A^\star$ search

Properties of IDA*:

- It is complete and optimal under the same conditions as $A^\star$.
- It is often good if we have step costs equal to 1.
- It does not require us to maintain a sorted queue of nodes.
- It only requires *space proportional to the longest path.*
- The time taken depends on the number of values h can take.

If h takes enough values to be problematic we can increase the limit on f by a fixed ϵ at each stage, guaranteeing a solution at most ϵ worse than the optimum.

Recursive best-first search (RBFS)

Another method by which we can attempt to overcome memory limitations is the *Recursive Best-First Search (RBFS)*.

Idea: try to use f, but only use *linear space* by doing a depth-first search with a few modifications:

1. We remember the $f(s')$ for the best alternative state s' we've seen so far on the way to the state s we're currently considering.
2. If s has $f(s) > f(s')$:
   - We go back and explore the best alternative...
   - ...and as we retrace our steps we replace the f cost of every state we've seen in the current path with $f(s)$. (See red text in pseudo-code.)

The replacement of f values as we retrace our steps provides a means of remembering how good a discarded path might be, so that we can easily return to it later.

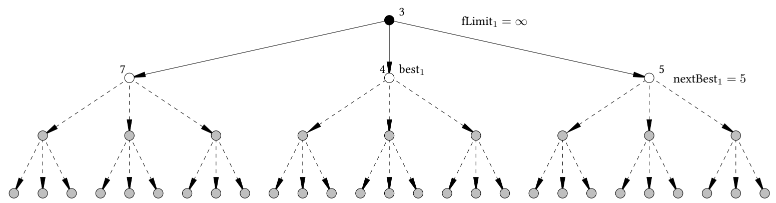
Recursive best-first search (RBFS)

```
function rbfs(s, fLimit)
  if goal(s) then
    return (SOME s, fLimit);
  if expand(s) = ∅ then
    return (NONE, ∞);
  for each s' ∈ expand(s) do
    f(s') = maximum(f(s'), f(s));
  while true do
    best = s' ∈ expand(s) with smallest f(s');
    if f(best) > fLimit then
      return (NONE, f(best));
    nextBest = s' ∈ expand(s) with second smallest f(s');
    (result, f') = rbfs (best, minimum(fLimit, f(nextBest)));
    f(best) = f';
    if result != NONE then
      return (result, f');
```


Recursive best-first search (RBFS): an example

This function is called using $\texttt{rbfs}(s_0, \infty)$ to begin the process.

Function call number 1:

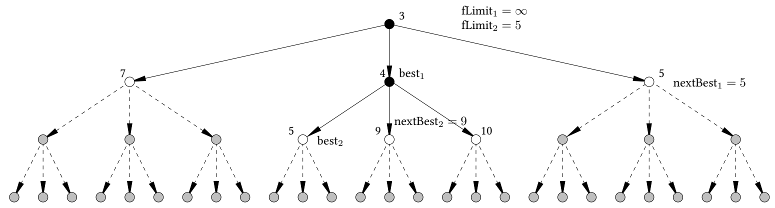


Now perform the recursive function call $(\text{result}_2, f') = \texttt{rbfs}(\text{best}_1, 5)$

so $f(\text{best}_1)$ takes the returned value f'

Recursive best-first search (RBFS): an example

Function call number 2:

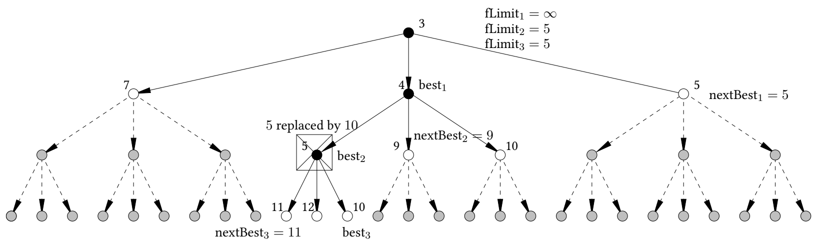


Now perform the recursive function call $(\text{result}_3, f') = \texttt{rbfs}(\text{best}_2, 5)$

so $f(\text{best}_2)$ takes the returned value f'

Recursive best-first search (RBFS): an example

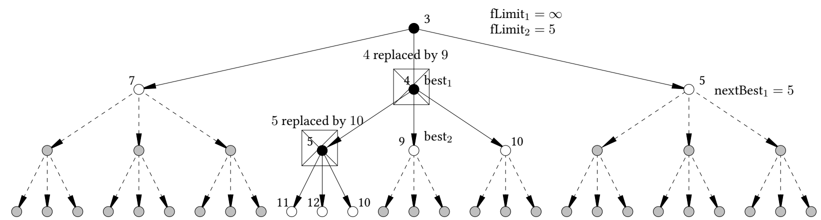
Function call number 3 :



Now $f(\text{best}_3) > \text{fLimit}_3$ so the function call returns $(\text{NONE}, 10)$ into (result_3, f') and $f(\text{best}_2) = 10$.

Recursive best-first search (RBFS): an example

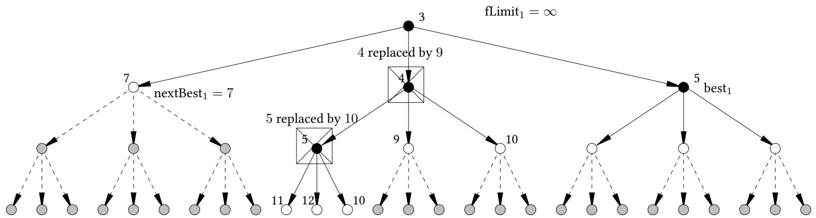
The while loop for function call 2 now repeats:



Now $f(\text{best}_2) > \text{fLimit}_2$ so the function call returns $(\text{NONE}, 9)$ into (result_2, f') and $f(\text{best}_1) = 9$.

Recursive best-first search (RBFS): an example

The while loop for function call 1 now repeats:



We do a further function call to expand the new best node, and so on...

Recursive best-first search (RBFS)

Some nice properties:

- If h is admissible then RBFS is optimal.
- Memory requirement is $O(bd)$
- Generally more efficient than IDA*.

And some less nice ones:

- Time complexity is hard to analyse, but can be exponential.
- Can spend a lot of time *re-generating nodes*.

To some extent IDA* and RBFS throw the baby out with the bathwater.

- They limit memory too harshly, so...
- ...we can try to use *all available memory*.

MA* and SMA* will not be covered in this course...

Local search

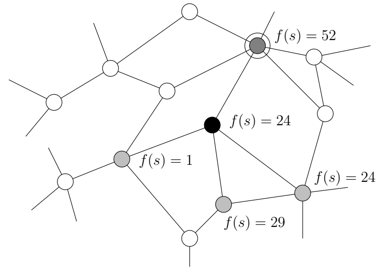
Sometimes, it's only the *goal* that we're interested in. The *path* needed to get there is irrelevant.

- For example: VLSI layout, factory design, automatic programming...
- We are now simply searching for a state that is in some sense *the best*.
- This is also known as *optimisation*.

This leads to the remarkably simple concept of *local search*.

Local search

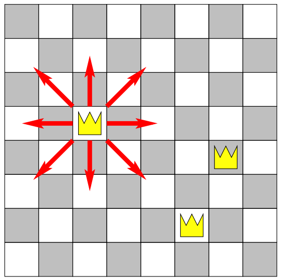
Instead of trying to find a path from start state to goal, we explore the *local area* of the graph, meaning those states one edge away from the one we're at:



We assume that we have a function $f(s)$ such that $f(s') > f(s)$ indicates s' is preferable to s.

The m-queens problem

You may be familiar with the m*-queens problem*.



Find an arrangement of m queens on an m by m board such that no queen is attacking another.

In the Prolog course you may have been tempted to generate permutations of row numbers and test for attacks.

This is a *hopeless strategy* for large m. (Imagine $m \simeq 1,000,000$.)

The m-queens problem

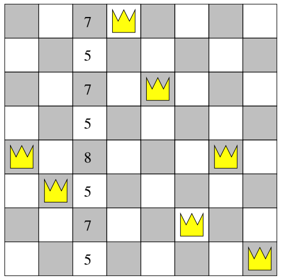
We might however consider the following:

- A state s for an m by m board is a sequence of m numbers drawn from the set $\{1, \dots, m\}$, possibly including repeats.
- We move from one state to another by moving a *single queen* to *any* alternative row.
- We define $f(s)$ to be the number of pairs of queens attacking one-another in the new position[2]. (Regardless of whether or not the attack is direct.)

[2] Note that we actually want to *minimize* f here. This is equivalent to maximizing $-f$, and I will generally use whichever seems more appropriate.

The m-queens problem

Here, we have $\{4, 3, ?, 8, 6, 2, 4, 1\}$ and the f values for the undecided queen are shown.



As we can choose which queen to move, each state in fact has 56 neighbours in the graph.

Hill-climbing search

Hill-climbing search is remarkably simple:

```
Generate a start state s;
while true do
    Generate the neighbours N = {s_1, ..., s_p} of s;
    N_f = {f(s_i)|s_i ∈ N};
    if max N_f ≤ f(s) then
        return s;
    s = s_i ∈ N with maximum f(s_i);
```

In fact, that looks so simple that it's amazing the algorithm is at all useful.

In this version we stop when we get to a node with no better neighbour.

Hill-climbing search: the reality

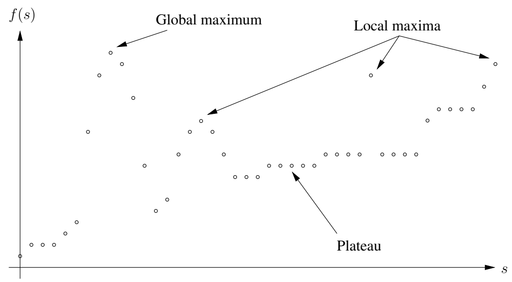
We might alternatively allow *sideways moves* by changing the stopping condition:

```
if max N_f < f(s) then
    return s;
```

Why would we consider doing this?

Hill-climbing search: the reality

In reality, nature has a number of ways of shaping f to complicate the search process.



Sideways moves allow us to move across *plateaus*.

However, should we ever find a *local maximum* then we'll return it: we won't keep searching to find a *global maximum*.

Hill-climbing search: the reality

Of course, the fact that we're dealing with a *general graph* means we need to think of something like the preceding figure, but in a *very large number of dimensions*, and this makes the problem *much harder*.

There is a body of techniques for trying to overcome such problems. For example:

- *Stochastic hill-climbing:* Choose a neighbour at random, perhaps with a probability depending on its f value. For example: let $N(s)$ denote the neighbours of s. Define

$$N^+(s) = \{s' \in N(s) | f(s') \geq f(s)\}$$
$$N^-(s) = \{s' \in N(s) | f(s') < f(s)\}.$$

Then

$$\Pr(s') = \begin{cases} 0 & \text{if } s' \in N^-(s) \\ \frac{1}{Z}(f(s') - f(s)) & \text{otherwise.} \end{cases}$$

Hill-climbing search: the reality

- *First choice:* Generate neighbours at random. Select the first one that is better than the current one. (Particularly good if nodes have *many neighbours.*)
- *Random restarts:* Run a procedure k times with a limit on the time allowed for each run.

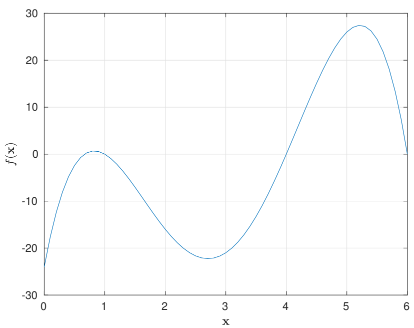
  Note: generating a start state at random may itself not be straightforward.
- *Simulated annealing:* Similar to stochastic hill-climbing, but start with lots of random variation and *reduce it over time.*

  Note: in some cases this is *provably* an effective procedure, although the time taken may be excessive if we want the proof to hold.
- *Beam search:* Maintain k states at any given time. At each search step, find the successors of each, and retain the best k from *all* the successors.

  Note: this is *not* the same as random restarts.

Gradient ascent and related methods

For some problems[3]—we do not have a search graph, but a *continuous search space.*



Typically, we have a function $f(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ and we want to find

$$\mathbf{x}_{\text{opt}} = \underset{\mathbf{x}}{\operatorname{argmax}} f(\mathbf{x})$$

[3]For the purposes of this course, the *training of neural networks* is a notable example.

Gradient ascent and related methods

In a single dimension we can clearly try to solve

$$\frac{df(x)}{dx} = 0$$

to find the *stationary points*, and use

$$\frac{d^2 f(x)}{dx^2}$$

to find a global *maximum*. In *multiple dimensions* the equivalent is to solve

$$\nabla f(\mathbf{x}) = \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} = \mathbf{0}$$

where

$$\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial f(\mathbf{x})}{\partial x_1} & \frac{\partial f(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f(\mathbf{x})}{\partial x_n} \end{bmatrix}.$$

and the equivalent of the second derivative is the *Hessian* matrix

$$\mathbf{H} = \begin{bmatrix} \frac{\partial f^2(\mathbf{x})}{\partial x_1^2} & \frac{\partial f^2(\mathbf{x})}{\partial x_1 \partial x_2} & \cdots & \frac{\partial f^2(\mathbf{x})}{\partial x_1 \partial x_n} \\ \frac{\partial f^2(\mathbf{x})}{\partial x_2 \partial x_1} & \frac{\partial f^2(\mathbf{x})}{\partial x_2^2} & \cdots & \frac{\partial f^2(\mathbf{x})}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f^2(\mathbf{x})}{\partial x_n \partial x_1} & \frac{\partial f^2(\mathbf{x})}{\partial x_n \partial x_2} & \cdots & \frac{\partial f^2(\mathbf{x})}{\partial x_n^2} \end{bmatrix}.$$

Gradient ascent and related methods

However this approach is usually *not analytically tractable* regardless of dimensionality.

The simplest way around this is to employ *gradient ascent*:

- Start with a randomly chosen point $\mathbf{x}_0$.
- Using a small *step size* ϵ, iterate using the equation

$$\mathbf{x}_{i+1} = \mathbf{x}_i + \epsilon \nabla f(\mathbf{x}_i).$$

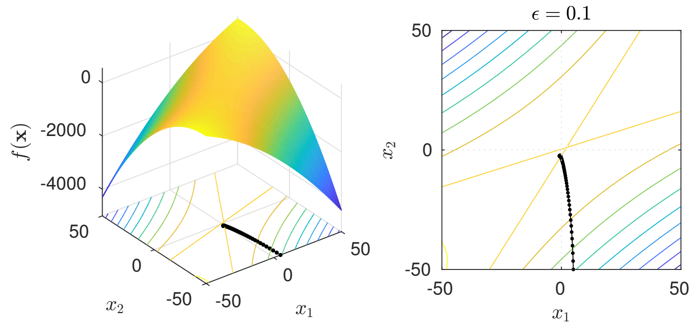
This can be understood as follows:

- At the current point $\mathbf{x}_i$ the gradient $\nabla f(\mathbf{x}_i)$ tells us the *direction* and *magnitude* of the slope at $\mathbf{x}_i$.
- Adding $\epsilon \nabla f(\mathbf{x}_i)$ therefore moves us a *small distance upward*.

This is perhaps more easily seen graphically...

Gradient ascent and related methods

Here we have a simple *parabolic surface*:

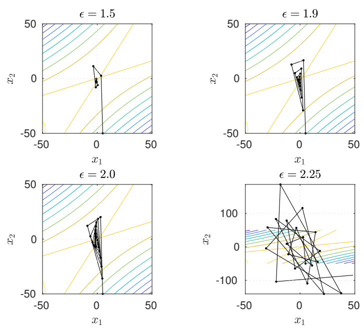


With $\epsilon = 0.1$ the procedure is clearly effective at finding the maximum.

Note however that *the steps are small*, and in a more realistic problem *it might take some time...*

Gradient ascent and related methods

Simply increasing the step size ϵ can lead to a different problem:



We can easily jump too far...

Gradient ascent and related methods

There is a large collection of more sophisticated methods. For example:

- *Line search:* increase ϵ until f *decreases* and maximise in the resulting interval. Then choose a new direction to move in. *Conjugate gradients*, the *Fletcher-Reeves* and *Polak-Ribiere* methods etc.
- Use $\mathbf{H}$ to exploit knowledge of the local shape of f. For example the *Newton-Raphson* and *Broyden-Fletcher-Goldfarb-Shanno (BFGS)* methods etc.

Artificial Intelligence

Games (adversarial search)

Reading: AIMA chapter 6.

Solving problems by search: playing games

How might an agent act when *the outcomes of its actions are not known* because an *adversary is trying to hinder it*?

- This is essentially a more realistic kind of search problem because we do not know the exact outcome of an action.
- This is a common situation when *playing games*: in chess, draughts, and so on an opponent *responds* to our moves.

Game playing has been of interest in AI because it provides an *idealisation* of a world in which two agents act to *reduce* each other's well-being.

We now look at:

- How game-playing can be modelled as *search*.
- The *minimax algorithm* for game-playing.
- Some problems inherent in the use of minimax.
- The concept of $\alpha - \beta$ *pruning*.

Playing games: search against an adversary

Despite the fact that games are an idealisation, game playing can be an excellent source of hard problems. For instance with chess:

- The average branching factor is roughly 35.
- Games can reach 50 moves per player.
- So a rough calculation gives the search tree 35^{100} nodes.
- Even if only different, legal positions are considered it's about 10^{40}.

So: in addition to the uncertainty due to the opponent:

- We can't make a complete search to find the best move...
- ... so we have to act even though we're not sure about the best thing to do.

And chess isn't even very hard: *Go* is *much* harder...

Note: yes, more advanced learning-based methods have conquered chess and Go, but that's an entirely different approach with its own pros and cons.

Perfect decisions in a two-person game

Say we have two players. Traditionally, they are called *Max* and *Min* for reasons that will become clear.

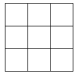
- We'll use *noughts and crosses* as an initial example.
- *Max* moves first.
- The players alternate until the game ends.
- At the end of the game, prizes are awarded. (Or punishments administered—EVIL ROBOT is starting up his favourite chainsaw...)

This is exactly the same game format as chess, Go, draughts and so on.

Perfect decisions in a two-person game

Games like this can be modelled as search problems as follows:

- There is an *initial state.*



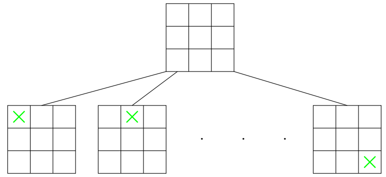
- There is a set of *operators*. Here, *Max* can place a cross in any empty square, or *Min* a nought.
- There is a *terminal test*. Here, the game ends when three noughts or three crosses are in a row, or there are no unused spaces.
- There is a *utility* or *payoff* function. This tells us, numerically, what the outcome of the game is.

This is enough to model the entire game.

Perfect decisions in a two-person game

We can *construct a tree* to represent a game.

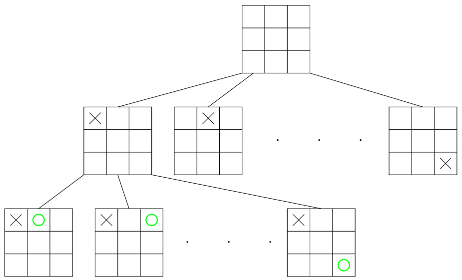
From the initial state *Max* can make nine possible moves:



Then it's *Min*'s turn...

Perfect decisions in a two-person game

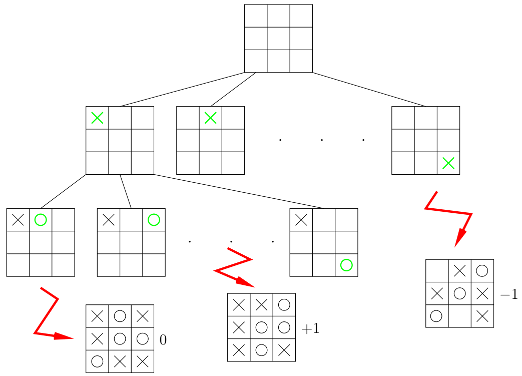
For each of *Max*'s opening moves *Min* has eight replies:



And so on...

This can be continued to represent *all* possibilities for the game.

Perfect decisions in a two-person game

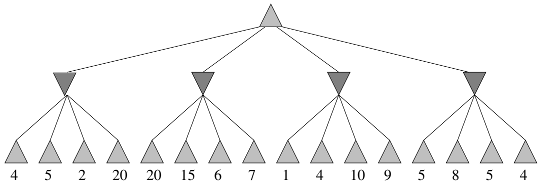


At the leaves a player has won or there are no spaces. Leaves are *labelled* using the utility function.

Perfect decisions in a two-person game

How can *Max* use this tree to decide on a first move?

Consider a much simpler tree:



Labels on the leaves denote utility.
High values are preferred by Max.
Low values are preferred by Min.

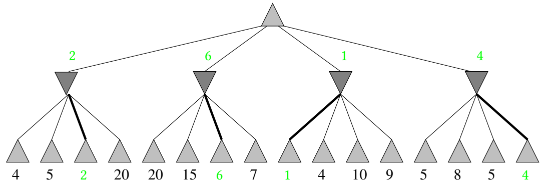
If *Max* is rational he will play to reach a position with the *biggest utility possible*

But if *Min* is rational she will play to *minimise* the utility available to *Max*.

The minimax algorithm

There are two moves: *Max* then *Min*. Game theorists would call this one move, or two *ply* deep.

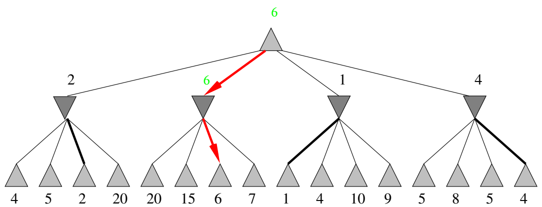
The *minimax algorithm* allows us to infer the best move that the current player can make, given the utility function, by working backward from the leaves.



As *Min* plays the last move, she *minimises* the utility available to *Max*.

The minimax algorithm

Moving one further step up the tree:



We can see that *Max*'s best opening move is move 2, as this leads to the node with highest utility.

The minimax algorithm

In general:

- Generate the complete tree and label the leaves according to the utility function.
- Working from the leaves of the tree upward, label the nodes depending on whether *Max* or *Min* is to move.
- If *Min* is to move label the current node with the *minimum* utility of any descendant.
- If *Max* is to move label the current node with the *maximum* utility of any descendant.

If the game is p ply and at each point there are q available moves then this process has (surprise, surprise) $O(q^p)$ time complexity and space complexity linear in p and q.

Making imperfect decisions

We need to avoid searching all the way to the end of the tree.

So:

- We generate only part of the tree: instead of testing whether a node is a leaf we introduce a *cut-off* test telling us when to stop.
- Instead of a utility function we introduce an *evaluation function* for the evaluation of positions for an incomplete game.

The evaluation function attempts to measure the expected utility of the current game position.

Making imperfect decisions

How can this be justified?

- This is a strategy that humans clearly sometimes make use of.
- For example, when using the concept of *material value* in chess.
- The effectiveness of the evaluation function is *critical*...
- ... but it must be computable in a reasonable time.
- (In principle it could just be done using minimax.)

The importance of the evaluation function can not be understated—it is probably the most important part of the design.

The evaluation function

Designing a good evaluation function can be extremely tricky:

- Let's say we want to design one for chess by giving each piece its material value: pawn = 1, knight/bishop = 3, rook = 5 and so on.
- Define the evaluation of a position to be the difference between the material value of black's and white's pieces

$$\text{eval}(\text{position}) = \sum_{\text{black's pieces } p_i} \text{value of } p_i - \sum_{\text{white's pieces } q_i} \text{value of } q_i$$

This seems like a reasonable first attempt. Why might it go wrong?

- Until the first capture the evaluation function gives 0, so in fact we have a *category* containing many different game positions with equal estimated utility.
- For example, all positions where white is one pawn ahead.

So in fact this seems highly naïve ...

The evaluation function

We can try to *learn* an evaluation function.

- For example, using material value, construct a *weighted linear evaluation function*

$$\text{eval}(\text{position}) = \sum_{i=1}^{n} w_i f_i$$

where the w_i are *weights* and the f_i represent *features* of the position—in this case, the value of the ith piece.
- Weights can be chosen by allowing the game to play itself and using *learning* techniques to adjust the weights to improve performance.

However in general

- Here we probably want to give *different evaluations* to *individual positions*.
- The design of an evaluation function can be highly *problem dependent* and might require significant *human input and creativity*.

$\alpha - \beta$ pruning

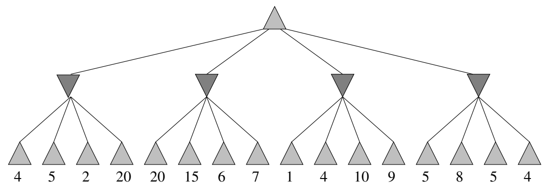
Even with a good evaluation function and cut-off test, the time complexity of the minimax algorithm makes it impossible to write a good chess program without some further improvement.

- Assuming we have 150 seconds to make each move, for chess we would be limited to a search of about 3 to 4 ply whereas...
- ...even an average human player can manage 6 to 8.

Luckily, it is possible to prune the search tree *without affecting the outcome* and *without having to examine all of it.*

$\alpha - \beta$ pruning

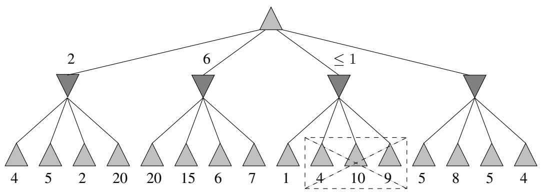
Returning for a moment to the earlier, simplified example:



The search is depth-first and left to right.

$\alpha - \beta$ pruning

The search continues as previously for the first 8 leaves.

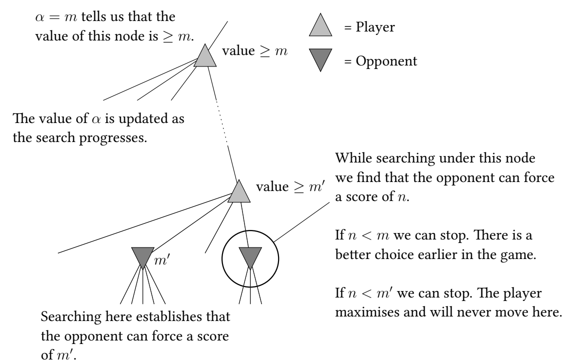


Then we note: if *Max* plays move 3 then *Min* can reach a leaf with utility at most 1.

So: *we don't need to search any further under Max's opening move* 3. This is because the search has *already established* that *Max* can do better by making opening move 2.

$\alpha - \beta$ pruning in general

Remember that this search is *depth-first*. We're only going to use knowledge of *nodes on the current path.*



So: once you've established that n is sufficiently small, you don't need to explore any more of the corresponding node's children.

$\alpha - \beta$ pruning in general

The situation is exactly analogous if we *swap player and opponent* in the previous diagram.

The search is depth-first, so we're only ever looking at *one path through the tree.*

We need to keep track of the values α and β where

$\alpha =$ the *highest* utility seen so far on the path for *Max*

$\beta =$ the *lowest* utility seen so far on the path for *Min*

Assume *Max begins.* Initial values for α and β are

$$\alpha = -\infty$$

and

$$\beta = +\infty.$$

$\alpha - \beta$ pruning in general

So: we start with the function call

$$\texttt{player}(-\infty, +\infty, \texttt{root})$$

The following function implements the procedure suggested by the previous diagram:

```
function player(α, β, n)
    if cutoff(n) then
        return eval(n);
    value = −∞;
    for each successor n′ of n do
        value = max(value, opponent(α, β, n′));
        if value ≥ β then
            return value;
        if value > α then
            α = value;
    return value;
```


$\alpha - \beta$ pruning in general

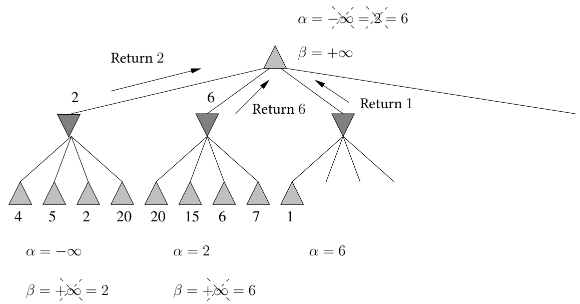
The function `opponent` is exactly analogous:

```
function opponent(α, β, n)
    if cutoff(n) then
        return eval(n);
    value = ∞;
    for each successor n′ of n do
        value = min(value, player(α, β, n′));
        if value ≤ α then
            return value;
        if value < β then
            β = value;
    return value;
```

Note: the semantics here is that parameters are passed to functions *by value.*

$\alpha - \beta$ pruning in general

Applying this to the earlier example and keeping track of the values for α and β you should obtain:



How effective is $\alpha - \beta$ pruning?

(Warning: the theoretical results that follow are somewhat idealised.)

A quick inspection should convince you that the *order* in which moves are arranged in the tree is critical.

So, it seems sensible to try good moves first:

- If you were to have a perfect move-ordering technique then $\alpha - \beta$ pruning would be $O(q^{p/2})$ as opposed to $O(q^p)$.
- Consequently the branching factor would effectively be $\sqrt{q}$ instead of q.
- We would therefore expect to be able to search ahead *twice as many moves as before*.

However, this is not realistic: if you had such an ordering technique you'd be able to play perfect games!

How effective is $\alpha - \beta$ pruning?

If moves are arranged at random then $\alpha - \beta$ pruning is:

- $O((q/\log q)^p)$ asymptotically when $q > 1000$ or...
- ...about $O(q^{3p/4})$ for reasonable values of q.

In practice *simple ordering techniques* can get *close to the best case*. For example, if we try captures, then threats, then moves forward *etc.*

Alternatively, we can implement an *iterative deepening* approach and use the order obtained at one iteration to drive the next.

A further optimisation: the transposition table

Finally, note that many games correspond to *graphs* rather than *trees* because the same state can be arrived at in different ways.

- This is essentially the same effect we saw in heuristic search: recall *graph search* versus *tree search*.
- It can be addressed in a similar way: store a state with its evaluation in a hash table—generally called a *transposition table*—the first time it is seen.

The transposition table is essentially equivalent to the *closed list* introduced as part of graph search.

This can vastly increase the effectiveness of the search process, because we don't have to evaluate a single state multiple times.

Artificial Intelligence

Constraint satisfaction problems (CSPs)

Reading: AIMA chapter 5.

Constraint satisfaction problems (CSPs)

The search scenarios examined so far seem in some ways unsatisfactory.

- States were represented using an *arbitrary* and *problem-specific* data structure.
- Heuristics were also *problem-specific*.
- It would be nice to be able to *transform* general search problems into a *standard format*.

CSPs *standardise* the manner in which states and goal tests are represented. By standardising like this we benefit in several ways:

- We can devise *general purpose* algorithms and heuristics.
- We can look at general methods for exploring the *structure* of the problem.
- Consequently it is possible to introduce techniques for *decomposing* problems.
- We can try to understand the relationship between the *structure* of a problem and the *difficulty of solving it*.

Introduction to constraint satisfaction problems

We now return to the idea of problem solving by search and examine it from this new perspective.

Aims:

- To introduce the idea of a constraint satisfaction problem (CSP) as a general means of representing and solving problems by search.
- To look at a *backtracking algorithm* for solving CSPs.
- To look at some *general heuristics* for solving CSPs.
- To look at *more intelligent ways of backtracking*.

Another method of interest in AI that allows us to do similar things involves transforming to a *propositional satisfiability* problem.

We'll see an example of this—and of the application of CSPs—when we discuss *planning*.

Constraint satisfaction problems

We have:

- A set of n *variables* $V_1, V_2, \ldots, V_n$.
- For each V_i a *domain* D_i specifying the values that V_i can take.
- A set of m *constraints* $C_1, C_2, \ldots, C_m$.

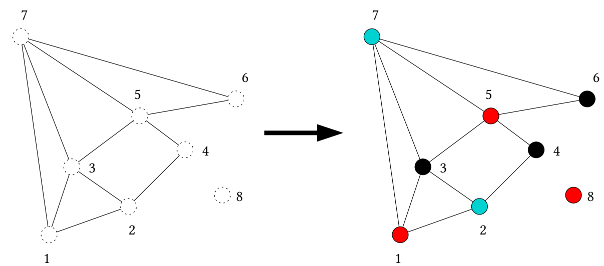
Each constraint C_i involves a set of variables and specifies an *allowable collection of values*.

- A *state* is an assignment of specific values to some or all of the variables.
- An assignment is *consistent* if it violates no constraints.
- An assignment is *complete* if it gives a value to every variable.

A *solution* is a consistent and complete assignment.

Example

We will use the problem of *colouring the nodes of a graph* as a running example.



Each node corresponds to a *variable*. We have three colours and directly connected nodes should have different colours.

Example

This translates easily to a CSP formulation:

- The variables are the nodes
$$V_i = \text{node } i$$
- The domain for each variable contains the values black, red and cyan
$$D_i = \{B, R, C\}$$
- The constraints enforce the idea that directly connected nodes must have different colours. For example, for variables V_1 and V_2 the constraints specify
$$(B, R), (B, C), (R, B), (R, C), (C, B), (C, R)$$
- Variable V_8 is unconstrained.

Different kinds of CSP

This is an example of the simplest kind of CSP: it is *discrete* with *finite domains*. We will concentrate on these.

We will also concentrate on *binary constraints*; that is, constraints between *pairs of variables*.

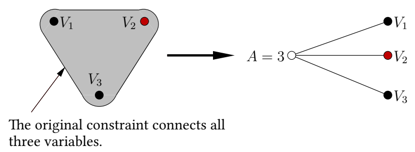
- Constraints on single variables—*unary constraints*—can be handled by adjusting the variable's domain. For example, if we don't want V_i to be *red*, then we just remove that possibility from D_i.
- *Higher-order constraints* applying to three or more variables can certainly be considered, but...
- ...when dealing with finite domains they can always be converted to sets of binary constraints by introducing extra *auxiliary variables*.

How does that work?

Auxiliary variables

Example: three variables each with domain $\{B, R, C\}$.

A single constraint
$$(C, C, C), (R, B, B), (B, R, B), (B, B, R)$$



New, binary constraints:

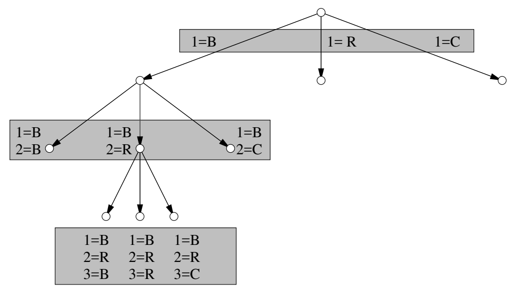
$(A = 1, V_1 = C), (A = 1, V_2 = C), (A = 1, V_3 = C)$
$(A = 2, V_1 = R), (A = 2, V_2 = B), (A = 2, V_3 = B)$
$(A = 3, V_1 = B), (A = 3, V_2 = R), (A = 3, V_3 = B)$
$(A = 4, V_1 = B), (A = 4, V_2 = B), (A = 4, V_3 = R)$

Introducing auxiliary variable A with domain $\{1, 2, 3, 4\}$ allows us to convert this to a set of binary constraints.

Backtracking search

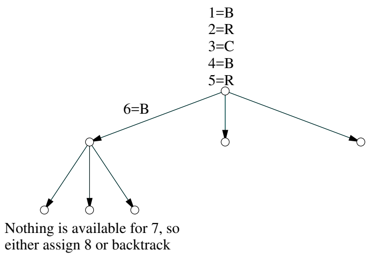
Backtracking search now takes on a very simple form: search depth-first, assigning a single variable at a time, and backtrack if no valid assignment is available.

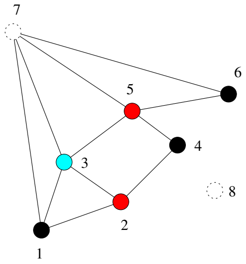
Using the graph colouring example, the search now looks something like this...



...and new possibilities appear.

Backtracking search





Rather than using problem-specific heuristics to try to improve searching, we can now explore heuristics applicable to *general* CSPs.

Backtracking search

Starting with:

$$\texttt{backtrack}([], \texttt{problemDescription})$$

```
function backTrack(assignmentList, problemDescription)
  if assignmentList is complete then
    return SOME assignmentList;
  nextVar = getNextVariable(assignmentList, problemDescription);
  for each v in orderValues(nextVar, assignmentList, problemDescription) do
    if v is consistent with assignmentList then
      add "nextVar = v" to assignmentList;
      solution = backTrack(assignmentList, problemDescription);
      if solution is not FAIL then
        return solution;
      remove "nextVar = v" from assignmentList;
  return FAIL;
```


Backtracking search: possible heuristics

There are several points we can examine in an attempt to obtain general CSP-based heuristics:

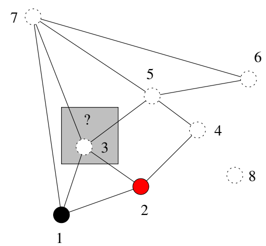
- In what order should we try to *assign variables*?
- In what order should we try to *assign possible values* to a variable?

Or being a little more subtle:

- What *effect* might the *values assigned so far* have on *later attempted assignments*?
- When *forced to backtrack*, is it possible to *avoid the same failure later on*?
- Can we try to force the search in a successful direction (remember the use of *heuristics*)?
- Can we try to force *failures/backtracks* to occur quickly?

Heuristics I: Choosing the order of variable assignments and values

Say we have $1 = B$ and $2 = R$



At this point there is *only one possible assignment* for 3, whereas the others have more flexibility.

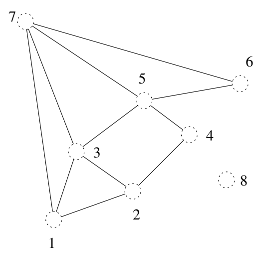
Assigning such variables *first* is called the *minimum remaining values (MRV)* heuristic.

(Alternatively, the *most constrained variable* or *fail first* heuristic.)

Heuristics I: Choosing the order of variable assignments and values

How do we choose a variable to begin with?

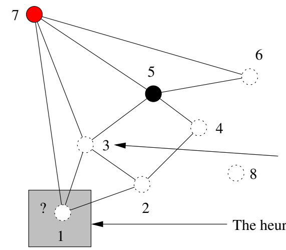
The *degree heuristic* chooses the variable involved in the most constraints on as yet unassigned variables.



MRV is usually better but the degree heuristic is a good tie breaker.

Heuristics I: Choosing the order of variable assignments and values

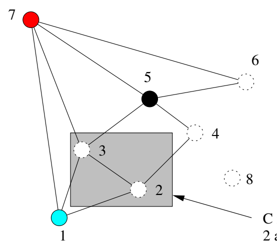
Once a variable is chosen, in *what order should values be assigned*?



The *least constraining value* heuristic chooses first the value that leaves the maximum possible freedom in choosing assignments for the variable's neighbours.

Heuristics II: forward checking and constraint propagation

Continuing the previous slide's progress, now add $1 = C$.



Each time we assign a value to a variable, it makes sense to delete that value from the collection of *possible assignments to its neighbours*.

This is called *forward checking*. It works nicely in conjunction with MRV.

Heuristics II: forward checking and constraint propagation

We can visualise this process as follows:

	1	2	3	4	5	6	7	8
Start	BRC	BRC	BRC	BRC	BRC	BRC	BRC	BRC
$2 = B$	RC	$= B$	RC	RC	BRC	BRC	BRC	BRC
$3 = R$	C	$= B$	$= R$	RC	BC	BRC	BC	BRC
$6 = B$	C	$= B$	$= R$	RC	C	$= B$	C	BRC
$5 = C$	C	$= B$	$= R$	R	$= C$	$= B$	!	BRC

At the fourth step 7 has *no possible assignments left*.

However, we could have detected a problem a little earlier...

Heuristics II: forward checking and constraint propagation

...by looking at step three.

	1	2	3	4	5	6	7	8
Start	BRC	BRC	BRC	BRC	BRC	BRC	BRC	BRC
$2=B$	RC	$=B$	RC	RC	BRC	BRC	BRC	BRC
$3=R$	C	$=B$	$=R$	RC	BC	BRC	BC	BRC
$6=B$	C	$=B$	$=R$	RC	C	$=B$	C	BRC
$5=C$	C	$=B$	$=R$	R	$=C$	$=B$	!	BRC

- At step three, 5 can be C only and 7 can be C only.
- But 5 and 7 are connected.
- So we can't progress, but this hasn't been detected.
- Ideally we want to do *constraint propagation*.

Trade-off: time to do the search, against time to explore constraints.

Constraint propagation

Arc consistency:

Consider a constraint as being *directed*. For example $4 \rightarrow 5$.

In general, say we have a constraint $i \rightarrow j$ and currently the domain of i is D_i and the domain of j is D_j.

$i \rightarrow j$ is *consistent* if

$$\forall d \in D_i, \exists d' \in D_j \text{ such that } i \rightarrow j \text{ is valid}$$

Example:

In step three of the table, $D_4 = \{R, C\}$ and $D_5 = \{C\}$.

- $5 \rightarrow 4$ in step three of the table *is consistent*.
- $4 \rightarrow 5$ in step three of the table *is not consistent*.

$4 \rightarrow 5$ can be made consistent by deleting C from D_4.

Or in other words, regardless of what you assign to i you'll be able to find something valid to assign to j.

Enforcing arc consistency

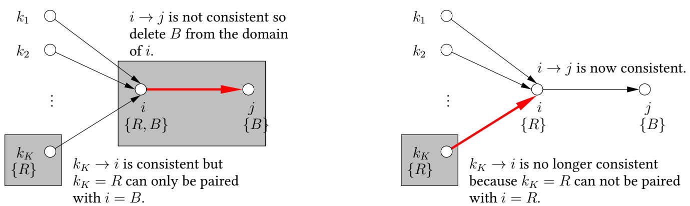
We can enforce arc consistency each time a variable i is assigned.

- We need to maintain a *collection of arcs to be checked*.
- Each time we alter a domain, we may have to include further arcs in the collection.

This is because if $i \rightarrow j$ is inconsistent resulting in a deletion from D_i we may as a consequence make some arc $k \rightarrow i$ inconsistent.

Why is this?

Enforcing arc consistency



- $i \rightarrow j$ inconsistent means removing a value from D_i.
- $\exists d \in D_i$ such that there is no valid $d' \in D_j$ *so delete* $d \in D_i$.

However some $d'' \in D_k$ may only have been pairable with d.

We need to continue until all consequences are taken care of.

The AC-3 algorithm

```
function AC-3(problemDescription)
    Queue toCheck = [ all arcs i → j ];
    while toCheck is not empty do
        i → j = next(toCheck);
        if removeInconsistencies(D_i, D_j) then
            if domain of i is empty then
                return FALSE;
            for each k that is a neighbour of i, where k ∉ {i, j} do
                add k → i to toCheck;
```

```
function removeInconsistencies(D_1, D_2)
    Bool result = FALSE;
    for each d ∈ D_1 do
        if no d' ∈ D_2 valid with d then
            remove d from D_1;
            result = TRUE;
    return result;
```


Enforcing arc consistency

Complexity:

- A binary CSP with n variables can have $O(n^2)$ directional constraints $i \to j$.
- Any $i \to j$ can be considered at most d times where $d = \max_k |D_k|$ because only d things can be removed from D_i.
- Checking any single arc for consistency can be done in $O(d^2)$.

So the complexity is $O(n^2 d^3)$.

Note: this setup includes 3SAT.

Consequence: we can't check for consistency in polynomial time, which suggests this doesn't guarantee to find all inconsistencies.

A more powerful form of consistency

We can define a stronger notion of consistency as follows:

- *Given:* any $k - 1$ variables and any consistent assignment to these.
- *Then:* We can find a consistent assignment to any kth variable.

This is known as k*-consistency*.

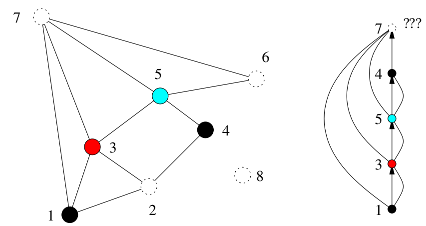
Strong k*-consistency* requires the we be k-consistent, $k-1$-consistent *etc* as far down as 1-consistent.

If we can demonstrate strong n-consistency (where as usual n is the number of variables) then an assignment can be found in $O(nd)$.

Unfortunately, demonstrating strong n-consistency will be *worst-case exponential.*

Backjumping

The basic backtracking algorithm backtracks to the *most recent assignment*. This is known as *chronological backtracking*. It is not always the best policy:



Say we've assigned $1 = B$, $3 = R$, $5 = C$ and $4 = B$ and now we want to assign something to 7. This isn't possible so we backtrack, however re-assigning 4 clearly doesn't help.

Backjumping

With some careful bookkeeping it is often possible to *jump back multiple levels* without sacrificing the ability to find a solution.

We need some definitions:

- When we set a variable V_i to some value $d \in D_i$ we refer to this as the *assignment* $A_i = (V_i \leftarrow d)$.
- A *partial instantiation* $I_k = \{A_1, A_2, \ldots, A_k\}$ is a *consistent* set of assignments to the first k variables...
- ... where *consistent* means that no constraints are violated.
- Conversely, I_k *conflicts* with some variable V if no value for V is consistent with I_k.

Henceforth we shall assume that variables are assigned in the order $V_1, V_2, \ldots, V_n$ when formally presenting algorithms.

Gaschnig's algorithm

Gaschnig's algorithm works as follows. Say we have a partial instantiation I_k:

- When choosing a value for V_{k+1} we need to check that any candidate value $d \in D_{k+1}$, is consistent with I_k.
- When testing potential values for d, we will generally discard one or more possibilities, because they conflict with some member of I_k
- We keep track of the *most recent assignment* A_j for which this has happened.

Finally, if *no* value for V_{k+1} is consistent with I_k then we backtrack to V_j.

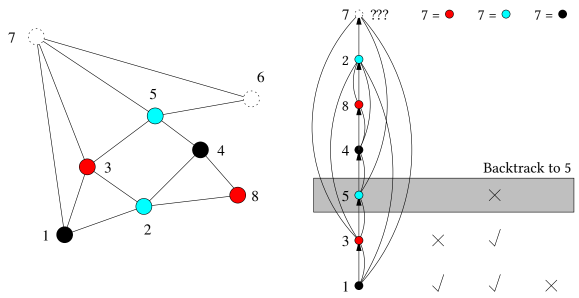
More formally: if I_k conflicts with V_{k+1} we backtrack to V_j where

$$j = \min\{j \leq k | I_j \text{ conflicts with } V_{k+1}\}.$$

If there are no possible values left to try for V_j then we backtrack *chronologically*.

Gaschnig's algorithm

Example:



If there's no value left to try for 5 then backtrack to 3 and so on.

Graph-based backjumping

This allows us to jump back multiple levels *when we initially detect a conflict*.

Can we do better than chronological backtracking *thereafter*?

Some more definitions:

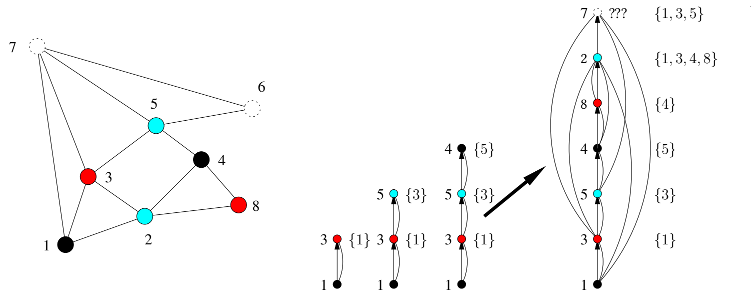
- We assume an ordering $V_1, V_2, \ldots, V_n$ for the variables.
- Given $V' = \{V_1, V_2, \ldots, V_k\}$ where $k < n$ the *ancestors* of V_{k+1} are the members of V' connected to V_{k+1} by a constraint.
- The *parent* $P(V_{k+1})$ of V_{k+1} is its most recent ancestor.

The ancestors for each variable can be accumulated as assignments are made.

Graph-based backjumping backtracks to the *parent* of V_{k+1}.

Note: Gaschnig's algorithm uses *assignments* whereas graph-based backjumping uses *constraints*.

Graph-based backjumping



At this point, backjump to the *parent* for 7, which is 5.

Backjumping and forward checking

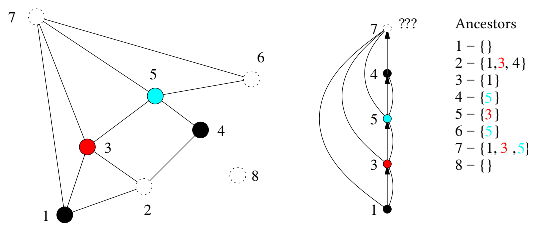
If we use *forward checking*: say we're assigning to V_{k+1} by making $V_{k+1} = d$:

- Forward checking removes d from the D_i of *all* V_i connected to V_{k+1} by a constraint.
- When doing graph-based backjumping, we'd also add V_{k+1} to the ancestors of V_i.

In fact, use of forward checking can make some forms of backjumping *redundant*.

Note: there are in fact many ways of combining *constraint propagation* with *backjumping*, and we will not explore them in further detail here.

Backjumping and forward checking

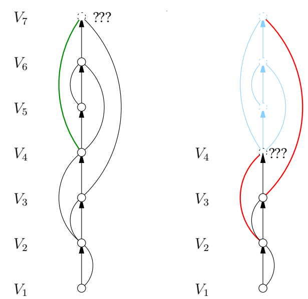


	1	2	3	4	5	6	7	8
Start	BRC	BRC	BRC	BRC	BRC	BRC	BRC	BRC
$1 = B$	$= B$	RC	RC	BRC	BRC	BRC	RC	BRC
$3 = R$	$= B$	C	$= R$	BRC	BC	BRC	C	BRC
$5 = C$	$= B$	C	$= R$	BR	$= C$	BR	!	BRC
$4 = B$	$= B$	C	$= R$	BR	$= C$	BR	!	BRC

Forward checking finds the problem *before backtracking does*.

Graph-based backjumping

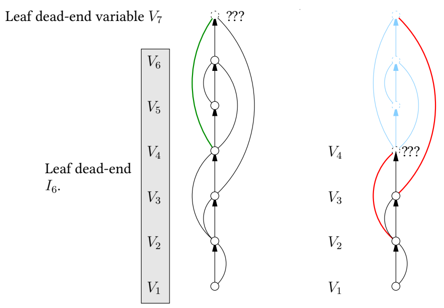
We're not quite done yet though. What happens when *there are no assignments left for the parent we just backjumped to*?



Backjumping from V_7 to V_4 is fine. However we shouldn't then just backjump to V_2, because changing V_3 could fix the problem at V_7.

Graph-based backjumping

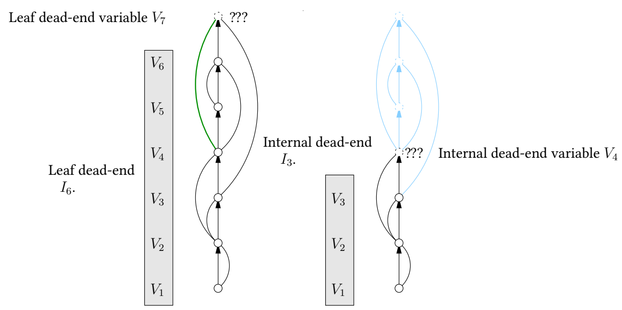
To describe an algorithm in this case is a little involved.



Given an instantiation I_k and V_{k+1}, if there is no consistent $d \in D_{k+1}$ we call I_k a *leaf dead-end* and V_{k+1} a *leaf dead-end variable*.

Graph-based backjumping

Also



If V_i was backtracked to from a later leaf dead-end and there are no more values to try for V_i then we refer to it as an *internal dead-end variable* and call I_{i-1} an *internal dead-end*.

Graph-based backjumping

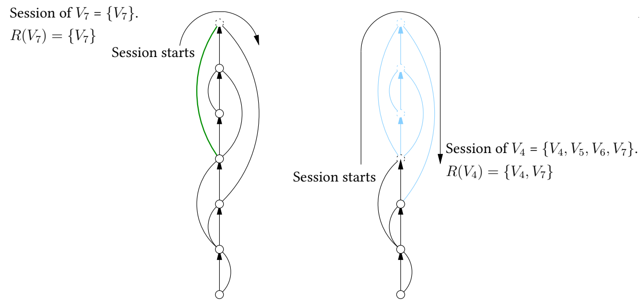
To keep track of exactly where to jump to we also need the definitions:

- The *session* of a variable V begins when the search algorithm visits it and ends when it backtracks through it to an earlier variable.
- The *current session* of a variable V is the set of all variables visiting during its session.
- In particular, the current session for any V contains V.
- The *relevant dead-ends for the current session* $R(V)$ for a variable V are:
  1. $R(V)$ is initialized to $\{V\}$ when V is first visited.
  2. If V is a leaf dead-end variable then $R(V) = \{V\}$.
  3. If V was backtracked to from a dead-end V' then $R(V) = R(V) \cup R(V')$.

And we're not done yet...

Graph-based backjumping

Example:



As expected, the relevant dead-ends for V_4 are $\{V_4\}$ and $\{V_7\}$.

Graph-based backjumping

One more bunch of definitions before the pain stops. Say V_k is a dead-end:

- The *induced ancestors* $\text{ind}(V_k)$ of V_k are defined as

$$\text{ind}(V_k) = \{V_1, V_2, \ldots, V_{k-1}\} \cap \left(\bigcup_{V \in R(V_k)} \text{ancestors}(V) \right)$$

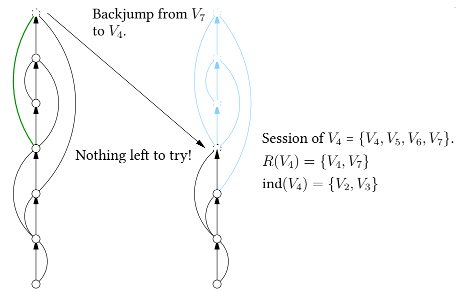
- The *culprit* for V_k is the most recent $V' \in \text{ind}(V_k)$.

Note that these definitions depend on $R(V_k)$.

FINALLY: graph-based backjumping *backjumps to the culprit*.

Graph-based backjumping

Example:



As expected, we back jump to V_3 instead of V_2. Hooray!

Gaschnig's algorithm and graph-based backjumping can be *combined* to produce *conflict-directed backjumping*.

We will not explore conflict-directed backjumping in this course.

Varieties of CSP

We have only looked at *discrete* CSPs with *finite domains*. These are the simplest. We could also consider:

1. Discrete CSPs with *infinite domains*:
   - We need a *constraint language*. For example

   $$V_3 \leq V_{10} + 5$$

   - Algorithms are available for integer variables and linear constraints.
   - There is *no algorithm* for integer variables and nonlinear constraints.
2. Continuous domains—using linear constraints defining convex regions we have *linear programming*. This is solvable in polynomial time in n.
3. We can introduce *preference constraints* in addition to *absolute constraints*, and in some cases an *objective function.*

Artificial Intelligence

Knowledge representation and reasoning

Reading: AIMA, chapters 7 to 10. (Lots of this covered in *Logic and Proof*.) AIFCA, chapters 14 and 15.

Knowledge representation and reasoning

We now look at how an agent might *represent* knowledge about its environment, and *reason* with this knowledge to achieve its goals.

Initially we'll represent and reason using first order logic (FOL). *Aims:*

- To show how FOL can be used to *represent knowledge* about an environment in the form of both *background knowledge* and *knowledge derived from percepts*.
- To show how this knowledge can be used to *derive non-perceived knowledge* about the environment using a *theorem prover*.
- To introduce the *situation calculus* and demonstrate its application in a simple environment as a means by which an agent can work out what to do next.

Using FOL in all its glory can be problematic.

Later we'll look at how some of the problems can be addressed using *semantic networks*, *frames*, *inheritance* and *rules*.

Knowledge representation and reasoning

Earlier in the course we looked at what an *agent* should be able to do.

It seems that all of us—and all intelligent agents—should use *logical reasoning* to help us interact successfully with the world.

Any intelligent agent should:

- Possess *knowledge* about the *environment* and about *how its actions affect the environment*.
- Use some form of *logical reasoning* to *maintain* its knowledge as *percepts* arrive.
- Use some form of *logical reasoning* to *deduce actions* to perform in order to achieve *goals*.

Knowledge representation and reasoning

This raises some important questions:

- How do we describe the current state of the world?
- How do we infer from our percepts, knowledge of unseen parts of the world?
- How does the world change as time passes?
- How does the world stay the same as time passes? (The *frame problem*.)
- How do we know the effects of our actions? (The *qualification* and *ramification problems*.)

We'll now look at one way of answering some of these questions.

FOL (arguably?) seems to provide a good way in which to represent the required kinds of knowledge: it is *expressive*, *concise*, *unambiguous*, it can be adapted to *different contexts*, and it has an *inference procedure*, although a semidecidable one.

In addition is has a well-defined *syntax* and *semantics*.

Logic for knowledge representation

Problem: it's quite easy to talk about things like *set theory* using FOL. For example, we can easily write axioms like

$$\forall S\ .\ \forall S'\ .\ ((\forall x\ .\ (x \in S \Leftrightarrow x \in S')) \Rightarrow S = S')$$

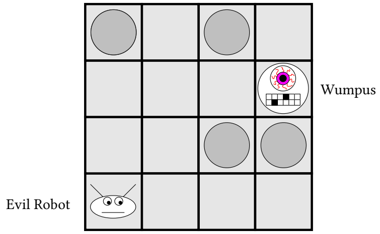
But how would we go about representing the proposition that *if you have a bucket of water and throw it at your friend they will get wet, have a bump on their head from being hit by a bucket, and the bucket will now be empty and dented*?

More importantly, how could this be represented within a wider framework for reasoning about the world?

It's time to introduce *The Wumpus*...

Wumpus world

As a simple test scenario for a knowledge-based agent we will make use of the *Wumpus World*.



The Wumpus World is a 4 by 4 grid-based cave.

EVIL ROBOT wants to enter the cave, find some gold, and get out again unscathed.

Wumpus world

The rules of *Wumpus World*:

- Unfortunately the cave contains a number of pits, which EVIL ROBOT can fall into. Eventually his batteries will fail, and that's the end of him.
- The cave also contains the Wumpus, who is armed with state-of-the-art *Evil Robot Obliteration Technology*.
- The Wumpus itself knows where the pits are and never falls into one.

Wumpus world

EVIL ROBOT can move around the cave at will and can perceive the following:

- In a position adjacent to the Wumpus, a stench is perceived. (Wumpuses are famed for their *lack of personal hygiene*.)
- In a position adjacent to a pit, a *breeze* is perceived.
- In the position where the gold is, a glitter is perceived.
- On trying to move into a wall, a *bump* is perceived.
- On killing the Wumpus a *scream* is perceived.

In addition, EVIL ROBOT has a single arrow, with which to try to kill the Wumpus.

"Adjacent" in the following does *not* include diagonals.

Wumpus world

So we have:

Percepts: `stench`, `breeze`, `glitter`, `bump`, `scream`.

Actions: `forward`, `turnLeft`, `turnRight`, `grab`, `release`, `shoot`, `climb`.

Of course, our aim now is *not* just to design an agent that can perform well in a single cave layout.

We want to design an agent that can *usually* perform well *regardless* of the layout of the cave.

Logic for knowledge representation

The fundamental aim is to construct a *knowledge base* KB containing a *collection of statements* about the world—expressed in FOL—such that *useful things can be derived* from it.

Our central aim is to generate sentences that are *true*, if *the sentences in the* KB *are true.*

This process is based on concepts familiar from your introductory logic courses:

- Entailment: $\texttt{KB} \models \alpha$ means that the KB entails α.
- Proof: $\texttt{KB} \vdash_i \alpha$ means that α is derived from the KB using inference procedure i. If i is *sound* then we have a *proof*.
- i is *sound* if it can generate only entailed α.
- i is *complete* if it can find a proof for *any* entailed α.

Example: Prolog

You have by now learned a little about programming in *Prolog*. For example:

$$\texttt{concat}([], L, L).$$
$$\texttt{concat}([H|T], L, [H|L2]) \text{ :- } \texttt{concat}(T, L, L2).$$

is a program to concatenate two lists. The query

$$\texttt{concat}([1,2,3],[4,5],X).$$

results in

$$X = [1,2,3,4,5].$$

What's happening here? Well, Prolog is just a *more limited form of FOL* so...

Example: Prolog

... we are in fact doing inference from a KB:

- The Prolog programme itself is the KB. It expresses some *knowledge about lists.*
- The query is expressed in such a way as to *derive some new knowledge.*

How does this relate to full FOL? First of all the list notation is nothing but *syntactic sugar*. It can be removed: we define a constant called `empty` and a function called `cons`.

Now $[1,2,3]$ just means

$$\texttt{cons}(1, \texttt{cons}(2, \texttt{cons}(3, \texttt{empty})))$$

which is a term in FOL.

I will assume the use of the syntactic sugar for lists from now on.

Prolog and FOL

The program when expressed in FOL, says

$$\forall x \,.\, \texttt{concat}(\texttt{empty}, x, x) \wedge$$
$$\forall h, t, l_1, l_2 \,.\, \texttt{concat}(t, l_1, l_2) \rightarrow \texttt{concat}(\texttt{cons}(h,t), l_1, \texttt{cons}(h, l_2))$$

The rule is simple—given a Prolog program:

- *Universally quantify all the unbound variables in each line of the program* and ...
- ... *form the conjunction of the results.*

If the universally quantified lines are $L_1, L_2, \ldots, L_n$ then the Prolog programme corresponds to the KB

$$\texttt{KB} = L_1 \wedge L_2 \wedge \cdots \wedge L_n$$

Now, what does the query mean?

Prolog and FOL

When you give the query

$$\texttt{concat}([1,2,3],[4,5],X).$$

to Prolog it responds by *trying to prove* the following statement

$$\texttt{KB} \to \exists X\,.\,\texttt{concat}([1,2,3],[4,5],X)$$

So: it tries to prove that the KB *implies the query*, and variables in the query are existentially quantified.

When a proof is found, it supplies a *value for X* that *makes the inference true.*

Prolog and FOL

Prolog differs from FOL in that, amongst other things:

- It restricts you to using *Horn clauses.*
- Its inference procedure is not a *full-blown proof procedure.*
- It does not deal with *negation* correctly.

However *the central idea also works for full-blown theorem provers.*

If you want to experiment, you can obtain *Prover9* from

www.cs.unm.edu/~mccune/mace4/

We'll see a brief example now, and a more extensive example of its use later, time permitting...

Prolog and FOL

Expressed in Prover9, the above Prolog program and query look like this:

```
set(prolog_style_variables).

% This is the translated Prolog program for list concatenation.
% Prover9 has its own syntactic sugar for lists.

formulas(assumptions).
   concat([], L, L).
   concat(T, L, L2) -> concat([H:T], L, [H:L2]).
end_of_list.

% This is the query.

formulas(goals).
   exists X concat([1, 2, 3], [4, 5], X).
end_of_list.
```

Note: it is assumed that *unbound variables are universally quantified.*

Prolog and FOL

You can try to infer a proof using

```
prover9 -f file.in
```

and the result is (in addition to a lot of other information):

```
1 concat(T,L,L2) -> concat([H:T],L,[H:L2]) # label(non_clause).  [assumption].
2 (exists X concat([1,2,3],[4,5],X)) # label(non_clause) # label(goal).  [goal].
3 concat([],A,A).  [assumption].
4 -concat(A,B,C) | concat([D:A],B,[D:C]).  [clausify(1)].
5 -concat([1,2,3],[4,5],A).  [deny(2)].
6 concat([A],B,[A:B]).  [ur(4,a,3,a)].
7 -concat([2,3],[4,5],A).  [resolve(5,a,4,b)].
8 concat([A,B],C,[A,B:C]).  [ur(4,a,6,a)].
9 $F.  [resolve(8,a,7,a)].
```

This shows that a proof is found but doesn't explicitly give a value for X—we'll see how to extract that later...

The fundamental idea

So the *basic idea* is: build a KB that encodes *knowledge about the world*, the *effects of actions* and so on.

The KB is a conjunction of pieces of knowledge, such that:

- A query regarding what our agent should do *can be posed in the form*

$$\exists \texttt{actionList} \,.\, \texttt{Goal}(\ldots\texttt{actionList}\ldots)$$

- Proving that

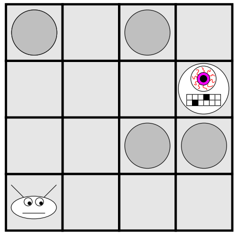
$$\texttt{KB} \rightarrow \exists \texttt{actionList} \,.\, \texttt{Goal}(\ldots\texttt{actionList}\ldots)$$

  instantiates `actionList` to an *actual list of actions* that will achieve a goal represented by the `Goal` predicate.

We sometimes use the notation `ask` and `tell` to refer to *querying* and *adding to the* KB.

Using FOL in AI: the triumphant return of the Wumpus

We want to be able to *speculate* about the past and about *possible futures*. So:



- We include *situations* in the logical language used by our KB.
- We include *axioms* in our KB that relate to situations.

This gives rise to *situation calculus*.

Situation calculus

In *situation calculus*:

- The world consists of sequences of *situations*.
- Over time, an agent moves from one situation to another.
- Situations are changed as a result of *actions*.

In Wumpus World the actions are: `forward`, `shoot`, `grab`, `climb`, `release`, `turnRight`, `turnLeft`.

- A *situation argument* is added to items that can change over time. For example

$$\texttt{At}(\texttt{location}, s)$$

  Items that can change over time are called *fluents*.
- A situation argument is not needed for things that don't change. These are sometimes referred to as *eternal* or *atemporal*.

Representing change as a result of actions

Situation calculus uses a function

$$\texttt{result}(\texttt{action}, s)$$

to denote the *new* situation arising as a result of performing the specified action in the specified situation.

$$\begin{aligned}
&\texttt{result}(\texttt{grab}, s_0) = s_1\\
&\texttt{result}(\texttt{turnLeft}, s_1) = s_2\\
&\texttt{result}(\texttt{shoot}, s_2) = s_3\\
&\texttt{result}(\texttt{forward}, s_3) = s_4\\
&\quad\vdots
\end{aligned}$$

Axioms I: possibility axioms

The first kind of axiom we need in a $\texttt{KB}$ specifies *when particular actions are possible*.

We introduce a predicate

$$\texttt{Poss}(\texttt{action}, s)$$

denoting that an action can be performed in situation s.

We then need a *possibility axiom* for each action. For example:

$$\texttt{At}(l, s) \wedge \texttt{Available}(\texttt{gold}, l, s) \rightarrow \texttt{Poss}(\texttt{grab}, s)$$

Remember that *unbound variables are universally quantified*.

Axioms II: effect axioms

Given that an action results in a new situation, we can introduce *effect axioms* to specify the properties of the new situation.

For example, to keep track of whether EVIL ROBOT has the gold we need *effect axioms* to describe the effect of picking it up:

$$\texttt{Poss}(\texttt{grab}, s) \rightarrow \texttt{Have}(\texttt{gold}, \texttt{result}(\texttt{grab}, s))$$

Effect axioms describe the way in which the world *changes*.

We would probably also include

$$\neg\texttt{Have}(\texttt{gold}, s_0)$$

in the $\texttt{KB}$, where s_0 is the *starting situation*.

Important: we are describing *what is true* in the *situation that results* from *performing an action* in a *given situation*.

Axioms III: frame axioms

We need *frame axioms* to describe *the way in which the world stays the same*.

Example:

$$\begin{aligned} &\texttt{Have}(o, s) \wedge \\ &\quad \neg(a = \texttt{release} \wedge o = \texttt{gold}) \;\wedge\; \neg(a = \texttt{shoot} \wedge o = \texttt{arrow}) \\ &\quad \rightarrow \texttt{Have}(o, \texttt{result}(a, s)) \end{aligned}$$

describes the effect of *having something and not discarding it*.

In a more general setting such an axiom might well look different. For example

$$\begin{aligned} &\neg\texttt{Have}(o, s) \wedge \\ &\quad (a \neq \texttt{grab}(o) \;\vee\; \neg(\texttt{Available}(o, s) \;\wedge\; \texttt{Portable}(o))) \\ &\quad \rightarrow \neg\texttt{Have}(o, \texttt{result}(a, s)) \end{aligned}$$

describes the effect of *not having something and not picking it up*.

The frame problem

The *frame problem* has historically been a major issue.

Representational frame problem: a large number of frame axioms are required to represent the many things in the world which will not change as the result of an action.

We will see how to solve this in a moment.

Inferential frame problem: when reasoning about a sequence of situations, all the unchanged properties still need to be carried through all the steps.

This can be alleviated using *planning systems* that allow us to reason efficiently when actions change only a small part of the world. There are also other remedies, which we will not cover.

Successor-state axioms

Effect axioms and frame axioms can be combined into *successor-state axioms*.

One is needed for each predicate that can change over time.

Action a is possible $\rightarrow$
(true in new situation $\iff$
(you did something to make it true $\vee$
it was already true and you didn't make it false))

For example

$$\begin{aligned}
&\texttt{Poss}(a, s) \rightarrow \\
&\quad (\texttt{Have}(o, \texttt{result}(a, s)) \iff ((a = \texttt{grab} \wedge \texttt{Available}(o, s)) \vee \\
&\quad (\texttt{Have}(o, s) \ \wedge \ \neg(a = \texttt{release} \ \wedge \ o = \texttt{gold}) \wedge \\
&\quad \neg(a = \texttt{shoot} \ \wedge \ o = \texttt{arrow}))))
\end{aligned}$$

Knowing where you are, and so on...

We now have considerable flexibility in adding further rules:

- If s_0 is the *initial situation* we know that $\texttt{At}((1, 1), s_0)$.
- We need to keep track of what way we're facing. Say north is 0, south is 2, east is 1 and west is 3. We might assume $\texttt{facing}(s_0) = 0$.
- We need to know how motion affects location

$$\texttt{forwardResult}((x, y), \texttt{north}) = (x, y + 1)$$

$$\texttt{forwardResult}((x, y), \texttt{east}) = (x + 1, y)$$

  and so on.
- The concept of adjacency is very important in the Wumpus world

$$\texttt{Adjacent}(l_1, l_2) \iff \exists d \ \texttt{forwardResult}(l_1, d) = l_2$$

- We also know that the cave is 4 by 4 and surrounded by walls

$$\texttt{WallHere}((x, y)) \iff (x = 0 \vee y = 0 \vee x = 5 \vee y = 5)$$

The qualification and ramification problems

Qualification problem: we are in general never completely certain what conditions are required for an action to be effective.

Consider for example turning the key to start your car.

This will lead to problems if important conditions are omitted from axioms.

Ramification problem: actions tend to have implicit consequences that are large in number.

For example, if I pick up a sandwich in a dodgy sandwich shop, I will also be picking up all the bugs that live in it. I don't want to model this explicitly.

Solving the ramification problem

The ramification problem can be solved by *modifying successor-state axioms*.

For example:

$$\begin{aligned}
&\texttt{Poss}(a, s) \rightarrow \\
&\quad (\texttt{At}(o, l, \texttt{result}(a, s)) \iff \\
&\quad (\exists l' \ . \ a = \texttt{go}(l', l) \wedge \\
&\qquad\quad [o = \texttt{robot} \ \vee \ \texttt{Has}(\texttt{robot}, o, s)]) \vee \\
&\qquad (\texttt{At}(o, l, s) \wedge \\
&\qquad\quad [\neg \exists l'' \ . \ a = \texttt{go}(l, l'') \ \wedge \ l \neq l'' \wedge \\
&\qquad\quad \{o = \texttt{robot} \ \vee \ \texttt{Has}(\texttt{robot}, o, s)\}]))
\end{aligned}$$

describes the fact that anything EVIL ROBOT is carrying moves around with him.

Deducing properties of the world: causal and diagnostic rules

If you know where you are, then you can think about *places* rather than just *situations*. *Synchronic rules* relate properties shared by a single state of the world.

There are two kinds: *causal* and *diagnostic*.

Causal rules: some properties of the world will produce percepts.

$$\texttt{WumpusAt}(l_1) \land \texttt{Adjacent}(l_1, l_2) \rightarrow \texttt{StenchAt}(l_2)$$

$$\texttt{PitAt}(l_1) \land \texttt{Adjacent}(l_1, l_2) \rightarrow \texttt{BreezeAt}(l_2)$$

Systems reasoning with such rules are known as *model-based* reasoning systems.

Diagnostic rules: infer properties of the world from percepts. For example:

$$\texttt{At}(l, s) \land \texttt{Breeze}(s) \rightarrow \texttt{BreezeAt}(l)$$

$$\texttt{At}(l, s) \land \texttt{Stench}(s) \rightarrow \texttt{StenchAt}(l)$$

These may not be very strong.

The difference between model-based and diagnostic reasoning can be important. For example, medical diagnosis can be done based on symptoms or based on a model of disease.

General axioms for situations and objects

Note: in FOL, if we have two constants `robot` and `gold` then an interpretation is free to assign them to be the same thing. This is not something we want to allow.

Unique names axioms state that each pair of distinct items in our model of the world must be different

$$\begin{aligned} &\texttt{robot} \neq \texttt{gold} \\ &\texttt{robot} \neq \texttt{arrow} \\ &\texttt{robot} \neq \texttt{wumpus} \\ &\vdots \end{aligned}$$

Unique actions axioms state that actions must share this property, so for each pair of actions

$$\begin{aligned} &\texttt{go}(l, l') \neq \texttt{grab} \\ &\texttt{go}(l, l') \neq \texttt{drop}(o) \\ &\vdots \end{aligned}$$

and in addition we need to define equality for actions, so for each action

$$\begin{aligned} \texttt{go}(l, l') = \texttt{go}(l'', l''') &\iff l = l'' \land l' = l''' \\ \texttt{drop}(o) = \texttt{drop}(o') &\iff o = o' \\ &\vdots \end{aligned}$$

General axioms for situations and objects

The situations are *ordered* so

$$s_0 \neq \texttt{result}(a, s)$$

and situations are *distinct* so

$$\texttt{result}(a, s) = \texttt{result}(a', s') \iff a = a' \land s = s'$$

Strictly speaking we should be using a *many-sorted* version of FOL.

In such a system variables can be divided into *sorts* which are implicitly separate from one another.

Finally, we're going to need to specify *what's true in the start state*.

For example

$$\begin{aligned} &\texttt{At}(\texttt{robot}, [1, 1], s_0) \\ &\texttt{At}(\texttt{wumpus}, [3, 4], s_0) \\ &\texttt{Has}(\texttt{robot}, \texttt{arrow}, s_0) \\ &\vdots \end{aligned}$$

and so on.

Sequences of situations

We know that the function `result` tells us about the situation resulting from performing an action in an earlier situation.

How can this help us find *sequences of actions to get things done*?

Define

$$\begin{aligned} \texttt{Sequence}([], s, s') &= s' = s \\ \texttt{Sequence}([a], s, s') &= \texttt{Poss}(a, s) \land s' = \texttt{result}(a, s) \\ \texttt{Sequence}(a :: as, s, s') &= \exists t \;.\; \texttt{Sequence}([a], s, t) \land \texttt{Sequence}(as, t, s') \end{aligned}$$

To obtain a *sequence of actions that achieves* $\texttt{Goal}(s)$ we can use the query

$$\exists a \; \exists s \;.\; \texttt{Sequence}(a, s_0, s) \land \texttt{Goal}(s)$$

Interesting reading

Knowledge representation based on logic is a vast subject and can't be covered in full in the lectures.

In particular:

- Techniques for representing *further kinds of knowledge.*
- Techniques for moving beyond the idea of a *situation.*
- Reasoning systems based on *categories.*
- Reasoning systems using *default information.*
- *Truth maintenance systems.*

Happy reading...

Knowledge representation and reasoning

It should be clear that generating sequences of actions by inference in FOL is highly non-trivial.

Ideally we'd like to maintain an *expressive* language while *restricting* it enough to be able to do inference *efficiently*.

Further aims:

- To give a brief introduction to *semantic networks* and *frames* for knowledge representation.
- To see how *inheritance* can be applied as a reasoning method.
- To look at the use of *rules* for knowledge representation, along with *forward chaining* and *backward chaining* for reasoning.

Further reading: *The Essence of Artificial Intelligence*, Alison Cawsey. Prentice Hall, 1998.

Frames and semantic networks

Frames and semantic networks represent knowledge in the form of *classes of objects* and *relationships between them*:

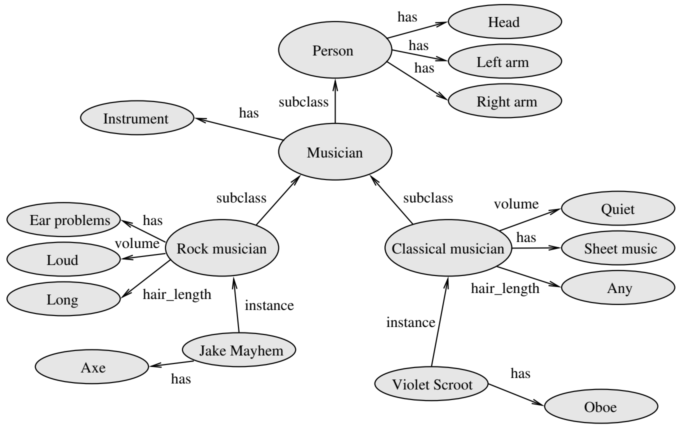
- The *subclass* and *instance* relationships are emphasised.
- We form *class hierarchies* in which *inheritance* is supported and provides the main *inference mechanism.*

As a result inference is quite limited.

We also need to be extremely careful about *semantics.*

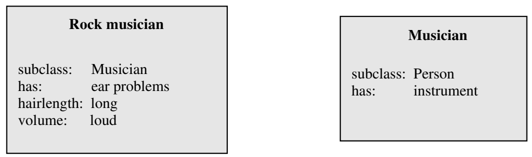
The only major difference between the two ideas is *notational.*

Example of a semantic network



Frames

Frames once again support inheritance through the *subclass relationship*.



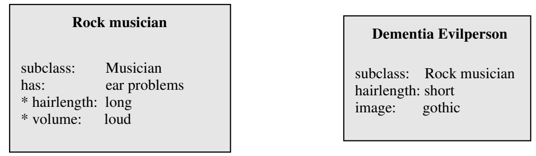
`has`, `hairlength`, `volume` *etc* are *slots*.

`long`, `loud`, `instrument` *etc* are *slot values*.

These are a direct predecessor of *object-oriented programming languages*.

Defaults

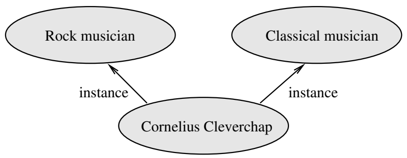
Both approaches to knowledge representation are able to incorporate *defaults*:



Starred slots are *typical values* associated with subclasses and instances, but *can be overridden.*

Multiple inheritance

Both approaches can incorporate *multiple inheritance*, at a cost:



- What is `hairlength` for `Cornelius` if we're trying to use inheritance to establish it?
- This can be overcome initially by specifying which class is inherited from *in preference* when there's a conflict.
- But the problem is still not entirely solved—what if we want to prefer inheritance of some things from one class, but inheritance of others from a different one?

Other issues

- Slots and slot values can themselves be frames. For example `Dementia` may have an instrument slot with the value `Electricharp`, which itself may have properties described in a frame.
- Slots can have *specified attributes*. For example, we might specify that:
  - `instrument` can have multiple values
  - Each value can only be an instance of `Instrument`
  - Each value has a slot called `owned_by`

  and so on.
- Slots may contain arbitrary pieces of program. This is known as *procedural attachment*. The fragment might be executed to return the slot's value, or update the values in other slots *etc.*

Rule-based systems

A rule-based system requires three things:

1. A set of `if — then` *rules*. These denote specific pieces of knowledge about the world.

   They should be interpreted similarly to logical implication.

   Such rules denote *what to do* or *what can be inferred* under given circumstances.
2. A collection of *facts* denoting what the system regards as currently true about the world.
3. An interpreter able to apply the current rules in the light of the current facts.

Forward chaining

The first of two basic kinds of interpreter *begins with established facts and then applies rules to them.*

This is a *data-driven* process. It is appropriate if we know the *initial facts* but not the required conclusion.

Example: XCON—used for configuring VAX computers.

In addition:

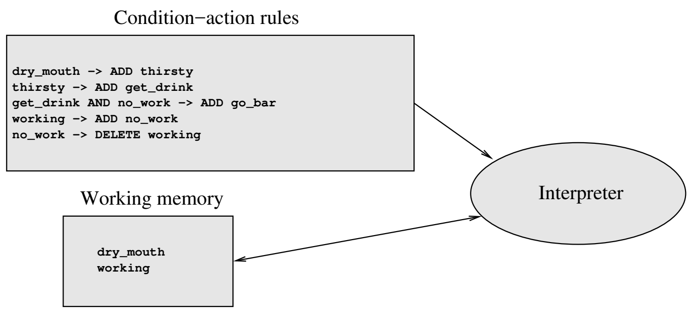
- We maintain a *working memory*, typically of what has been inferred so far.
- Rules are often *condition-action rules*, where the right-hand side specifies an action such as adding or removing something from working memory, printing a message *etc.*
- In some cases actions might be entire program fragments.

Forward chaining

The basic algorithm is:

1. Find all the rules that can fire, based on the current working memory.
2. Select a rule to fire. This requires a *conflict resolution strategy*.
3. Carry out the action specified, possibly updating the working memory.

Repeat this process until either *no rules can be used* or a *halt* appears in the working memory.



Example

Progress is as follows:

1. The rule
$$\texttt{dry\_mouth} \rightarrow \text{ADD } \texttt{thirsty}$$
fires adding thirsty to working memory.
2. The rule
$$\texttt{thirsty} \rightarrow \text{ADD } \texttt{get\_drink}$$
fires adding get_drink to working memory.
3. The rule
$$\texttt{working} \rightarrow \text{ADD } \texttt{no\_work}$$
fires adding no_work to working memory.
4. The rule
$$\texttt{get\_drink} \text{ AND } \texttt{no\_work} \rightarrow \text{ADD } \texttt{go\_bar}$$
fires, and we establish that it's time to go to the bar.

Conflict resolution

Clearly in any more realistic system we expect to have to deal with a scenario where *two or more rules can be fired at any one time*:

- Which rule we choose can clearly affect the outcome.
- We might also want to attempt to avoid inferring an abundance of useless information.

We therefore need a means of *resolving such conflicts*. Common *conflict resolution strategies* are:

- Prefer rules involving more recently added facts.
- Prefer rules that are *more specific*. For example
$$\texttt{patient\_coughing} \rightarrow \text{ADD } \texttt{lung\_problem}$$
is more general than
$$\texttt{patient\_coughing} \text{ AND } \texttt{patient\_smoker} \rightarrow \text{ADD } \texttt{lung\_cancer}.$$
- Allow the designer of the rules to specify priorities.
- Fire all rules *simultaneously*—this essentially involves following all chains of inference at once.

Reason maintenance

Some systems will allow information to be removed from the working memory if it is no longer *justified*.

For example, we might find that
$$\texttt{patient\_coughing}$$
and
$$\texttt{patient\_smoker}$$
are in working memory, and hence fire
$$\texttt{patient\_coughing} \text{ AND } \texttt{patient\_smoker} \rightarrow \text{ADD } \texttt{lung\_cancer}$$
but later infer something that causes patient_coughing to be *withdrawn* from working memory.

The justification for lung_cancer has been removed, and so it should perhaps be removed also.

Pattern matching

In general rules may be expressed in a slightly more flexible form involving *variables* which can work in conjunction with *pattern matching*.

For example the rule
$$\texttt{coughs}(X) \text{ AND } \texttt{smoker}(X) \rightarrow \text{ADD } \texttt{lung\_cancer}(X)$$
contains the variable X.

If the working memory contains $\texttt{coughs}(\texttt{neddy})$ and $\texttt{smoker}(\texttt{neddy})$ then
$$X = \texttt{neddy}$$
provides a match and
$$\texttt{lung\_cancer}(\texttt{neddy})$$
is added to the working memory.

Backward chaining

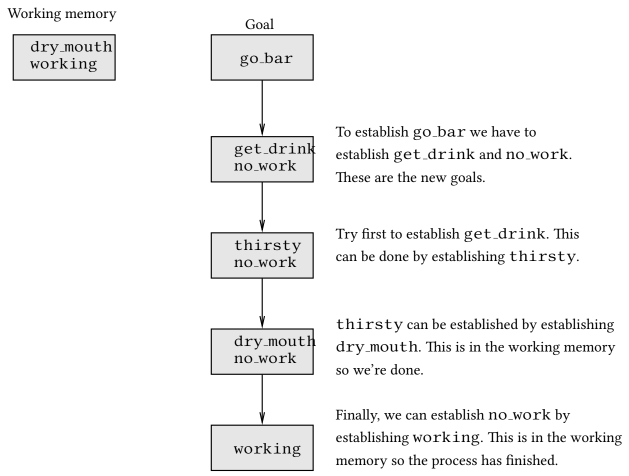
The second basic kind of interpreter begins with a *goal* and finds a rule that would achieve it.

It then works *backwards*, trying to achieve the resulting earlier goals in the succession of inferences.

Example: MYCIN—medical diagnosis with a small number of conditions.

This is a *goal-driven* process. If you want to *test a hypothesis* or you have some idea of a likely conclusion it can be more efficient than forward chaining.

Example



Example with backtracking

If at some point more than one rule has the required conclusion then we can *backtrack*.

Example: *Prolog* backtracks, and incorporates pattern matching. It orders attempts according to the order in which rules appear in the program.

Example: having added

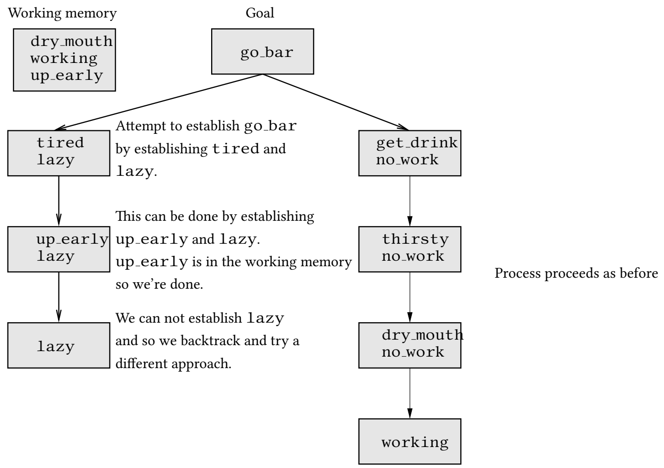
$$\texttt{up\_early} \rightarrow \text{ADD } \texttt{tired}$$

and

$$\texttt{tired} \text{ AND } \texttt{lazy} \rightarrow \text{ADD } \texttt{go\_bar}$$

to the rules, and `up_early` to the working memory:

Example with backtracking



Artificial Intelligence I

Planning algorithms

Reading: AIMA, chapter 11. AIFCA, chapter 6.

Problem solving is different to planning

In *search problems* we:

- *Represent states*: and a state representation contains *everything* that's relevant about the environment.
- *Represent actions*: by describing a new state obtained from a current state.
- *Represent goals*: all we know is how to test a state either to see if it's a goal, or using a heuristic.
- *A sequence of actions is a 'plan'*: but we only consider *sequences of consecutive actions.*

Search algorithms are good for solving problems that fit this framework. However for more complex problems they may fail completely...

Problem solving is different to planning

Representing a problem such as: *'go out and buy some pies'* is hopeless:

- There are *too many possible actions* at each step.
- A heuristic can only help you rank states. In particular it does not help you *ignore* useless actions.
- We are forced to start at the initial state, but you have to work out *how to get the pies*—that is, go to town and buy them, get online and find a web site that sells pies *etc*—*before you can start to do it*.

Knowledge representation and reasoning might not help either: although we end up with a sequence of actions—a plan—there is so much flexibility that complexity might well become an issue.

Our aim now is to look at how an agent might *construct a plan* enabling it to achieve a goal.

- We look at how we might update our concept of *knowledge representation and reasoning* to apply more specifically to planning tasks.
- We look in detail at the *partial-order planning algorithm.*

Planning algorithms work differently

Difference 1:

- Planning algorithms use a *special purpose language*—often based on FOL or a subset— to represent states, goals, and actions.
- States and goals are described by sentences, as might be expected, but...
- ...actions are described by stating their *preconditions* and their *effects.*

So if you know the goal includes (maybe among other things)

$$\texttt{Have}(\texttt{pie})$$

and action $\texttt{Buy}(x)$ has an effect $\texttt{Have}(x)$ then you know that a plan *including*

$$\texttt{Buy}(\texttt{pie})$$

might be reasonable.

Planning algorithms work differently

Difference 2:

- Planners can add actions at *any relevant point at all between the start and the goal*, not just at the end of a sequence starting at the start state.
- This makes sense: I may determine that $\texttt{Have}(\texttt{carKeys})$ is a good state to be in without worrying about what happens before or after finding them.
- By making an important decision like requiring $\texttt{Have}(\texttt{carKeys})$ early on we may reduce branching and backtracking.
- State descriptions are not complete—$\texttt{Have}(\texttt{carKeys})$ describes a *class of states*—and this adds flexibility.

So: you have the potential to search both *forwards* and *backwards* within the same problem.

Planning algorithms work differently

Difference 3:

It is assumed that most elements of the environment are *independent of most other elements.*

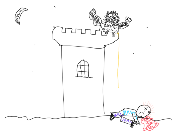
- A goal including several requirements can be attacked with a divide-and-conquer approach.
- Each individual requirement can be fulfilled using a subplan...
- ...and the subplans then combined.

This works provided there is not significant interaction between the subplans.

Remember: the *frame problem.*

Running example: gorilla-based mischief

We will use a simple example, based on one from Russell and Norvig.



The intrepid little scamps in the *Cambridge University Roof-Climbing Society* wish to attach an *inflatable gorilla* to the spire of a *Famous College*. To do this they need to leave home and obtain:

- *An inflatable gorilla*: these can be purchased from all good joke shops.
- *Some rope*: available from a hardware store.
- *A first-aid kit*: also available from a hardware store.

They need to return home after they've finished their shopping. How do they go about planning their *jolly escapade*?

The STRIPS language

STRIPS: *"Stanford Research Institute Problem Solver"* (1970).

States: are *conjunctions* of *ground literals.* They must not include *function symbols.*

$$\begin{aligned}\texttt{At}(\texttt{home}) &\wedge \neg\texttt{Have}(\texttt{gorilla})\\ &\wedge \neg\texttt{Have}(\texttt{rope})\\ &\wedge \neg\texttt{Have}(\texttt{kit})\end{aligned}$$

Goals: are *conjunctions* of *literals* where variables are assumed *existentially quantified.*

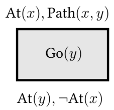
$$\texttt{At}(x) \wedge \text{Sells}(x, \texttt{gorilla})$$

A planner finds a sequence of actions that when performed makes the goal true.

We are no longer employing a full theorem-prover.

The STRIPS language

STRIPS represents actions using *operators*. For example



$$\mathrm{Op}(\text{Action: } \mathrm{Go}(y), \text{Pre: } \mathrm{At}(x) \wedge \mathrm{Path}(x, y), \text{Effect: } \mathrm{At}(y) \wedge \neg\mathrm{At}(x))$$

All variables are implicitly universally quantified. An operator has:

- An *action description*: what the action does.
- A *precondition*: what must be true before the operator can be used. A *conjunction of positive literals.*
- An *effect*: what is true after the operator has been used. A *conjunction of literals.*

The space of plans

We now make a change in perspective—we search in *plan space*:

- Start with an *empty plan.*
- *Operate on it* to obtain new plans. Incomplete plans are called *partial plans. Refinement operators* add constraints to a partial plan. All other operators are called *modification operators.*
- Continue until we obtain a plan that solves the problem.

Operations on plans can be:

- *Adding a step.*
- *Instantiating a variable.*
- *Imposing an ordering* that places a step in front of another.
- and so on...

Representing a plan: partial order planners

When putting on your shoes and socks:

- It *does not matter* whether you deal with your left or right foot first.
- It *does matter* that you place a sock on *before* a shoe, for any given foot.

It makes sense in constructing a plan *not* to make any *commitment* to which side is done first *if you don't have to.*

Principle of least commitment: do not commit to any specific choices until you have to. This can be applied both to ordering and to instantiation of variables.

A *partial order planner* allows plans to specify that some steps must come before others but others have no ordering.

A *linearisation* of such a plan imposes a specific sequence on the actions therein.

Representing a plan: partial order planners

A plan consists of:

1. A set $\{S_1, S_2, \ldots, S_n\}$ of *steps.* Each of these is one of the available *operators.*
2. A set of *ordering constraints.* An ordering constraint $S_i < S_j$ denotes the fact that step S_i must happen before step S_j. $S_i < S_j < S_k$ and so on has the obvious meaning. $S_i < S_j$ does *not* mean that S_i must *immediately* precede S_j.
3. A set of variable bindings $v = x$ where v is a variable and x is either a variable or a constant.
4. A set of *causal links* or *protection intervals* $S_i \xrightarrow{c} S_j$. This denotes the fact that the purpose of S_i is to achieve the precondition c for S_j.

A causal link is *always* paired with an equivalent ordering constraint.

Representing a plan: partial order planners

The *initial plan* has:

- Two steps, called Start and Finish.
- A single ordering constraint Start $<$ Finish.
- No *variable bindings*.
- No *causal links*.

In addition to this:

- The step Start has no preconditions, and its effect is the start state for the problem.
- The step Finish has no effect, and its precondition is the goal.
- Neither Start or Finish has an associated action.

We now need to consider what constitutes a *solution*...

Solutions to planning problems

A solution to a planning problem is any *complete* and *consistent* partially ordered plan.

Complete: each precondition of each step is *achieved* by another step in the solution.

A precondition c for S is achieved by a step S' if:

1. The precondition is an effect of the step

$$S' < S \text{ and } c \in \text{Effects}(S')$$

   and...

2. ... there is *no other* step that *could* cancel the precondition. That is, no S'' exists where:
   - The existing ordering constraints allow S'' to occur *after* S' but *before* S.
   - $\neg c \in \text{Effects}(S'')$.

Solutions to planning problems

Consistent: no contradictions exist in the binding constraints or in the proposed ordering. That is:

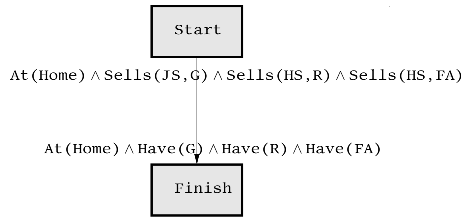
1. For binding constraints, we never have $v = X$ and $v = Y$ for distinct constants X and Y.
2. For the ordering, we never have $S < S'$ and $S' < S$.

Returning to the roof-climbers' shopping expedition, here is the basic approach:

- Begin with only the `Start` and `Finish` steps in the plan.
- At each stage add a new step.
- Always add a new step such that a *currently non-achieved precondition is achieved*.
- Backtrack when necessary.

An example of partial-order planning

Here is the *initial plan*:



Thin arrows denote ordering.

An example of partial-order planning

There are *two actions available*:

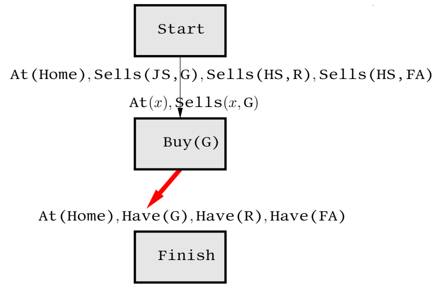
A planner might begin, for example, by adding a Buy(G) action in order to achieve the Have(G) precondition of Finish.

Note: the following order of events is by no means the only one available to a planner.

It has been chosen for illustrative purposes.

An example of partial-order planning

Incorporating the suggested step into the plan:

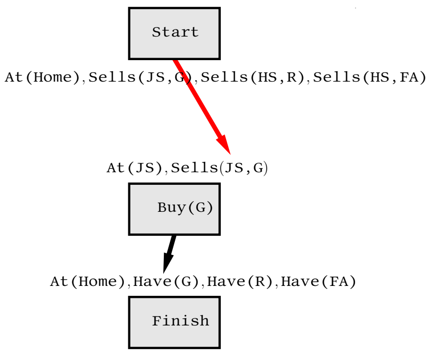


Thick arrows denote causal links. They always have a thin arrow underneath.

Here the new Buy step achieves the Have(G) precondition of Finish.

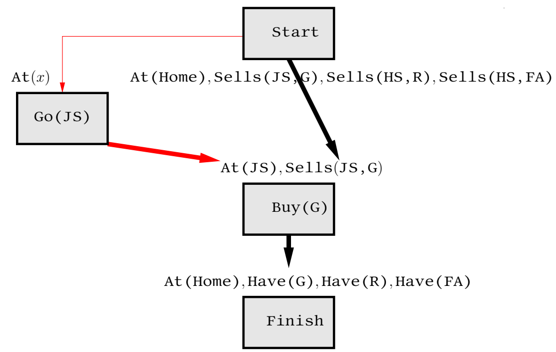
An example of partial-order planning

The planner can now introduce a second causal link from Start to achieve the Sells(x, G) precondition of Buy(G).



An example of partial-order planning

The planner's next obvious move is to introduce a Go step to achieve the At(JS) precondition of Buy(G).



And we continue...

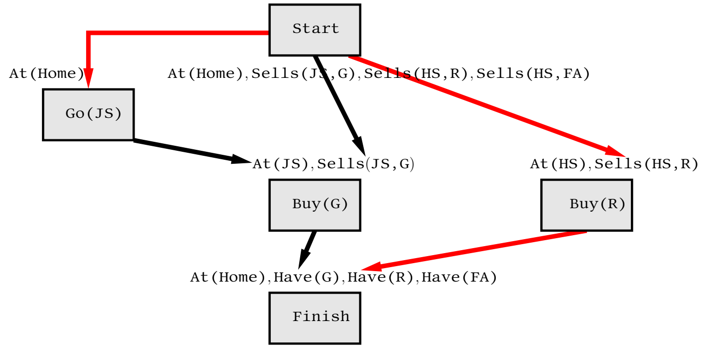
An example of partial-order planning

Initially the planner can continue quite easily in this manner:

- Add a causal link from Start to Go(JS) to achieve the At(x) precondition.
- Add the step Buy(R) with an associated causal link to the Have(R) precondition of Finish.
- Add a causal link from Start to Buy(R) to achieve the Sells(HS, R) precondition.

But then things get more interesting...

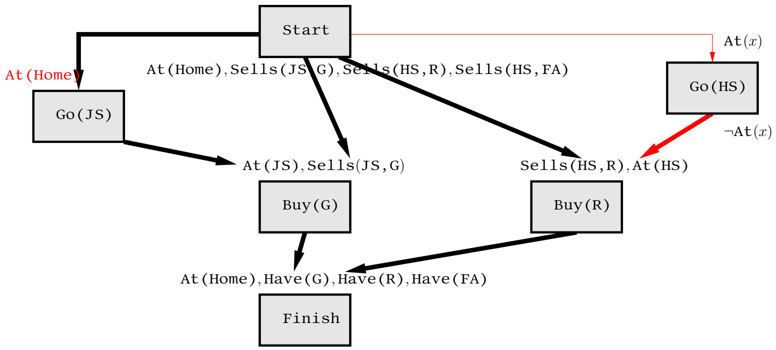
An example of partial-order planning



At this point it starts to get tricky...

The At(HS) precondition in Buy(R) is not achieved.

An example of partial-order planning

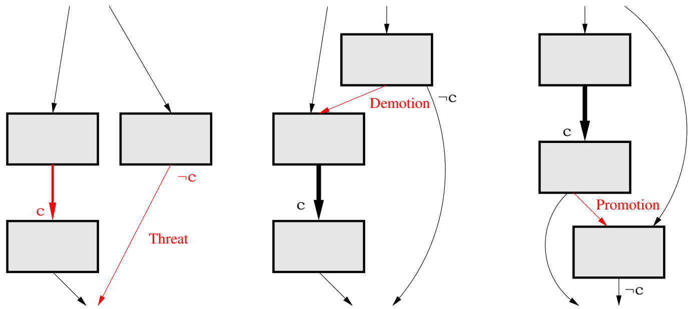


The At(HS) precondition is easy to achieve.

But if we introduce a causal link from Start *to* Go(HS) *then we risk invalidating the precondition for* Go(JS).

An example of partial-order planning

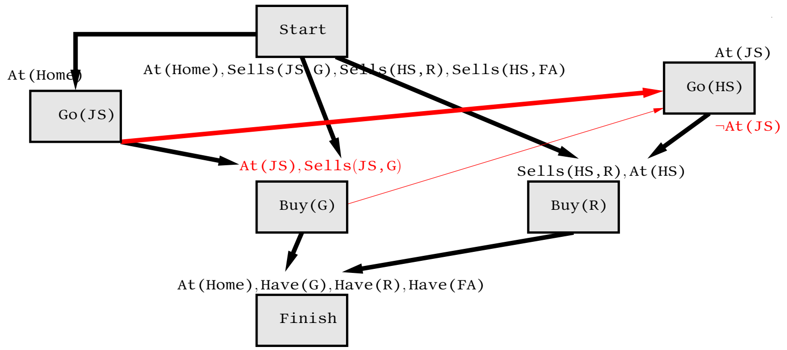
A step that might invalidate (sometimes the word *clobber* is employed) a previously achieved precondition is called a *threat*.



A planner can try to fix a threat by introducing an ordering constraint.

An example of partial-order planning

The planner could backtrack and try to achieve the $\texttt{At}(x)$ precondition using the existing $\texttt{Go}(\texttt{JS})$ step.



This involves a threat, but one that can be fixed using promotion.

The algorithm

Simplifying slightly to the case where there are *no variables*.

Say we have a partially completed plan and a set of the preconditions that have yet to be achieved.

- Select a precondition p that has not yet been achieved and is associated with an action B.
- At each stage *the partially complete plan is expanded into a new collection of plans.*
- To expand a plan, we can try to achieve p *either* by using an action that's already in the plan or by adding a new action to the plan. In either case, call the action A.

We then try to construct consistent plans where A achieves p.

The algorithm

This works as follows:

- For *each possible way of achieving* p:
  - Add Start $< A$, $A <$ Finish, $A < B$ and the causal link $A \xrightarrow{p} B$ to the plan.
  - If the resulting plan is consistent we're done, otherwise *generate all possible ways of removing inconsistencies* by promotion or demotion and *keep any resulting consistent plans.*

At this stage:

- If you have *no further preconditions that haven't been achieved* then *any plan obtained is valid.*

The algorithm

But how do we try to *enforce consistency*?

When you attempt to achieve p using A:

- Find all the existing causal links $A' \xrightarrow{\neg p} B'$ that are *clobbered* by A.
- For each of those you can try adding $A < A'$ or $B' < A$ to the plan.
- Find all existing actions C in the plan that clobber the *new* causal link $A \xrightarrow{p} B$.
- For each of those you can try adding $C < A$ or $B < C$ to the plan.
- Generate *every possible combination* in this way and retain any consistent plans that result.

Possible threats

What about dealing with *variables*?

If at any stage an effect $\neg\texttt{At}(x)$ appears, is it a threat to $\texttt{At}(\texttt{JS})$?

Such an occurrence is called a *possible threat* and we can deal with it by introducing *inequality constraints*: in this case $x \neq \texttt{JS}$.

- Each partially complete plan now has a set I of inequality constraints associated with it.
- An inequality constraint has the form $v \neq X$ where v is a variable and X is a variable or a constant.
- Whenever we try to make a substitution we check I to make sure we won't introduce a conflict.

If we *would* introduce a conflict then we discard the partially completed plan as inconsistent.

Planning II

Unsurprisingly, this process can become complex.

How might we improve matters?

One way would be to introduce *heuristics*. We now consider:

- The way in which *basic heuristics* might be defined for use in planning problems.
- The construction of *planning graphs* and their use in obtaining more sensible heuristics.
- Planning graphs as the basis of the *GraphPlan* algorithm.

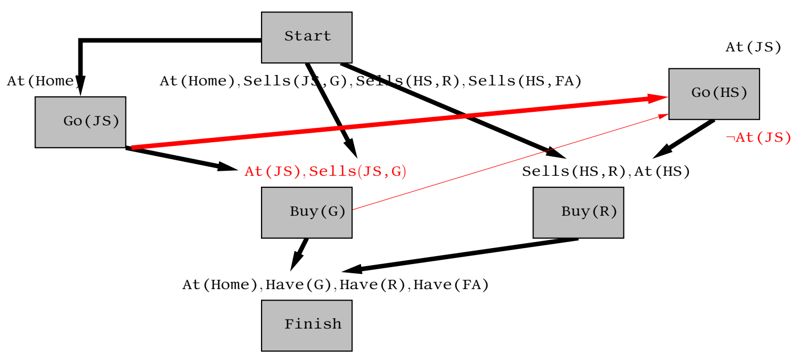
Another is to translate into the language of a *general-purpose* algorithm exploiting its own heuristics. We now consider:

- Planning using *propositional logic*.
- Planning using *constraint satisfaction*.

An example of partial-order planning

We left our example problem here:

The planner could backtrack and try to achieve the $\texttt{At}(x)$ precondition using the existing $\texttt{Go}(\texttt{JS})$ step.



This involves a threat, but one that can be fixed using promotion.

Using heuristics in planning

We found in looking at search problems that *heuristics* were a helpful thing to have.

Note that now there is no simple representation of a *state*, and consequently it is harder to measure the *distance to a goal*.

Defining heuristics for planning is therefore more difficult than it was for search problems. Simple possibilities:

$$h = \text{number of unsatisfied preconditions}$$

or

$$\begin{aligned} h = &\text{number of unsatisfied preconditions} \\ &- \text{number satisfied by the start state} \end{aligned}$$

These can lead to underestimates or overestimates:

- Underestimates if *actions can affect one another in undesirable ways*.
- Overestimates if *actions achieve many preconditions*.

Using heuristics in planning

We can go a little further by learning from *Constraint Satisfaction Problems* and adopting the *most constrained variable* heuristic:

- Prefer the precondition *satisfiable in the smallest number of ways.*

This can be computationally demanding but two special cases are helpful:

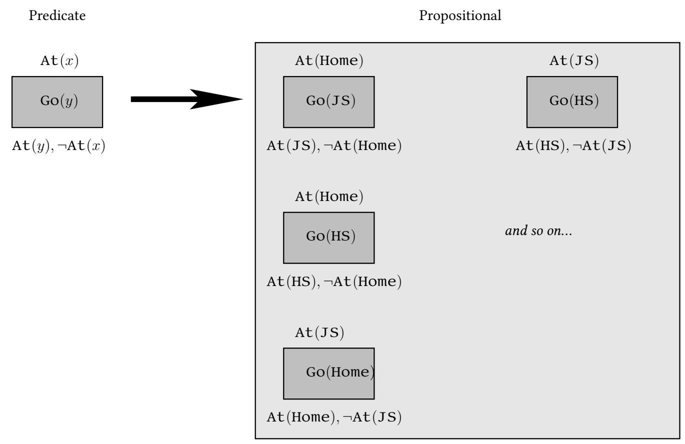
- Choose preconditions for which *no action will satisfy them.*
- Choose preconditions that *can only be satisfied in one way*.

But these still seem somewhat basic.

We can do better using *Planning Graphs.* These are *easy to construct* and can also be used to generate *entire plans.*

Planning graphs

Planning Graphs apply when it is possible to work entirely using *propositional* representations of plans. Luckily, STRIPS can always be propositionalized...



Planning graphs

A planning graph is constructed in levels:

- Level 0 corresponds to the *start state.*
- At each level we keep *approximate* track of all things that *could* be true at the corresponding time.
- At each level we keep *approximate* track of what actions *could* be applicable at the corresponding time.

The approximation is due to the fact that not all conflicts between actions are tracked. *So*:

- The graph can *underestimate* how long it might take for a particular proposition to appear, and therefore ...
- ...a heuristic can be extracted.

For example: the triumphant return of the gorilla-purchasing roof-climbers...

Planning graphs: a simple example

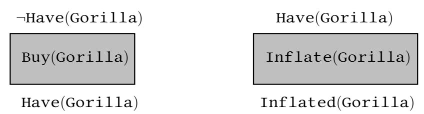
Our intrepid student adventurers will of course need to inflate their *gorilla* before attaching it to a *distinguished roof* . It has to be purchased before it can be inflated.

Start state: Empty.

We assume that anything not mentioned in a state is false. So the state is actually

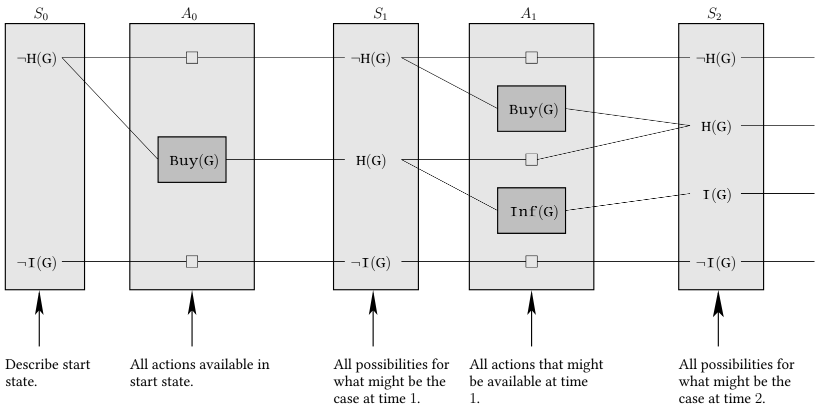
$$\neg\texttt{Have}(\texttt{Gorilla}) \text{ and } \neg\texttt{Inflated}(\texttt{Gorilla})$$

Actions:



Goal: $\texttt{Have}(\texttt{Gorilla})$ and $\texttt{Inflated}(\texttt{Gorilla})$.

Planning graphs



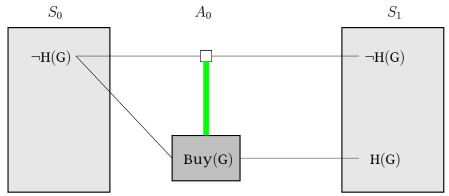
□ = a *persistence action*—what happens if no action is taken.

An action level A_i contains *all* actions that *could* happen given the propositions in S_i.

Mutex links

We also record, using *mutual exclusion (mutex) links* which pairs of actions could not occur together.

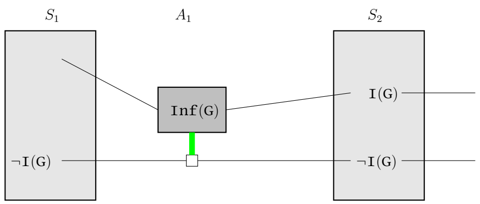
Mutex links 1: Effects are inconsistent.



The effect of one action negates the effect of another.

Mutex links

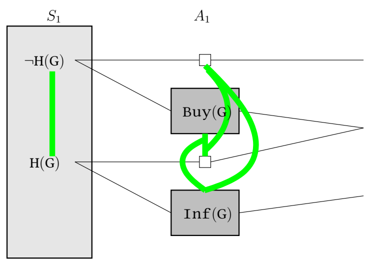
Mutex links 2: The actions interfere.



The effect of an action negates the precondition of another.

Mutex links

Mutex links 3: Competing for preconditions.



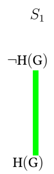
The precondition for an action is mutually exclusive with the precondition for another. (See next slide!)

Mutex links

A state level S_i contains *all* propositions that *could* be true, given the possible preceding actions.

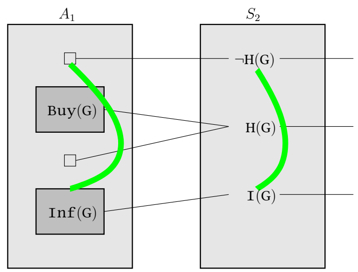
We also use mutex links to record pairs that can not be true simultaneously:

Possibility 1: pair consists of a proposition and its negation.



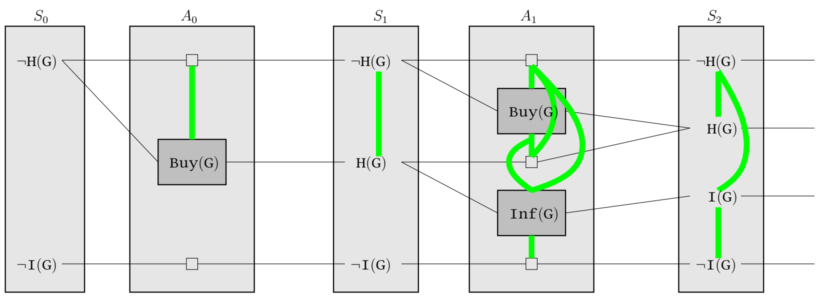
Mutex links

Possibility 2: all pairs of actions that could achieve the pair of propositions are mutex.



The construction of a planning graph is continued until two identical levels are obtained.

Planning graphs



Obtaining heuristics from a planning graph

To estimate the cost of reaching a single proposition:

- Any proposition not appearing in the final level has *infinite cost* and *can never be reached*.
- The *level cost* of a proposition is the level at which it first appears *but* this may be inaccurate as several actions can apply at each level and this cost does not count the *number of actions*. (It is however *admissible*.)
- A *serial planning graph* includes mutex links between all pairs of actions except persistence actions.

Level cost in serial planning graphs can be quite a good measurement.

Obtaining heuristics from a planning graph

How about estimating the cost to achieve a *collection* of propositions?

- *Max-level*: use the maximum level in the graph of any proposition in the set. Admissible but can be inaccurate.
- *Level-sum*: use the sum of the levels of the propositions. Inadmissible but sometimes quite accurate if goals tend to be decomposable.
- *Set-level*: use the level at which *all* propositions appear with none being mutex. Can be accurate if goals tend *not* to be decomposable.

Other points about planning graphs

A planning graph guarantees that:

1. *If* a proposition appears at some level, there *may* be a way of achieving it.
2. *If* a proposition does *not* appear, it can *not* be achieved.

The first point here is a loose guarantee because only *pairs* of items are linked by mutex links.

Looking at larger collections can strengthen the guarantee, but in practice the gains are outweighed by the increased computation.

Graphplan

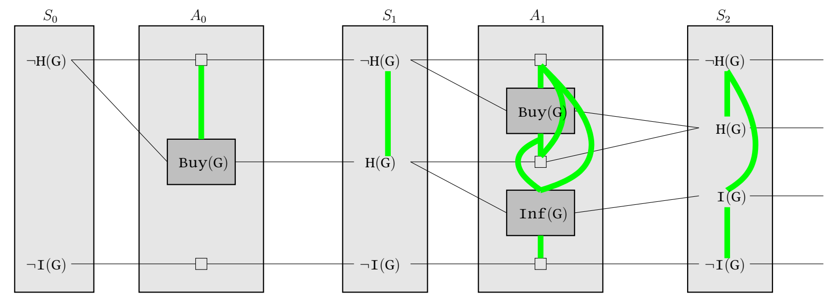
The *GraphPlan* algorithm goes beyond using the planning graph as a source of heuristics.

```
function GraphPlan()
    Start at level 0;
    while true do
        if All goal propositions appear in the current level AND no pair has a mutex link then
            Attempt to extract a plan;
            if A solution is obtained then
                return SOME solution;
            if Graph indicates there is no solution then
                return NONE;
        Expand the graph to the next level;
```

We *extract a plan* directly from the planning graph. Termination can be proved but will not be covered here.

Graphplan in action

Here, at levels S_0 and S_1 we do not have both H(G) and I(G) available with no mutex links, and so we expand first to S_1 and then to S_2.



At S_2 we try to extract a solution (plan).

Extracting a plan from the graph

Extraction of a plan can be formalised as a *search problem*.

States contain a *level*, and a collection of *unsatisfied goal propositions*.

Start state: the current final level of the graph, along with the relevant goal propositions.

Goal: a state at level S_0 containing the initial propositions.

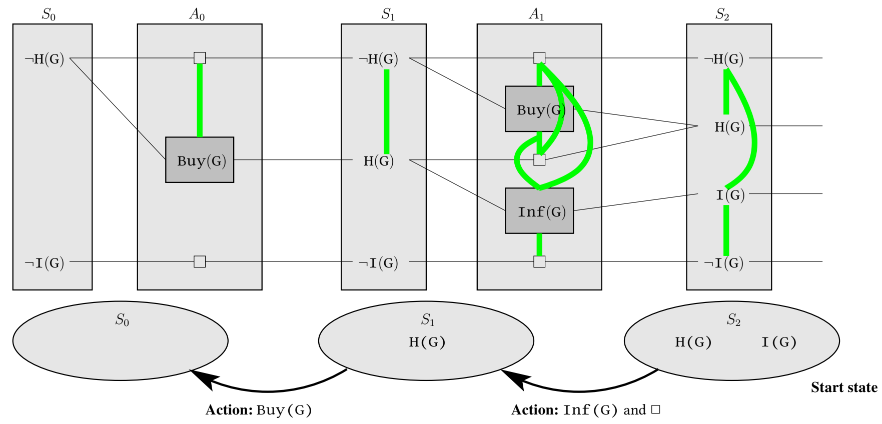
Actions: For a state S with level S_i, a valid action is to select any set X of actions in A_{i-1} such that:

1. no pair has a mutex link;
2. no pair of their preconditions has a mutex link;
3. the effects of the actions in X achieve the propositions in S.

The effect of such an action is a state having level S_{i-1}, and containing the preconditions for the actions in X.

Each action has a cost of 1.

Graphplan in action



Heuristics for plan extraction

We can of course also apply *heuristics* to this part of the process.

For example, when dealing with a *set of propositions*:

- Choose the proposition having *maximum level cost* first.
- For that proposition, attempt to achieve it using the action for which the *maximum/sum level cost of its preconditions is minimum.*

Planning III: planning using propositional logic

We've seen that plans might be extracted from a knowledge base via *theorem proving*, using *first order logic (FOL)* and *situation calculus*.

BUT: this might be computationally infeasible for realistic problems.

Sophisticated techniques are available for testing *satisfiability* in *propositional logic*, and these have also been applied to planning.

The basic idea is to attempt to find a model of a sentence having the form

$$\begin{aligned}&\text{description of start state}\\ &\qquad\wedge \text{ descriptions of the possible actions}\\ &\qquad\wedge \text{ description of goal}\end{aligned}$$

We attempt to construct this sentence such that:

- If M is a model of the sentence then M assigns `true` to a proposition if and only if it is in the plan.
- Any assignment denoting an incorrect plan will not be a model as the goal description will not be `true`.
- The sentence is unsatisfiable if no plan exists.

Propositional logic for planning

Two roof-climbers want to *swap places*:

Start state:

$$\begin{aligned} S = \mathtt{At}^0(\mathtt{a}, \mathtt{spire}) \wedge \mathtt{At}^0(\mathtt{b}, \mathtt{ground}) \\ \wedge \neg \mathtt{At}^0(\mathtt{a}, \mathtt{ground}) \wedge \neg \mathtt{At}^0(\mathtt{b}, \mathtt{spire}) \end{aligned}$$

Remember that an expression such as $\mathtt{At}^0(\mathtt{a}, \mathtt{spire})$ is a *proposition*. The superscripted number now denotes time.

Propositional logic for planning

Goal:

$$\begin{aligned} G = \mathtt{At}^i(\mathtt{a}, \mathtt{ground}) \wedge \mathtt{At}^i(\mathtt{b}, \mathtt{spire}) \\ \wedge \neg \mathtt{At}^i(\mathtt{a}, \mathtt{spire}) \wedge \neg \mathtt{At}^i(\mathtt{b}, \mathtt{ground}) \end{aligned}$$

Actions: can be introduced using the equivalent of successor-state axioms

$$\begin{aligned} &\mathtt{At}^1(\mathtt{a,ground}) \leftrightarrow \\ &\quad (\mathtt{At}^0(\mathtt{a}, \mathtt{ground}) \wedge \neg \mathtt{Move}^0(\mathtt{a}, \mathtt{ground}, \mathtt{spire})) \\ &\quad \vee (\mathtt{At}^0(\mathtt{a}, \mathtt{spire}) \wedge \mathtt{Move}^0(\mathtt{a}, \mathtt{spire}, \mathtt{ground})) \end{aligned} \tag{1}$$

Denote by A the collection of all such axioms.

Propositional logic for planning

We will now find that $S \wedge A \wedge G$ has a model in which $\mathtt{Move}^0(\mathtt{a}, \mathtt{spire}, \mathtt{ground})$ and $\mathtt{Move}^0(\mathtt{b}, \mathtt{ground}, \mathtt{spire})$ are $\mathtt{true}$ while all remaining actions are $\mathtt{false}$.

In more realistic planning problems we will clearly not know in advance at what time the goal might expect to be achieved.

We therefore:

- Loop through possible final times T.
- Generate a goal for time T and actions up to time T.
- Try to find a model and extract a plan.
- Until a plan is obtained or we hit some maximum time.

Propositional logic for planning

Unfortunately there is a problem—we may, if considerable care is not applied, also be able to obtain less sensible plans.

In the current example

$$\begin{aligned} \mathtt{Move}^0(\mathtt{b}, \mathtt{ground}, \mathtt{spire}) &= \mathtt{true} \\ \mathtt{Move}^0(\mathtt{a}, \mathtt{spire}, \mathtt{ground}) &= \mathtt{true} \\ \boxed{\mathtt{Move}^0(\mathtt{a}, \mathtt{ground}, \mathtt{spire})} &= \mathtt{true} \end{aligned}$$

is a model, because the successor-state axiom (1) does not in fact preclude the application of $\mathtt{Move}^0(\mathtt{a}, \mathtt{ground}, \mathtt{spire})$.

We need a *precondition axiom*

$$\mathtt{Move}^i(\mathtt{a}, \mathtt{ground}, \mathtt{spire}) \rightarrow \mathtt{At}^i(\mathtt{a}, \mathtt{ground})$$

and so on.

Propositional logic for planning

Life becomes more complicated still if a third location is added: `hospital`.

$$\texttt{Move}^0(\texttt{a}, \texttt{spire}, \texttt{ground}) \wedge \texttt{Move}^0(\texttt{a}, \texttt{spire}, \texttt{hospital})$$

is perfectly valid and so we need to specify that he can't move to two places simultaneously

$$\begin{gathered}\neg(\texttt{Move}^i(\texttt{a}, \texttt{spire}, \texttt{ground}) \wedge \texttt{Move}^i(\texttt{a}, \texttt{spire}, \texttt{hospital}))\\ \neg(\texttt{Move}^i(\texttt{a}, \texttt{ground}, \texttt{spire}) \wedge \texttt{Move}^i(\texttt{a}, \texttt{ground}, \texttt{hospital}))\\ \vdots\end{gathered}$$

and so on.

These are *action-exclusion* axioms.

Unfortunately they will tend to produce *totally-ordered* rather than *partially-ordered* plans.

Propositional logic for planning

Alternatively:

1. Prevent actions occurring together if one negates the effect or precondition of the other.
2. Or, specify that something can't be in two places simultaneously

$$\neg(\texttt{At}^i(x, \texttt{l1}) \wedge \texttt{At}^i(x, \texttt{l2}))$$

for all combinations of x, i and $\texttt{l1} \neq \texttt{l2}$.

This is an example of a *state constraint*.

Clearly this process can become very complex, but there are techniques to help deal with this.

Review of constraint satisfaction problems (CSPs)

Recall that in a CSP we have:

- A set of n *variables* $V_1, V_2, \ldots, V_n$.
- For each V_i a *domain* D_i specifying the values that V_i can take.
- A set of m *constraints* $C_1, C_2, \ldots, C_m$.

Each constraint C_i involves a set of variables and specifies an *allowable collection of values*.

- A *state* is an assignment of specific values to some or all of the variables.
- An assignment is *consistent* if it violates no constraints.
- An assignment is *complete* if it gives a value to every variable.

A *solution* is a consistent and complete assignment.

The state-variable representation

Another planning language: the *state-variable representation*.

Things of interest such as people, places, objects *etc* are divided into *domains*:

$$\begin{aligned}\mathscr{D}_1 &= \{\texttt{climber1}, \texttt{climber2}\}\\ \mathscr{D}_2 &= \{\texttt{home}, \texttt{jokeShop}, \texttt{hardwareStore}, \texttt{pavement}, \texttt{spire}, \texttt{hospital}\}\\ \mathscr{D}_3 &= \{\texttt{rope}, \texttt{gorilla}\}\end{aligned}$$

Part of the specification of a planning problem involves stating which domain a particular item is in. For example

$$\mathscr{D}_1(\texttt{climber1})$$

and so on.

Relations and functions have arguments chosen from unions of these domains.

$$\texttt{above} \subseteq \mathscr{D}_1^{\texttt{above}} \times \mathscr{D}_2^{\texttt{above}}$$

is a relation. The $\mathscr{D}_i^{\texttt{above}}$ are unions of one or more $\mathscr{D}_i$.

Note: $\mathscr{D}$ is used for domains in the state-variable representation. D is used for domains in CSPs.

The state-variable representation

The relation above is in fact a *rigid relation (RR)*, as it is unchanging: it does not depend upon *state*. (Remember *fluents* in situation calculus?)

Similarly, we have *functions*

$$\mathtt{at}(x_1, s) : \mathscr{D}_1^{\mathtt{at}} \times S \to \mathscr{D}^{\mathtt{at}}.$$

Here, $\mathtt{at}(x, s)$ is a *state-variable*. The domain $\mathscr{D}_1^{\mathtt{at}}$ and range $\mathscr{D}^{\mathtt{at}}$ are unions of one or more $\mathscr{D}_i$. In general these can have multiple parameters

$$\mathtt{sv}(x_1, \dots, x_n, s) : \mathscr{D}_1^{\mathtt{sv}} \times \cdots \times \mathscr{D}_n^{\mathtt{sv}} \times S \to \mathscr{D}^{\mathtt{sv}}.$$

A state-variable denotes assertions such as

$$\mathtt{at}(\mathtt{gorilla}, s) = \mathtt{jokeShop}$$

where s denotes a *state* and the set S of all states will be defined later.

The state variable allows things such as locations to change—again, much like *fluents* in the situation calculus.

Variables appearing in relations and functions are considered to be *typed*.

The state-variable representation

Note:

- For properties such as a *location* a function might be considerably more suitable than a relation.
- For locations, everything has to be *somewhere* and it can only be in *one place at a time*.

So a function is perfect and immediately solves some of the problems seen earlier.

The state-variable representation

Actions as usual, have a *name*, a *set of preconditions* and a *set of effects*.

- *Names* are unique, and followed by a list of variables involved in the action.
- *Preconditions* are expressions involving state variables and relations.
- *Effects* are assignments to state variables.

For example:

$\mathtt{buy}(x, y, l)$	
Preconditions	$\mathtt{at}(x, s) = l$ $\mathtt{sells}(l, y)$ $\mathtt{has}(y, s) = l$
Effects	$\mathtt{has}(y, s) = x$

The state-variable representation

Goals are sets of *expressions* involving *state variables*.

For example:

Goal:
$\mathtt{at}(\mathtt{climber}, s) = \mathtt{home}$
$\mathtt{has}(\mathtt{rope}, s) = \mathtt{climber}$
$\mathtt{at}(\mathtt{gorilla}, s) = \mathtt{spire}$

From now on we will generally suppress the state s when writing state variables.

The state-variable representation

A *state* is just a statement of what values the state variables take at a given time.

$$
\begin{aligned}
s = \{ \quad & \texttt{has}(\texttt{gorilla}) = \texttt{jokeShop} \\
& \texttt{has}(\texttt{firstAidKit}) = \texttt{climber2} \\
& \texttt{has}(\texttt{rope}) = \texttt{climber2} \\
& \quad \vdots \\
& \texttt{at}(\texttt{climber1}) = \texttt{jokeShop} \\
& \texttt{at}(\texttt{climber2}) = \texttt{spire} \\
& \quad \vdots \\
& \qquad \}
\end{aligned}
$$

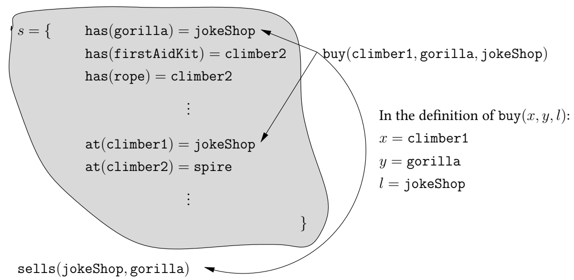
- For each state variable sv consider all ground instances, such as $\texttt{sv}(\texttt{climber}, \texttt{rope})$, with arguments *consistent* with the *rigid relations*.
  Define X to be the set of all such ground instances.
- A state s is then just a set
  $$s = \{(v = c) | v \in X\}$$
  where c is in the range of v.

This allows us to define the *effect of an action.*

A planning problem also needs a *start state* s_0, which can be defined in this way.

The state-variable representation

Considering all the *ground actions consistent with the rigid relations*:

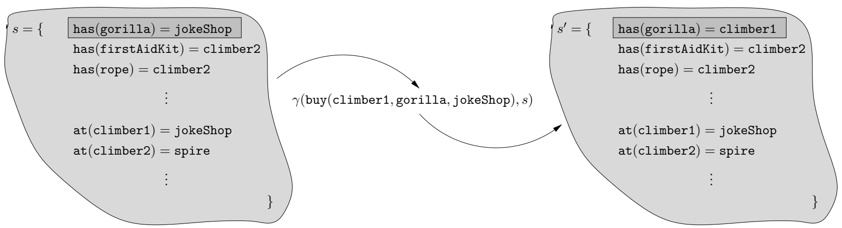


- An action is *applicable in* s if all expressions $v = c$ appearing in the set of preconditions also appear in s.
- As there is no rigid relation $sells(\texttt{jokeShop}, \texttt{fruitBats})$ we would *not* consider an action such as $\texttt{buy}(\texttt{climber1}, \texttt{fruitBats}, \texttt{jokeShop})$—it is not *consistent with the rigid relations.*

The state-variable representation

Finally, there is a function γ that maps a state and an action to a new state

$$\gamma(s, a) = s'$$



Specifically, we have

$$\gamma(s, a) = \{(v = c) | v \in X\}$$

where either c is specified in an effect of a, or otherwise $v = c$ is a member of s.

Note: the definition of γ implicitly solves the *frame problem.*

The state-variable representation

A *solution* to a planning problem is a sequence $(a_0, a_1, \ldots, a_n)$ of actions such that...

- a_0 is applicable in s_0 and for each i, a_i is applicable in $s_i = \gamma(s_{i-1}, a_{i-1})$.
- For each goal g we have
  $$g \in \gamma(s_n, a_n).$$

What we need now is a method for *transforming* a problem described in this language into a CSP.

We'll once again do this for a fixed upper limit T on the number of steps in the plan.

Converting to a CSP

Step 1: encode *actions* as *CSP variables.*

For each time step t where $0 \leq t \leq T - 1$, the CSP has a variable

$$\texttt{action}^t$$

with domain

$$D^{\texttt{action}^t} = \{\texttt{a} | \texttt{a} \text{ is the ground instance of an action}\} \cup \{\texttt{none}\}$$

Example: at some point in searching for a plan we might attempt to find the solution to the corresponding CSP involving

$$\texttt{action}^5 = \texttt{attach}(\texttt{gorilla}, \texttt{spire})$$

WARNING: be careful in what follows to distinguish between *state variables, actions etc* in the planning problem and *variables* in the CSP.

Converting to a CSP

Step 2: encode *ground state variables* as *CSP variables*, with a complete copy of all the state variables *for each time step.*

So, for each t where $0 \leq t \leq T$ we have a CSP variable

$$\texttt{sv}_i^t(c_1, \dots, c_n)$$

with domain $D = \mathscr{D}^{\texttt{sv}_i}$. (That is, the *domain* of the CSP variable is the *range* of the state variable.)

Example: at some point in searching for a plan we might attempt to find the solution to the corresponding CSP involving

$$\texttt{location}^9(\texttt{climber1}) = \texttt{hospital}.$$

Converting to a CSP

Step 3: encode the *preconditions for actions in the planning problem* as *constraints in the CSP problem.*

For each time step t and for each ground action $\texttt{a}(c_1, \dots, c_n)$ with arguments *consistent with the rigid relations in its preconditions*:

For a precondition of the form $\texttt{sv}_i = v$ include constraint pairs

$$(\texttt{action}^t = \texttt{a}(c_1, \dots, c_n),$$
$$\texttt{sv}_i^t = v)$$

Example: consider the action $\texttt{buy}(x, y, l)$ introduced above, and having the preconditions $\texttt{at}(x) = l$, $\texttt{sells}(l, y)$ and $\texttt{has}(y) = l$.

Assume $\texttt{sells}(y, l)$ is only true for

$$l = \texttt{jokeShop}$$

and

$$y = \texttt{gorilla}$$

so we only consider these values for l and y. Then for each time step t we have the constraints...

Converting to a CSP

$\texttt{action}^t = \texttt{buy}(\texttt{climber1}, \texttt{gorilla}, \texttt{jokeShop})$ paired with $\texttt{at}^t(\texttt{climber1}) = \texttt{jokeShop}$
$\texttt{action}^t = \texttt{buy}(\texttt{climber1}, \texttt{gorilla}, \texttt{jokeShop})$ paired with $\texttt{has}^t(\texttt{gorilla}) = \texttt{jokeShop}$
$\texttt{action}^t = \texttt{buy}(\texttt{climber2}, \texttt{gorilla}, \texttt{jokeShop})$ paired with $\texttt{at}^t(\texttt{climber2}) = \texttt{jokeShop}$
$\texttt{action}^t = \texttt{buy}(\texttt{climber2}, \texttt{gorilla}, \texttt{jokeShop})$ paired with $\texttt{has}^t(\texttt{gorilla}) = \texttt{jokeShop}$
and so on...

Converting to a CSP

Step 4: encode the *effects of actions in the planning problem* as *constraints in the CSP problem.*

For each time step t and for each ground action $\texttt{a}(c_1, \ldots, c_n)$ with arguments *consistent with the rigid relations in its preconditions*:

For an effect of the form $\texttt{sv}_i = v$ include constraint pairs

$$(\texttt{action}^t = \texttt{a}(c_1, \ldots, c_n),\\ \texttt{sv}_i^{t+1} = v)$$

Example: continuing with the previous example, we will include constraints

$\texttt{action}^t = \texttt{buy}(\texttt{climber1}, \texttt{gorilla}, \texttt{jokeShop})$ paired with $\texttt{has}^{t+1}(\texttt{gorilla}) = \texttt{climber1}$
$\texttt{action}^t = \texttt{buy}(\texttt{climber2}, \texttt{gorilla}, \texttt{jokeShop})$ paired with $\texttt{has}^{t+1}(\texttt{gorilla}) = \texttt{climber2}$
and so on...

Converting to a CSP

Step 5: encode the *frame axioms* as *constraints in the CSP problem.*

An action must not change things not appearing in its effects. So:

For:

1. Each time step t.
2. Each ground action $\texttt{a}(c_1, \ldots, c_n)$ with arguments *consistent with the rigid relations in its preconditions.*
3. Each $\texttt{sv}_\texttt{i}$ that *does not appear in the effects of* a, and each $v \in \mathscr{D}^{\texttt{sv}_i}$

include in the CSP the ternary constraint

$$(\texttt{action}^t = \texttt{a}(c_1, \ldots, c_n),\\ \texttt{sv}_i^t = v,\\ \texttt{sv}_i^{t+1} = v).$$

Finding a plan

Finally, having encoded a planning problem into a CSP, we solve the CSP.

The scheme has the following property:

A solution to the planning problem with at most T steps exists if and only if there is a a solution to the corresponding CSP.

Assume the CSP has a solution.

Then we can extract a plan simply by looking at the values assigned to the $\texttt{action}^t$ variables in the solution of the CSP.

It is also the case that:

There is a solution to the planning problem with at most T steps if and only if there is a solution to the corresponding CSP from which the solution can be extracted in this way.

For a proof see:

Automated Planning: Theory and Practice

Malik Ghallab, Dana Nau and Paolo Traverso. Morgan Kaufmann 2004.

Artificial Intelligence I

Machine learning using neural networks

Reading: AIMA, chapters 19 and 22.

Did you heed the DIRE WARNING?

At the beginning of the course I suggested making sure you can answer the following two questions:

1. Let
$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} a_i x_i^2$$
where the a_i are constants. Compute $\partial f / \partial x_j$ where $1 \leq j \leq n$?
Answer: As only one term in the sum depends on x_j, all the other terms differentiate to give 0 and
$$\frac{\partial f}{\partial x_j} = 2a_j x_j.$$
2. Let $f(x_1, \ldots, x_n)$ be a function. Now assume $x_i = g_i(y_1, \ldots, y_m)$ for each x_i and some collection of functions g_i. Assuming all requirements for differentiability and so on are met, can you write down an expression for $\partial f / \partial y_j$ where $1 \leq j \leq m$?
Answer: this is just the *chain rule* for partial differentiation
$$\frac{\partial f}{\partial y_j} = \sum_{i=1}^{n} \frac{\partial f}{\partial g_i} \frac{\partial g_i}{\partial y_j}.$$

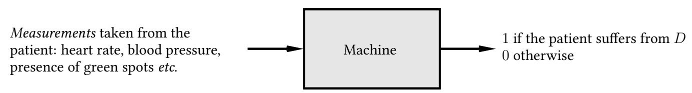
Supervised learning with neural networks

We now consider how an agent might *learn* to solve a general problem by seeing *examples*:

- I present an outline of *supervised learning*.
- I introduce the classical *perceptron*.
- I introduce *multilayer perceptrons* and the *backpropagation algorithm* for training them.

To begin, a common source of problems in AI is *medical diagnosis*.

Imagine that we want to automate the diagnosis of an Embarrassing Disease (call it D) by constructing a machine:



Could we do this by *explicitly writing a program* that examines the measurements and outputs a diagnosis? Experience suggests that this is unlikely.

An example, continued...

An alternative approach: each collection of measurements can be written as a vector,
$$\mathbf{x}^T = \left(x_1 \;\; x_2 \;\; \cdots \;\; x_n \right)$$
where,
$$\begin{aligned} x_1 &= \text{heart rate} \\ x_2 &= \text{blood pressure} \\ x_3 &= 1 \text{ if the patient has green spots, and } 0 \text{ otherwise} \\ &\vdots \end{aligned}$$
and so on.

(*Note*: it's a common convention that vectors are *column vectors* by default. This is why the above is written as a *transpose*.)

An example, continued...

A vector of this kind contains all the measurements for a single patient and is called a *feature vector* or *instance*.

The measurements are *attributes* or *features*.

Attributes or features generally appear as one of three basic types:

- *Continuous*: $x_i \in [x_{\text{min}}, x_{\text{max}}]$ where $x_{\text{min}}, x_{\text{max}} \in \mathbb{R}$.
- *Binary*: $x_i \in \{0, 1\}$ or $x_i \in \{-1, +1\}$.
- *Discrete*: x_i can take one of a finite number of values, say $x_i \in \{X_1, \ldots, X_p\}$.

An example, continued...

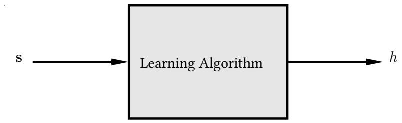
Now imagine that we have a large collection of patient histories (m in total) and for each of these we know whether or not the patient suffered from D.

- The ith patient history gives us an instance $\mathbf{x}_i$.
- This can be paired with a single bit—0 or 1—denoting whether or not the ith patient suffers from D. The resulting pair is called an *example* or a *labelled example*.
- Collecting all the examples together we obtain a *training sequence*

$$\mathbf{s} = ((\mathbf{x}_1, 0), (\mathbf{x}_2, 1), \ldots, (\mathbf{x}_m, 0)).$$

An example, continued...

In supervised machine learning we aim to design a *learning algorithm* which takes $\mathbf{s}$ and produces a *hypothesis* h.



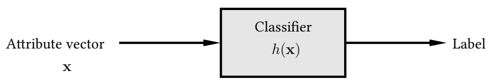
Intuitively, a hypothesis is something that lets us diagnose *new* patients.

This is *IMPORTANT*: we want to diagnose patients that *the system has never seen.*

The ability to do this successfully is called *generalisation.*

An example, continued...

In fact, a hypothesis is just a *function* that maps *instances* to *labels.*



As h is a *function* it assigns a label to *any* $\mathbf{x}$ and *not just the ones that were in the training sequence.*

What we mean by a *label* here depends on whether we're doing *classification* or *regression.*

Supervised learning: classification and regression

In *classification* we're assigning $\mathbf{x}$ to one of a set $\{\omega_1, \ldots, \omega_c\}$ of c *classes.* For example, if $\mathbf{x}$ contains measurements taken from a patient then there might be three classes:

$$\omega_1 = \text{patient has disease}$$
$$\omega_2 = \text{patient doesn't have disease}$$
$$\omega_3 = \text{don't ask me buddy, I'm just a computer!}$$

The *binary* case above also fits into this framework, and we'll often specialise to the case of two classes, denoted C_1 and C_2.

In *regression* we're assigning $\mathbf{x}$ to a *real number* $h(\mathbf{x}) \in \mathbb{R}$. For example, if $\mathbf{x}$ contains measurements taken regarding today's weather then we might have

$$h(\mathbf{x}) = \text{estimate of amount of rainfall expected tomorrow.}$$

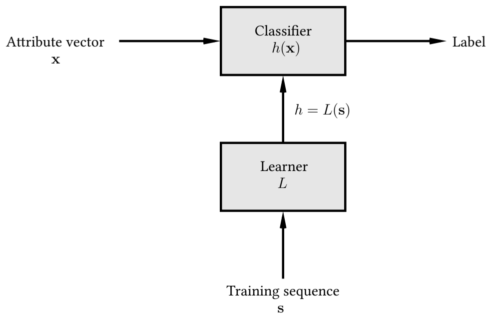
For the *two-class classification problem* we will also refer to a situation somewhat between the two, where

$$h(\mathbf{x}) = \Pr(\mathbf{x} \text{ is in } C_1)$$

and so we would typically assign $\mathbf{x}$ to class C_1 if $h(\mathbf{x}) > 1/2$.

Summary

We don't want to design h explicitly.



So we use a *learner* L to infer it on the basis of a sequence $\mathbf{s}$ of *training examples*.

Neural networks

There is generally a set $\mathcal{H}$ of hypotheses from which L is allowed to select h

$$L(\mathbf{s}) = h \in \mathcal{H}$$

$\mathcal{H}$ is called the *hypothesis space*.

The learner can output a hypothesis explicitly or—as in the case of a *neural network*—it can output a vector

$$\mathbf{w}^T = \begin{pmatrix} w_1 & w_2 & \cdots & w_W \end{pmatrix}$$

of *weights* which in turn specify h

$$h(\mathbf{x}) = f(\mathbf{w}; \mathbf{x})$$

where $\mathbf{w} = L(\mathbf{s})$.

Types of learning

The form of machine learning described is called *supervised learning*. The literature also discusses *unsupervised learning*, *semisupervised learning*, learning using *membership queries* and *equivalence queries*, and *reinforcement learning*. (More about some of this next year...)

Supervised learning has multiple applications:

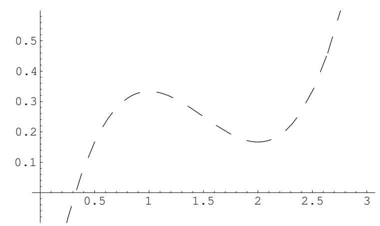
- *Speech recognition.*
- Deciding *whether or not to give credit*.
- Detecting *credit card fraud*.
- Deciding whether to *buy or sell a stock option.*
- Deciding whether a *tumour is benign.*
- *Data mining*: extracting interesting but hidden knowledge from existing, large databases. For example, databases containing *financial transactions* or *loan applications.*
- *Automatic driving*. (See Pomerleau, 1989, in which a car is driven for 90 miles at 70 miles per hour, on a public road with other cars present, but with no assistance from humans.)

This is very similar to curve fitting

This process is in fact very similar to *curve fitting*. Think of the process as follows:

- Nature picks an $h' \in \mathcal{H}$ but doesn't reveal it to us.
- Nature then shows us a training sequence $\mathbf{s}$ where each $\mathbf{x}_i$ is labelled as $h'(\mathbf{x}_i) + \epsilon_i$ where ϵ_i is noise of some kind.

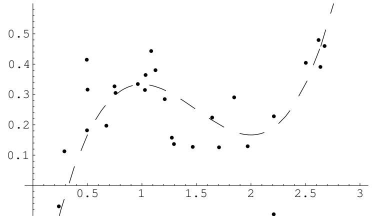
Our job is to try to infer what h' is *on the basis of* $\mathbf{s}$ *only*. *Example*: if $\mathcal{H}$ is the set of all polynomials of degree 3 then nature might pick $h'(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x - \frac{1}{2}$.



The line is dashed to emphasise the fact that *we don't get to see it*.

Curve fitting

We can now use h' to obtain a training sequence $\mathbf{s}$ in the manner suggested..



Here we have,

$$\mathbf{s}^T = ((x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m))$$

where each x_i and y_i is a real number.

Curve fitting

We'll use a *learning algorithm* L that operates in a reasonable-looking way: it picks an $h \in \mathcal{H}$ minimising the following quantity,

$$E = \sum_{i=1}^{m} (h(x_i) - y_i)^2.$$

In other words

$$h = L(\mathbf{s}) = \operatorname*{argmin}_{h \in \mathcal{H}} \sum_{i=1}^{m} (h(x_i) - y_i)^2.$$

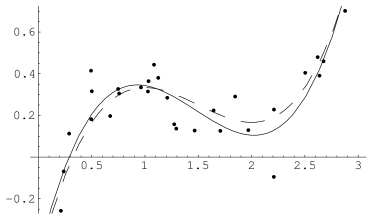
Why is this sensible?

1. Each term in the sum is 0 if $h(x_i)$ is *exactly* y_i.
2. Each term *increases* as the difference between $h(x_i)$ and y_i increases.
3. We add the terms for all examples.

WARNING! This is a weak justification! In the *Exercise Sheet* you are guided to produce a much better justification. Attempt this and discuss it with your Supervisors.

Curve fitting

If we pick h using this method then we get:



The chosen h is close to the target h', even though it was chosen *using only a small number of noisy examples.*

It is not quite identical to the target concept.

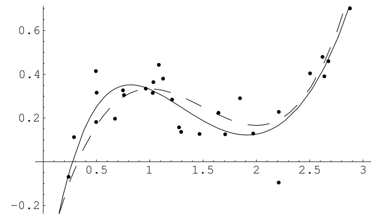
However if we were given a new point $\mathbf{x}'$ and asked to guess the value $h'(\mathbf{x}')$ then guessing $h(\mathbf{x}')$ might be expected to do quite well.

Curve fitting

Problem: we don't know *what $\mathcal{H}$ nature is using*. What if the one we choose doesn't match? We can make *our* $\mathcal{H}$ 'bigger' by defining it as

$$\mathcal{H} = \{h : h \text{ is a polynomial of degree at most } 5\}.$$

If we use the same learning algorithm then we get:



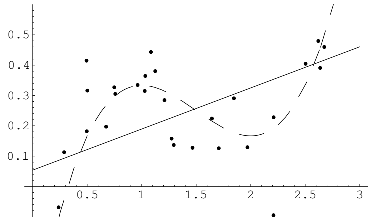
The result in this case is similar to the previous one: h is again quite close to h', but not quite identical.

Curve fitting

So what's the problem? Repeating the process with,

$$\mathcal{H} = \{h : h \text{ is a polynomial of degree at most } 1\}$$

gives the following:



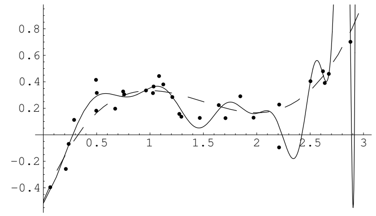
In effect, we have made *our* $\mathcal{H}$ too 'small'. It does not in fact contain any hypothesis similar to h'.

Curve fitting

So we have to make $\mathcal{H}$ *huge, right? WRONG!!!* With

$$\mathcal{H} = \{h : h \text{ is a polynomial of degree at most } 25\}$$

we get:



BEWARE!!! This is known as *overfitting*.

The perceptron

The example just given illustrates much of what we want to do. However in practice we deal with *more than a single dimension*, so

$$\mathbf{x}^T = (\, x_1 \;\; x_2 \;\; \cdots \;\; x_n \,).$$

The simplest form of hypothesis used is the *linear discriminant*, also known as the *perceptron*. Here

$$h(\mathbf{w}; \mathbf{x}) = \sigma\left(w_0 + \sum_{i=1}^{n} w_i x_i\right) = \sigma\left(w_0 + w_1 x_1 + w_2 x_2 + \cdots + w_n x_n\right).$$

So: we have a *linear function* modified by the *activation function* σ.

The perceptron's influence continues to be felt in the recent and ongoing development of *support vector machines*, and forms the basis for most of the field of supervised learning.

The perceptron activation function I

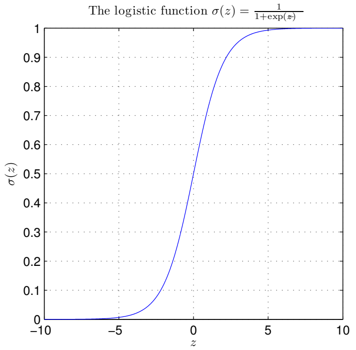
There are three standard forms for the activation function:

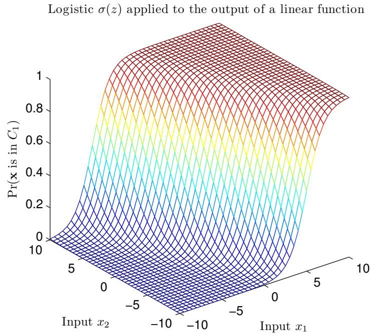
1. *Linear*: for *regression problems* we often use
$$\sigma(z) = z.$$
2. *Step*: for *two-class classification problems* we often use
$$\sigma(z) = \begin{cases} C_1 & \text{if } z > 0 \\ C_2 & \text{otherwise.} \end{cases}$$
3. *Sigmoid/Logistic*: for *probabilistic classification* we often use
$$\Pr(\mathbf{x} \text{ is in } C_1) = \sigma(z) = \frac{1}{1 + \exp(-z)}.$$

The *step function* is important but the algorithms involved are somewhat different to those we'll be seeing. We won't consider it further.

The *sigmoid/logistic function* plays a major role in what follows.

The sigmoid/logistic function





Different functions for $E(\mathbf{w})$.

The function $\sigma(z)$ has range $(0, 1)$. So for 2-class classification we might try to learn

$$\Pr(\mathbf{x} \in C_1) = h(\mathbf{w}; \mathbf{x}) = \sigma(f(\mathbf{w}; \mathbf{x}))$$

where C_1, C_2 denote the two classes and $f(\mathbf{w}; \mathbf{x})$ is a real-valued function.

It then follows that

$$\Pr(\mathbf{x} \in C_2) = 1 - h(\mathbf{w}; \mathbf{x}).$$

Now, say we have a training set

$$\mathbf{s} = ((\mathbf{x}_1, y_1), (\mathbf{x}_2, y_2), \ldots, (\mathbf{x}_m, y_m))$$

using labels in $y_i \in \{0, 1\}$.

Assume the feature vectors are fixed.

Given a weight vector $\mathbf{w}$ *what is the probability that* $h(\mathbf{w}; \mathbf{x})$ *would reproduce the labels in* $\mathbf{s}$?

Different functions for $E(\mathbf{w})$.

Write

$$\Pr(\mathbf{x}_i \text{ is labelled } y_i) = \begin{cases} h(\mathbf{w}; \mathbf{x}_i) & \text{if } y_i = 1 \\ 1 - h(\mathbf{w}; \mathbf{x}_i) & \text{if } y_i = 0 \end{cases}$$

or more neatly

$$\Pr(\mathbf{x}_i \text{ is labelled } y_i) = h(\mathbf{w}; \mathbf{x}_i)^{y_i}(1 - h(\mathbf{w}; \mathbf{x}_i))^{1-y_i}.$$

Assume labels are independent.

Then

$$\Pr(\mathbf{s} \text{ gets labelled correctly}) = \prod_{i=1}^{m} h(\mathbf{w}; \mathbf{x}_i)^{y_i}(1 - h(\mathbf{w}; \mathbf{x}_i))^{1-y_i}.$$

It makes sense to *choose* $\mathbf{w}$ *by maximizing this probability*.

Different functions for $E(\mathbf{w})$.

We don't like products, but taking the $\log$ turns it into a sum.

So: *minimize*

$$E(\mathbf{w}) = -\sum_{i=1}^{m} y_i \log h(\mathbf{w}; \mathbf{x}_i) + (1 - y_i)\log(1 - h(\mathbf{w}; \mathbf{x}_i)).$$

This is completely different from the sum of squares we used for regression.

What about more than two classes?

Say we have classes $C_1, C_2, \ldots, C_n$.

Learn corresponding weight vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n$.

The idea now is that

$$\Pr(\mathbf{x} \in C_j) = \text{prob}(j, \mathbf{x}) = \frac{\exp(f(\mathbf{w}_j; \mathbf{x}))}{\sum_{i=1}^{n} \exp(f(\mathbf{w}_i; \mathbf{x}))}.$$

The $\exp$ insures that probabilities are *positive*.

The denominator forces the probabilities to *sum to* 1.

Different functions for $E(\mathbf{w})$.

You can now carry through an analogous derivation to obtain a suitable expression $E(\mathbf{w}_1, \ldots, \mathbf{w}_n)$ for use with multiple classes.

This is another problem in the *Exercise Sheet* for the course.

You should find that

$$E(\mathbf{w}_1, \ldots, \mathbf{w}_n) = -\sum_{i=1}^{m} \sum_{j=1}^{n} \mathbb{I}[y_i = C_j] \log \text{prob}(j, \mathbf{x}_i).$$

Try it, and if necessary *discuss with your Supervisor*.

Gradient descent

A method for *training a basic perceptron* works as follows. Assume we're dealing with a *regression problem* and using $\sigma(z) = z$.

We define a measure of *error* for a given collection of weights. For example

$$E(\mathbf{w}) = \sum_{i=1}^{m} (y_i - h(\mathbf{w}; \mathbf{x}_i))^2.$$

Modifying our notation slightly so that

$$\mathbf{x}^T = \begin{pmatrix} 1 & x_1 & x_2 & \cdots & x_n \end{pmatrix}$$
$$\mathbf{w}^T = \begin{pmatrix} w_0 & w_1 & w_2 & \cdots & w_n \end{pmatrix}$$

lets us write

$$E(\mathbf{w}) = \sum_{i=1}^{m} (y_i - \mathbf{w}^T \mathbf{x}_i)^2.$$

We want to *minimise* $E(\mathbf{w})$.

Gradient descent

One way to approach this is to start with a random $\mathbf{w}_0$ and update it as follows:

$$\mathbf{w}_{t+1} = \mathbf{w}_t - \eta \left. \frac{\partial E(\mathbf{w})}{\partial \mathbf{w}} \right|_{\mathbf{w}_t}$$

where

$$\frac{\partial E(\mathbf{w})}{\partial \mathbf{w}} = \begin{pmatrix} \frac{\partial E(\mathbf{w})}{\partial w_0} & \frac{\partial E(\mathbf{w})}{\partial w_1} & \cdots & \frac{\partial E(\mathbf{w})}{\partial w_n} \end{pmatrix}^T$$

and η is some small positive number.

The vector

$$-\frac{\partial E(\mathbf{w})}{\partial \mathbf{w}}$$

tells us the *direction of the steepest decrease in* $E(\mathbf{w})$.

Gradient descent

With

$$E(\mathbf{w}) = \sum_{i=1}^{m} (y_i - \mathbf{w}^T \mathbf{x}_i)^2$$

we have

$$\begin{aligned}
\frac{\partial E(\mathbf{w})}{\partial w_j} &= \frac{\partial}{\partial w_j} \left(\sum_{i=1}^{m} (y_i - \mathbf{w}^T \mathbf{x}_i)^2 \right) \\
&= \sum_{i=1}^{m} \left(\frac{\partial}{\partial w_j} (y_i - \mathbf{w}^T \mathbf{x}_i)^2 \right) \\
&= \sum_{i=1}^{m} \left(2(y_i - \mathbf{w}^T \mathbf{x}_i) \frac{\partial}{\partial w_j} \left(-\mathbf{w}^T \mathbf{x}_i \right) \right) \\
&= -2 \sum_{i=1}^{m} \mathbf{x}_i^{(j)} \left(y_i - \mathbf{w}^T \mathbf{x}_i \right)
\end{aligned}$$

where $\mathbf{x}_i^{(j)}$ is the jth element of $\mathbf{x}_i$.

Gradient descent

The method therefore gives the algorithm

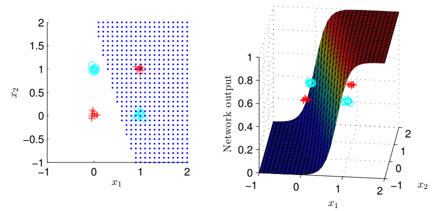
$$\mathbf{w}_{t+1} = \mathbf{w}_t + 2\eta \sum_{i=1}^{m} \left(y_i - \mathbf{w}_t^T \mathbf{x}_i\right) \mathbf{x}_i$$

Some things to note:

- In this case $E(\mathbf{w})$ is *parabolic* and has a *unique global minimum* and *no local minima* so this works well.
- *Gradient descent* in some form is a very common approach to this kind of problem.
- We can perform a similar calculation for *other activation functions* and for *other definitions for* $E(\mathbf{w})$.
- Such calculations lead to *different algorithms*.

Perceptrons aren't very powerful: the parity problem

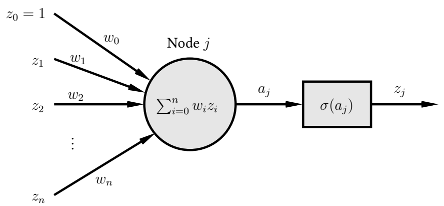
There are many problems a perceptron can't solve.



We need a network that computes *more interesting functions*.

The multilayer perceptron

Each *node* in the network is itself a perceptron:



Weights w_i connect nodes together, and a_j is the weighted sum or *activation* for node j. σ is the *activation function* and the *output* is $z_j = \sigma(a_j)$.

Reminder: we'll continue to use the notation

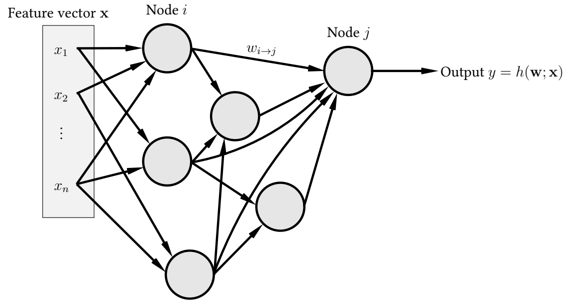
$$\mathbf{z}^T = \begin{pmatrix} 1 & z_1 & z_2 & \cdots & z_n \end{pmatrix}$$
$$\mathbf{w}^T = \begin{pmatrix} w_0 & w_1 & w_2 & \cdots & w_n \end{pmatrix}$$

so that

$$\sum_{i=0}^{n} w_i z_i = w_0 + \sum_{i=1}^{n} w_i z_i = \mathbf{w}^T \mathbf{z}.$$

The multilayer perceptron

In the general case we have a *completely unrestricted feedforward structure*:



Each node is a perceptron. *No specific layering* is assumed.

$w_{i\to j}$ connects node i to node j. w_0 for node j is denoted $w_{0\to j}$.

Backpropagation

As usual we have:

- Instances $\mathbf{x}^T = (x_1, \ldots, x_n)$.
- A training sequence $\mathbf{s} = ((\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_m, y_m))$.

We also define a measure of training error

$$E(\mathbf{w}) = \text{measure of the error of the network on } \mathbf{s}$$

where $\mathbf{w}$ is the vector of *all the weights in the network*.

Our aim is to find a set of weights that *minimises* $E(\mathbf{w})$ using *gradient descent*.

Backpropagation: the general case

The *central task* is therefore to calculate

$$\frac{\partial E(\mathbf{w})}{\partial \mathbf{w}}$$

To do that we need to calculate the individual quantities

$$\frac{\partial E(\mathbf{w})}{\partial w_{i \to j}}$$

for *every weight* $w_{i \to j}$ *in the network.*

Often $E(\mathbf{w})$ is the sum of separate components, one for each example in $\mathbf{s}$

$$E(\mathbf{w}) = \sum_{p=1}^{m} E_p(\mathbf{w})$$

in which case

$$\frac{\partial E(\mathbf{w})}{\partial \mathbf{w}} = \sum_{p=1}^{m} \frac{\partial E_p(\mathbf{w})}{\partial \mathbf{w}}$$

We can therefore consider examples individually.

Backpropagation: the general case

Place example p at the input and calculate a_j and z_j for *all nodes* including the output y. This is *forward propagation.*

We have

$$\frac{\partial E_p(\mathbf{w})}{\partial w_{i \to j}} = \frac{\partial E_p(\mathbf{w})}{\partial a_j} \frac{\partial a_j}{\partial w_{i \to j}}$$

where $a_j = \sum_k w_{k \to j} z_k$.

Here the sum is over *all the nodes connected to node* j. As

$$\frac{\partial a_j}{\partial w_{i \to j}} = \frac{\partial}{\partial w_{i \to j}} \left(\sum_k w_{k \to j} z_k \right) = z_i$$

we can write

$$\frac{\partial E_p(\mathbf{w})}{\partial w_{i \to j}} = \delta_j z_i$$

where we've defined

$$\delta_j = \frac{\partial E_p(\mathbf{w})}{\partial a_j}.$$

Backpropagation: the general case

So we now need to calculate the values for δ_j. When j is the *output node*—that is, the one producing the output $y = h(\mathbf{w}; \mathbf{x}_p)$ of the network—this is easy as $z_j = y$ and

$$\begin{aligned} \delta_j &= \frac{\partial E_p(\mathbf{w})}{\partial a_j} \\ &= \frac{\partial E_p(\mathbf{w})}{\partial y} \frac{\partial y}{\partial a_j} \\ &= \frac{\partial E_p(\mathbf{w})}{\partial y} \sigma'(a_j) \end{aligned}$$

using the fact that $y = \sigma(a_j)$. *The first term is in general easy to calculate* for a given E as the error is generally just a measure of the distance between y and the label y_p in the training sequence.

Example: when

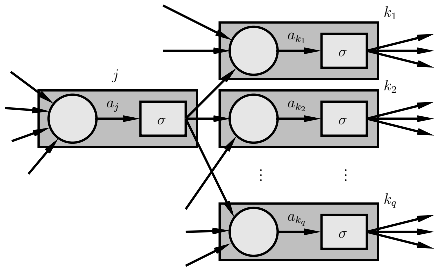
$$E_p(\mathbf{w}) = (y - y_p)^2$$

we have

$$\begin{aligned} \frac{\partial E_p(\mathbf{w})}{\partial y} &= 2(y - y_p) \\ &= 2(h(\mathbf{w}; \mathbf{x}_p) - y_p). \end{aligned}$$

Backpropagation: the general case

When j is *not an output node* we need something different:

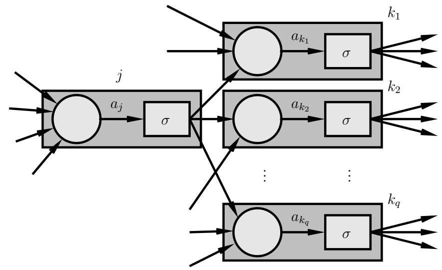


We're interested in

$$\delta_j = \frac{\partial E_p(\mathbf{w})}{\partial a_j}$$

Altering a_j can affect *several other nodes* $k_1, k_2, \ldots, k_q$ *each of which can in turn affect* $E_p(\mathbf{w})$.

Backpropagation: the general case

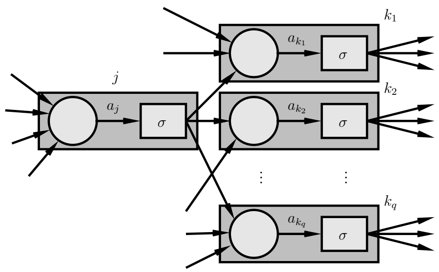


We have

$$\delta_j = \frac{\partial E_p(\mathbf{w})}{\partial a_j} = \sum_{k \in \{k_1, k_2, \ldots, k_q\}} \frac{\partial E_p(\mathbf{w})}{\partial a_k} \frac{\partial a_k}{\partial a_j} = \sum_{k \in \{k_1, k_2, \ldots, k_q\}} \delta_k \frac{\partial a_k}{\partial a_j}$$

where $k_1, k_2, \ldots, k_q$ are the nodes to which node j sends a connection.

Backpropagation: the general case

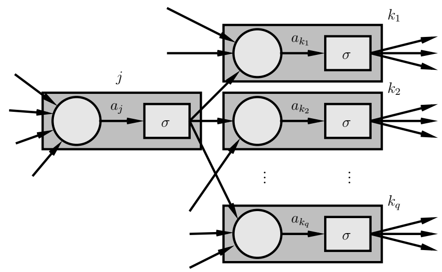


Because we know how to compute δ_j *for the output node* we can *work backwards* computing further δ values.

We will *always know all the values* δ_k *for nodes ahead of where we are.*

Hence the term *backpropagation.*

Backpropagation: the general case



$$\frac{\partial a_k}{\partial a_j} = \frac{\partial}{\partial a_j}\left(\sum_i w_{i \to k} \sigma(a_i)\right) = w_{j \to k} \sigma'(a_j)$$

and

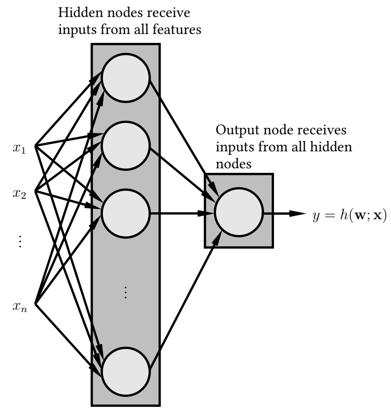
$$\delta_j = \sum_{k \in \{k_1, k_2, \ldots, k_q\}} \delta_k w_{j \to k} \sigma'(a_j) = \sigma'(a_j) \sum_{k \in \{k_1, k_2, \ldots, k_q\}} \delta_k w_{j \to k}.$$

Backpropagation: the general case

Summary: to calculate $\frac{\partial E_p(\mathbf{w})}{\partial \mathbf{w}}$ for the pth pattern:

1. *Forward propagation*: apply $\mathbf{x}_p$ and calculate outputs *etc* for *all the nodes in the network.*
2. *Backpropagation 1*: for the *output* node
$$\frac{\partial E_p(\mathbf{w})}{\partial w_{i\to j}} = z_i\delta_j = z_i\sigma'(a_j)\frac{\partial E_p(\mathbf{w})}{\partial y}$$
where $y = h(\mathbf{w}; \mathbf{x}_p)$.
3. *Backpropagation 2*: For other nodes
$$\frac{\partial E_p(\mathbf{w})}{\partial w_{i\to j}} = z_i\sigma'(a_j)\sum_k \delta_k w_{j\to k}$$
where the δ_k were calculated at an earlier step.

Backpropagation: a specific example



For the output: $\sigma(a) = a$. For the hidden nodes $\sigma(a) = \frac{1}{1+\exp(-a)}$.

Backpropagation: a specific example

For the output: $\sigma(a) = a$ so $\sigma'(a) = 1$.

For the hidden nodes:
$$\sigma(a) = \frac{1}{1+\exp(-a)}$$
so
$$\sigma'(a) = \sigma(a)\left[1 - \sigma(a)\right].$$
We'll continue using the same definition for the error
$$E(\mathbf{w}) = \sum_{p=1}^{m}(y_p - h(\mathbf{w}; \mathbf{x}_p))^2$$
$$E_p(\mathbf{w}) = (y_p - h(\mathbf{w}; \mathbf{x}_p))^2.$$

Backpropagation: a specific example

For the output: the equation is
$$\frac{\partial E_p(\mathbf{w})}{\partial w_{i\to \text{output}}} = z_i\delta_{\text{output}} = z_i\sigma'(a_{\text{output}})\frac{\partial E_p(\mathbf{w})}{\partial y}$$
where $y = h(\mathbf{w}; \mathbf{x}_p)$. So as
$$\begin{aligned}\frac{\partial E_p(\mathbf{w})}{\partial y} &= \frac{\partial}{\partial y}\left((y_p - y)^2\right)\\ &= 2(y - y_p)\\ &= 2\left[h(\mathbf{w}; \mathbf{x}_p) - y_p\right]\end{aligned}$$
and $\sigma'(a) = 1$ so
$$\delta_{\text{output}} = 2\left[h(\mathbf{w}; \mathbf{x}_p) - y_p\right]$$
and
$$\boxed{\frac{\partial E_p(\mathbf{w})}{\partial w_{i\to \text{output}}} = 2z_i(h(\mathbf{w}; \mathbf{x}_p) - y_p)}$$

Backpropagation: a specific example

For the hidden nodes: the equation is

$$\frac{\partial E_p(\mathbf{w})}{\partial w_{i\to j}} = z_i \sigma'(a_j) \sum_k \delta_k w_{j\to k}.$$

However *there is only one output* so

$$\frac{\partial E_p(\mathbf{w})}{\partial w_{i\to j}} = z_i \sigma(a_j) \left[1 - \sigma(a_j)\right] \delta_{\text{output}} w_{j\to \text{output}}$$

and we know that

$$\delta_{\text{output}} = 2\left[h(\mathbf{w}; \mathbf{x}_p) - y_p\right]$$

so

$$\boxed{\begin{aligned}\frac{\partial E_p(\mathbf{w})}{\partial w_{i\to j}} &= 2 z_i \sigma(a_j) \left[1 - \sigma(a_j)\right] \left[h(\mathbf{w}; \mathbf{x}_p) - y_p\right] w_{j\to \text{output}} \\ &= 2 x_i z_j (1 - z_j) \left[h(\mathbf{w}; \mathbf{x}_p) - y_p\right] w_{j\to \text{output}}.\end{aligned}}$$

Putting it all together

We can then use the derivatives in one of two basic ways:

Batch: (as described previously)

$$\frac{\partial E(\mathbf{w})}{\partial \mathbf{w}} = \sum_{p=1}^{m} \frac{\partial E_p(\mathbf{w})}{\partial \mathbf{w}}$$

then

$$\mathbf{w}_{t+1} = \mathbf{w}_t - \eta \left. \frac{\partial E(\mathbf{w})}{\partial \mathbf{w}} \right|_{\mathbf{w}_t}.$$

Sequential: using just one pattern at once

$$\mathbf{w}_{t+1} = \mathbf{w}_t - \eta \left. \frac{\partial E_p(\mathbf{w})}{\partial \mathbf{w}} \right|_{\mathbf{w}_t}$$

selecting patterns *in sequence or at random.*

Example: the parity problem revisited

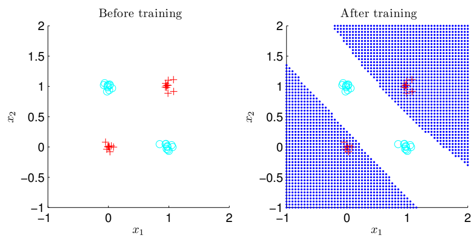
As an example we show the result of training a network with:

- Two inputs.
- One output.
- One hidden layer containing 5 units.
- $\eta = 0.01$.
- All other details as above.

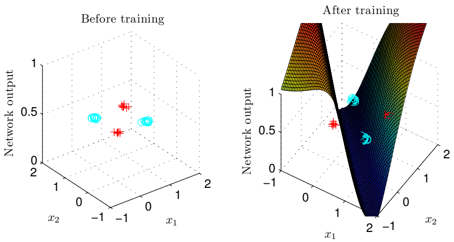
The problem is the parity problem. There are 40 noisy examples.

The sequential approach is used, with 1000 repetitions through the entire training sequence.

Example: the parity problem revisited



Example: the parity problem revisited



Example: the parity problem revisited