

Definition. A register machine is specified by:

(Recall)

- ▶ finitely many registers R_0, R_1, \dots, R_n
(each capable of storing a natural number);
- ▶ a program consisting of a finite list of instructions of the form *label : body*, where for $i = 0, 1, 2, \dots$, the $(i + 1)^{\text{th}}$ instruction has label L_i .

Instruction **body** takes one of three forms:

$R^+ \rightarrow L'$	add 1 to contents of register R and jump to instruction labelled L'
$R^- \rightarrow L', L''$	if contents of R is > 0 , then subtract 1 from it and jump to L' , else jump to L''
HALT	stop executing instructions

Recall:

Computable functions

Definition. $f \in \mathbb{N}^n \rightarrow \mathbb{N}$ is (**register machine**) **computable** if there is a register machine M with at least $n + 1$ registers R_0, R_1, \dots, R_n (and maybe more) such that for all $(x_1, \dots, x_n) \in \mathbb{N}^n$ and all $y \in \mathbb{N}$,
the computation of M starting with $R_0 = 0$,
 $R_1 = x_1, \dots, R_n = x_n$ and all other registers set to 0 , halts with $R_0 = y$
if and only if $f(x_1, \dots, x_n) = y$.

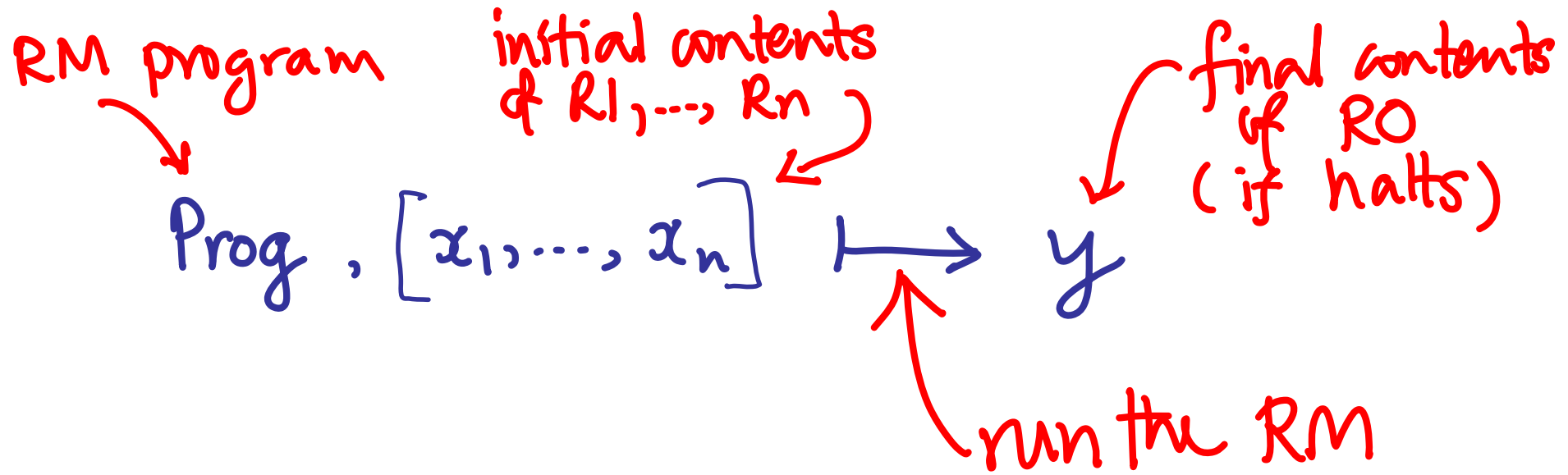
N.B. there may be many different M that compute the same partial function f .

Coding programs as numbers

Turing/Church solution of the Entscheidungsproblem uses the idea that (formal descriptions of) algorithms can be the data on which algorithms act.

To realize this idea with Register Machines we have to be able to code RM programs as numbers. (In general, such codings are often called Gödel numberings.)

"Effective" numerical codes



"Effective" numerical codes

Prog, $[x_1, \dots, x_n] \mapsto y$

code \downarrow \uparrow decode

$\langle \ulcorner \text{Prog} \urcorner, \ulcorner [x_1, \dots, x_n] \urcorner \rangle$
a number

Want numerical codings

$\langle -, - \rangle, \ulcorner - \urcorner, \ulcorner [-, \dots, -] \urcorner$

So that

$\cdot \xrightarrow{\text{decode}} \cdot \xrightarrow{\text{run}} \cdot$

is RM computable

Numerical coding of pairs

$\{0, 1, 2, 3, \dots\}$

For $x, y \in \mathbb{N}$, define

$$\begin{cases} \langle\langle x, y \rangle\rangle \triangleq 2^x(2y + 1) \\ \langle x, y \rangle \triangleq 2^x(2y + 1) - 1 \end{cases}$$

left-hand side is equal to
the right-hand side by definition

Numerical coding of pairs

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$\langle\langle x, y \rangle\rangle$	0	1	2	...
0	1	3	5	...
1	2	6	10	...
2	4	12	20	...
\vdots	\vdots	\vdots	\vdots	

$\langle x, y \rangle$	0	1	2	...
0	0	2	4	...
1	1	5	9	...
2	3	11	19	...
\vdots	\vdots	\vdots	\vdots	

Numerical coding of pairs

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So

$$\begin{aligned} \boxed{0b\langle\langle x, y \rangle\rangle} &= \boxed{0by} \mid \boxed{1} \mid \boxed{0 \cdots 0} \\ \boxed{0b\langle x, y \rangle} &= \boxed{0by} \mid \boxed{0} \mid \boxed{1 \cdots 1} \end{aligned}$$

$\underbrace{\quad\quad\quad}_{x \text{ 0s}}$
 $\underbrace{\quad\quad\quad}_{x \text{ 1s}}$

(Notation: $0bx \triangleq x$ in binary.)

E.g. $27 = 0b11011 = \langle\langle 0, 13 \rangle\rangle = \langle 2, 3 \rangle$

Numerical coding of pairs

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So

$$\begin{array}{l} \boxed{0b\langle\langle x, y \rangle\rangle} = \boxed{0by} \boxed{1} \boxed{0 \dots 0} \\ \boxed{0b\langle x, y \rangle} = \boxed{0by} \boxed{0} \boxed{1 \dots 1} \end{array}$$

$\langle -, - \rangle$ gives a bijection (one-one correspondence) between $\mathbb{N} \times \mathbb{N}$ and \mathbb{N} .

$\langle\langle -, - \rangle\rangle$ gives a bijection between $\mathbb{N} \times \mathbb{N}$ and $\{n \in \mathbb{N} \mid n \neq 0\}$.

Numerical coding of lists

list $\mathbb{N} \triangleq$ set of all finite lists of natural numbers, using ML notation for lists:

- ▶ empty list: $[]$
- ▶ list-cons: $x :: \ell \in \textit{list } \mathbb{N}$ (given $x \in \mathbb{N}$ and $\ell \in \textit{list } \mathbb{N}$)
- ▶ $[x_1, x_2, \dots, x_n] \triangleq x_1 :: (x_2 :: (\dots x_n :: [] \dots))$

Numerical coding of lists

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For $l \in list\ \mathbb{N}$, define $\lceil l \rceil \in \mathbb{N}$ by induction on the length of the list l :

$$\begin{cases} \lceil [] \rceil \triangleq 0 \\ \lceil x :: l \rceil \triangleq \langle\langle x, \lceil l \rceil \rangle\rangle = 2^x(2 \cdot \lceil l \rceil + 1) \end{cases}$$

Thus $\lceil [x_1, x_2, \dots, x_n] \rceil = \langle\langle x_1, \langle\langle x_2, \dots \langle\langle x_n, 0 \rangle\rangle \dots \rangle\rangle \rangle$

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For example:

$$\lceil [3] \rceil = \lceil 3 :: [] \rceil = \langle\langle 3, 0 \rangle\rangle = 2^3(2 \cdot 0 + 1) = 8 = \text{0b1000}$$

$$\lceil [1, 3] \rceil = \langle\langle 1, \lceil [3] \rceil \rangle\rangle = \langle\langle 1, 8 \rangle\rangle = 34 = \text{0b100010}$$

$$\lceil [2, 1, 3] \rceil = \langle\langle 2, \lceil [1, 3] \rceil \rangle\rangle = \langle\langle 2, 34 \rangle\rangle = 276 = \text{0b100010100}$$

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$$0b\lceil [x_1, x_2, \dots, x_n] \rceil = \boxed{1} \boxed{0 \dots 0} \boxed{1} \boxed{0 \dots 0} \dots \boxed{1} \boxed{0 \dots 0}$$

x_n 0s x_{n-1} 0s x_1 0s

Numerical coding of lists

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For $l \in \mathit{list} \mathbb{N}$, define $\lceil l \rceil \in \mathbb{N}$ by induction on the length of the list l :

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$$\text{0b}\lceil [x_1, x_2, \dots, x_n] \rceil = \boxed{1} \boxed{0 \dots 0} \boxed{1} \boxed{0 \dots 0} \dots \boxed{1} \boxed{0 \dots 0}$$

Hence $l \mapsto \lceil l \rceil$ gives a bijection from $\mathit{list} \mathbb{N}$ to \mathbb{N} .

Numerical coding of programs

If P is the RM program

$$\begin{array}{l} L_0 : \mathit{body}_0 \\ L_1 : \mathit{body}_1 \\ \vdots \\ L_n : \mathit{body}_n \end{array}$$

then its numerical code is

$$\lceil P \rceil \triangleq \lceil \lceil \mathit{body}_0 \rceil, \dots, \lceil \mathit{body}_n \rceil \rceil$$

where the numerical code $\lceil \mathit{body} \rceil$ of an instruction body is

$$\text{defined by: } \begin{cases} \lceil R_i^+ \rightarrow L_j \rceil \triangleq \langle\langle 2i, j \rangle\rangle \\ \lceil R_i^- \rightarrow L_j, L_k \rceil \triangleq \langle\langle 2i + 1, \langle j, k \rangle \rangle\rangle \\ \lceil \text{HALT} \rceil \triangleq 0 \end{cases}$$

Any $x \in \mathbb{N}$ decodes to a unique instruction $body(x)$:

if $x = 0$ then $body(x)$ is HALT,

else ($x > 0$ and) let $x = \langle\langle y, z \rangle\rangle$ in

if $y = 2i$ is even, then

$body(x)$ is $R_i^+ \rightarrow L_z$,

else $y = 2i + 1$ is odd, let $z = \langle j, k \rangle$ in

$body(x)$ is $R_i^- \rightarrow L_j, L_k$

So any $e \in \mathbb{N}$ decodes to a unique program $prog(e)$,
called the register machine **program with index e** :

$prog(e) \triangleq \begin{array}{|l} L_0 : body(x_0) \\ \vdots \\ L_n : body(x_n) \end{array}$ where $e = \ulcorner [x_0, \dots, x_n] \urcorner$

Example of $prog(e)$

- ▶ $786432 = 2^{19} + 2^{18} = 0b110\underbrace{\dots 0}_{18 \text{ "0"s}} = \lceil [18, 0] \rceil$
- ▶ $18 = 0b10010 = \langle\langle 1, 4 \rangle\rangle = \langle\langle 1, \langle 0, 2 \rangle \rangle\rangle = \lceil R_0^- \rightarrow L_0, L_2 \rceil$
- ▶ $0 = \lceil \text{HALT} \rceil$

So $prog(786432) =$

$L_0 : R_0^- \rightarrow L_0, L_2$
$L_1 : \text{HALT}$

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N.B. jump to label with no body (erroneous halt)

What function is computed by a RM with $prog(786432)$ as its program?

$$666 = 0b1010011010$$

$$= \lceil [1, 1, 0, 2, 1] \rceil$$

prog(666) =

$$L_0 : R_0^+ \rightarrow L_0$$

$$L_1 : R_0^+ \rightarrow L_0$$

$$L_2 : \text{HALT}$$

$$L_3 : R_0^- \rightarrow L_0, L_0$$

$$L_4 : R_0^+ \rightarrow L_0$$

(never halts!)

What partial function does this compute?

Example of $prog(e)$

- ▶ $786432 = 2^{19} + 2^{18} = 0b110\underbrace{\dots 0}_{18 \text{ "0"s}} = \lceil [18, 0] \rceil$
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- ▶ $0 = \lceil \text{HALT} \rceil$

So $prog(786432) =$

$L_0 : R_0^- \rightarrow L_0, L_2$
$L_1 : \text{HALT}$

N.B. In case $e = 0$ we have $0 = \lceil [] \rceil$, so $prog(0)$ is the program with an empty list of instructions, which by convention we regard as a RM that does nothing (i.e. that halts immediately).

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So that

$\cdot \xrightarrow{\text{decode}} \cdot \xrightarrow{\text{run}} \cdot$

is RM computable

Universal register machine, *U*

High-level specification

Universal RM U carries out the following computation, starting with $R_0 = \mathbf{0}$, $R_1 = e$ (code of a program), $R_2 = a$ (code of a list of arguments) and all other registers zeroed:

- ▶ decode e as a RM program P
- ▶ decode a as a list of register values a_1, \dots, a_n
- ▶ carry out the computation of the RM program P starting with $R_0 = \mathbf{0}, R_1 = a_1, \dots, R_n = a_n$ (and any other registers occurring in P set to $\mathbf{0}$).

Mnemonics for the registers of **U** and the role they play in its program:

$R_1 \equiv P$ code of the RM to be simulated

$R_2 \equiv A$ code of current register contents of simulated RM

$R_3 \equiv PC$ program counter—number of the current instruction (counting from **0**)

$R_4 \equiv N$ code of the current instruction body

$R_5 \equiv C$ type of the current instruction body

$R_6 \equiv R$ current value of the register to be incremented or decremented by current instruction (if not **HALT**)

$R_7 \equiv S$, $R_8 \equiv T$ and $R_9 \equiv Z$ are auxiliary registers.

Overall structure of U 's program

1 copy PC th item of list in P to N (~~halting if $PC >$ length of list~~); goto 2

2 if $N = 0$ then copy 0th item of list in A to R_0 and halt, else (decode N as $\langle\langle y, z \rangle\rangle$; $C ::= y$; $N ::= z$; goto 3)

{at this point either $C = 2i$ is even and current instruction is $R_i^+ \rightarrow L_z$, or $C = 2i + 1$ is odd and current instruction is $R_i^- \rightarrow L_j, L_k$ where $z = \langle j, k \rangle$ }

3 copy i th item of list in A to R ; goto 4

4 execute current instruction on R ; update PC to next label; restore register values to A ; goto 1

Overall structure of U 's program

1 copy PC th item of list in P to N (~~halting if $PC > \text{length of list}$~~); goto 2

2 if $N = 0$ then copy 0th item of list in A to R_0 and halt, else (decode N as $\langle\langle y, z \rangle\rangle$; $C ::= y$; $N ::= z$; goto 3)

{at this point either $C = 2i$ is even and current instruction is $R_i^+ \rightarrow L_z$, or $C = 2i + 1$ is odd and current instruction is $R_i^- \rightarrow L_j, L_k$ where $z = \langle j, k \rangle$ }

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To implement this, we need RMs for manipulating (codes of) lists of numbers...