$$
\begin{array}{c}
b_1b_2b_3b_4b_5b_6b_7 \\
\bar{b_1}b_2b_3b_4b_5b_6b_7 \\
b_1\bar{b_2}b_3b_4b_5b_6b_7 \\
b_1b_2\bar{b_3}b_4b_5b_6b_7 \\
b_1b_2b_3\bar{b_4}b_5b_6b_7 \\
b_1b_2b_3b_4\bar{b_5}b_6b_7 \\
b_1b_2b_3b_4b_5\bar{b_6}b_7 \\
b_1b_2b_3b_4b_5b_6\bar{b_7}
\end{array}
$$

Then considering the information capacity per symbol:

$$
\begin{aligned}
C &= max(H(\mathcal{Y}) - H(\mathcal{Y}|\mathcal{X})) \\
&= \frac{1}{7}\left(7 - \sum_j \sum_k p(y_k|x_j)\log\left(\frac{1}{p(y_k|x_j)}\right)p(x_j)\right) \\
&= \frac{1}{7}\left(7 + \sum_j 8(\frac{1}{8}\log\frac{1}{8})\frac{1}{N}\right) \\
&= \frac{1}{7}\left(7 + N(\frac{8}{8}\log\frac{1}{8})\frac{1}{N}\right) \\
&= \frac{4}{7}
\end{aligned}
$$

The capacity of the channel is 4/7 information bits per binary digit of the channel coding. Can we find a mechanism to encode 4 information bits in 7 channel bits subject to the error property described above?

The (7/4) Hamming Code provides a *systematic* code to perform this – a systematic code is one in which the obvious binary encoding of the source symbols is present in the channel encoded form. For our source which emits at each time interval 1 of 16 symbols, we take the binary representation of this and copy it to bits b_3, b_5, b_6 and b_7 of the encoded block; the remaining bits are given by b_4, b_2, b_1, and *syndromes* by s_4, s_2, s_1:

$$
\begin{aligned}
b_4 &= b_5 \oplus b_6 \oplus b_7 \text{ and,} \\
s_4 &= b_4 \oplus b_5 \oplus b_6 \oplus b_7 \\
b_2 &= b_3 \oplus b_6 \oplus b_7 \text{ and,} \\
s_2 &= b_2 \oplus b_3 \oplus b_6 \oplus b_7 \\
b_1 &= b_3 \oplus b_5 \oplus b_7 \text{ and,} \\
s_1 &= b_1 \oplus b_3 \oplus b_5 \oplus b_7
\end{aligned}
$$

On reception if the binary number $s_4s_2s_1 = 0$ then there is no error, else $b_{s_4s_2s_1}$ is the bit in error.

This Hamming code uses 3 bits to correct 7 ($= 2^3 - 1$) error patterns and transfer 4 useful bits. In general a Hamming code uses m bits to correct $2^m - 1$ error patterns and transfer $2^m - 1 - m$ useful bits. The Hamming codes are called *perfect* as they use m bits to correct $2^m - 1$ errors.