Proof of the Kraft-McMillan Inequality

26th October 2001

Peter J. Taylor
Andrew D. Rogers

Consider a set of codewords $C_1, C_2, \ldots, C_N$ of lengths $n_1, n_2, \ldots, n_N$, such that:

$$n_1 \leq n_2 \leq \ldots \leq n_N$$

Now consider the finite binary tree representing these codes, $\mathrm{T_C}$. Some of the nodes are labelled as codewords. We have the restriction that the subtree rooted at a codeword contains only that one codeword.

Tripartition the nodes of the tree into codewords, prefixes and NCPs (Neither Codeword nor Prefix). If the subtree rooted at node X contains at least one codeword, and X is not a codeword, then X is a prefix. If the subtree rooted at X contains no codewords at all, X is an NCP.

Now define:

$$d_{i,\mathrm{T}} \stackrel{\mathrm{def}}{=} \begin{cases} \text{depth of node } C_i \text{ in tree T} & C_i \in \mathrm{T} \\ 0 & C_i \notin \mathrm{T} \end{cases}$$

Define the cost, C(T), of the tree T as:

$$\mathrm{C(T)} \stackrel{\mathrm{def}}{=} \sum_{i \in \{j \in \mathbb{N} \mid C_j \in \mathrm{T}\}} \frac{1}{2^{d_{i,\mathrm{T}}}}$$

If the root of T is a codeword, C(T) = 1, by definitions of codeword and C.

If the root of T is an NCP, C(T) = 0, by definitions of NCP and C.

If the root of T is a prefix, and the subtrees are T_1 and T_2, then:

$$\mathrm{C(T)} = (\mathrm{C(T_1)} + \mathrm{C(T_2)}) \ / \ 2$$

(as $d_{i,\mathrm{T}} = d_{i,\mathrm{T}_1} + 1$ for $i \in \{j \in \mathbb{N} \,|\, C_j \in \mathrm{T}_1\}$, $d_{i,\mathrm{T}} = d_{i,\mathrm{T}_2} + 1$ for $i \in \{k \in \mathbb{N} \,|\, C_k \in \mathrm{T}_2\}$ and $(C_i \in \mathrm{T}) \Rightarrow (C_i \in \mathrm{T}_1) \oplus (C_i \in \mathrm{T}_2)$).

Then by structural induction on a finite tree T, $\mathrm{C(T)} \leq 1$.

Case 1: The root of T is a codeword. Then C(T) = 1
Case 2: The root of T is an NCP. Then C(T) = 0
Case 3: The root of T is a prefix, and the subtrees are T_1 and T_2. By the inductive hypothesis, $\mathrm{C(T_1)} \leq 1$ and $\mathrm{C(T_2)} \leq 1$. Therefore $\mathrm{C(T)} = (\mathrm{C(T_1)} + \mathrm{C(T_2)}) \ / \ 2 \leq 1$.

□

Therefore:

$$\mathrm{C(T)} = \sum_{i \in \{j \in \mathbb{N} \mid C_j \in \mathrm{T}\}} \frac{1}{2^{d_{i,\mathrm{T}}}} \leq 1$$

But for $\mathrm{T_C}$, $\{j \in \mathbb{N} \,|\, C_j \in \ \mathrm{T_C}\} = \{1, 2, \ldots, \mathrm{N}\}$ and $d_{i,\mathrm{T_C}} = n_i$, so:

$$\sum_{1 \leq i \leq N} \frac{1}{2^{n_i}} = \mathrm{C(T_C)} \leq 1$$

$$\sum_i \frac{1}{2^{n_i}} \leq 1$$

□