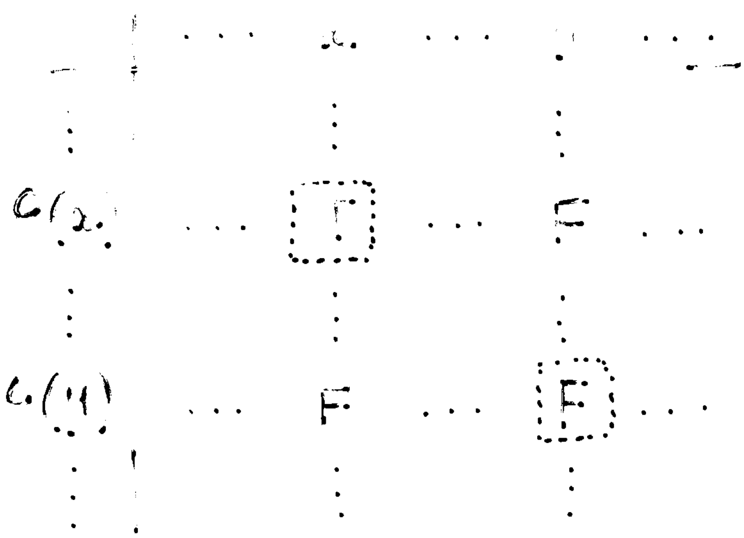
Cantor's diagonal argument revisited

Let X be a set. There is no bijection $\theta : X \to \mathcal{P}(X)$.

Proof By contradiction. Suppose
$$\theta : X \to \mathcal{P}(X)$$
is a bijection.

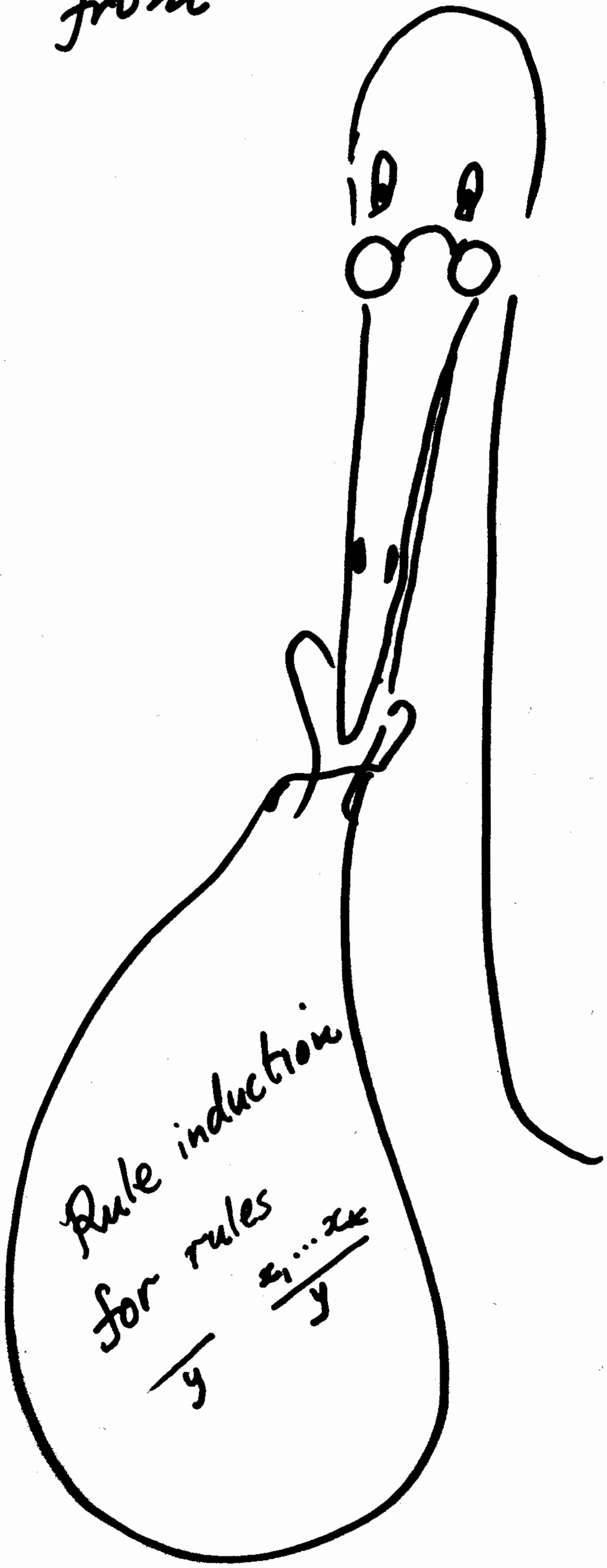
Define
$$Y = \{ x \in X \mid x \notin \theta(x) \}.$$



As θ is a bijection, there is $y \in X$ such that $\theta(y) = Y$. But ...

Ch. 4 Inductive definitions

Where induction principles come from



Boolean propositions from rules

$$A, B, \dots ::= a, b, c, \dots \mid true \mid false \mid A \wedge B \mid A \vee B \mid \neg A$$

$$a, b, c, \dots \in Var$$

$$\frac{}{a}\ a \in Var \qquad \frac{}{true} \qquad \frac{}{false}$$

$$\frac{A \qquad B}{A \wedge B} \qquad \frac{A \qquad B}{A \vee B} \qquad \frac{A}{\neg A}$$

A derivation:

$$\dfrac{\dfrac{\dfrac{}{a}}{\neg a} \qquad \dfrac{\dfrac{}{b} \qquad \dfrac{}{true}}{b \vee true}}{\neg a \wedge (b \vee true)}$$

Non-negative integers $\mathbb{N}_0$ from rules

- $0 \in \mathbb{N}_0$
- If $n \in \mathbb{N}_0$, then $n+1 \in \mathbb{N}_0$

$$\frac{}{0} \qquad\qquad \frac{n}{n+1}$$

$\mathbb{N}_0$ is the least set closed under the rules.

Strings Σ^*

Σ is a set of symbols, the alphabet

empty string $\varepsilon \in \Sigma^*$

concatenation If $x \in \Sigma^*$ and $a \in \Sigma$,
then $ax \in \Sigma^*$

$$\frac{}{\varepsilon} \qquad \frac{x}{a\,x} \; a \in \Sigma$$

An instance of a rule:

$$\frac{x_1,\ x_2,\ \cdots,\ x_i,\ \cdots}{y}$$

Premise

Conclusion

a pair (X/y) where

$$X = \{x_1, x_2, \cdots, x_i, \cdots\}.$$

When X is finite, the rule is finitary.

NB. Can have $X = \emptyset$.

R collection of rules

A set Q is R-closed iff

$\forall (X/y) \in R.\ X \subseteq Q \Rightarrow y \in Q$

Define set inductively defined by R

$$I_R = \bigcap \{ Q \mid Q \text{ is } R\text{-closed} \}$$

need non-empty i.e. R is bounded

Proposition 4.3 P.55

(i) I_R is R-closed

(ii) Q is R-closed $\Rightarrow$ $I_R \subseteq Q$.

Rule induction:

$\forall x \in I_R.\ P(x)$ if

for all rules $(X/y) \in R$ s.t. $X \subseteq I_R$

$(\forall x \in X.\ P(x)) \Rightarrow P(y)$.

Transitive closure of a relation

Let $R \subseteq U \times U$.

Its transitive closure $R^+ \subseteq U \times U$ is given by:

$$\frac{}{(a,b)}\ (a,b) \in R \qquad\qquad \frac{(a,b) \quad (b,c)}{(a,c)}$$

$$R^+ = \left\{ (a,b) \in U \times U \;\middle|\; \begin{array}{l} \text{there is an } R\text{-chain} \\ \text{from } a \text{ to } b. \end{array} \right\}$$

$$a = a_1 \, R \, a_2 \, R \, a_3 \cdots R \, a_n = b$$