Lemma 7 (Substitution) *If* $\Gamma \vdash e{:}T$ *and* $\Gamma, x{:}T \vdash e'{:}T'$ *with* $x \notin dom(\Gamma)$ *then* $\Gamma \vdash \{e/x\}e'{:}T'$.

Proof Rule induction on derivations of $\Gamma, x{:}T \vdash e'{:}T'$.

Take $\Phi(\Gamma', e', T') = \forall \Gamma, e, T, x.((\Gamma \vdash e{:}T) \wedge (x \notin \mathrm{dom}(\Gamma)) \wedge (\Gamma' = \Gamma, x{:}T)) \Rightarrow \Gamma \vdash \{e/x\}e'{:}T'$.

(so the conclusion of the principle of rule induction, ie $\forall \Gamma', e', T'.\ \ \Gamma' \vdash e'{:}T' \Rightarrow \Phi(\Gamma', e', T')$, implies our Lemma)

- Base case: type rule (var):

  $$\text{(var)} \quad \Gamma \vdash x{:}T \quad \text{if } \Gamma(x) = T$$

  This has no premises, so we have to show simply $\Phi(\Gamma', e', T')$ for all concrete instances of the rule.

  (looking at the structure of Φ) Take arbitrary Γ, e, T, x.

  Assume $(\Gamma \vdash e{:}T) \wedge (x \notin \mathrm{dom}(\Gamma)) \wedge (\Gamma' = \Gamma, x{:}T)$.

  Now we have to show $\Gamma \vdash \{e/x\}e'{:}T'$.

  There are two kinds of instance of the rule, one where $e' = x$ and one where $e' = y$ for some variable $y \neq x$.

  - case $e' = x$.

    $$\text{(var)} \quad \Gamma' \vdash x{:}T' \quad \text{if } \Gamma'(x) = T'$$

    Here must have $T' = T$, by the third conjunct of the assumption. Now by the definition of substitution $\{e/x\}x = e$, so we have to show $\Gamma \vdash e{:}T$ — but we know that from the first conjunct of the assumption.

  - case $e' = y$ for some variable $y \neq x$.

    $$\text{(var)} \quad \Gamma' \vdash y{:}T' \quad \text{if } \Gamma'(y) = T'$$

    Now by the definition of substitution $\{e/x\}y = y$, so we have to show $\Gamma \vdash y{:}T'$. That is simply an instance of (var), as we know $\Gamma'(y) = T'$ and $\Gamma' = \Gamma, x{:}T$ so $\Gamma(y) = T'$.

- Inductive case: type rule (app):

  $$\text{(app)} \quad \frac{\Gamma \vdash e_1{:}T \to T' \qquad \Gamma \vdash e_2{:}T}{\Gamma \vdash e_1\ e_2{:}T'}$$

  This has two premises, so we have to show

  $\Phi(\Gamma', e', T') \wedge \Phi(\Gamma'', e'', T'') \Rightarrow \Phi(\Gamma''', e''', T''')$

  for all concrete instances of the rule with premises $\Gamma' \vdash e'{:}T'$ and $\Gamma'' \vdash e''{:}T''$ and conclusion $\Gamma''' \vdash e'''{:}T'''$.

  Take an arbitrary instance of the rule, ie Γ, e_1, T, T', e_2. Assume $\Phi(\Gamma, e_1, T \to T')$ and $\Phi(\Gamma, e_2, T)$ (*).

  We have to show $\Phi(\Gamma, e_1\ e_2, T')$, ie (alpha-renaming Φ)

  $\forall \Gamma', e', T', x'.((\Gamma' \vdash e'{:}T') \wedge (x' \notin \mathrm{dom}(\Gamma')) \wedge (\Gamma = \Gamma', x'{:}T')) \Rightarrow \Gamma' \vdash \{e'/x'\}(e_1\ e_2){:}T'$

  Take arbitrary Γ', e', T', x' and assume $(\Gamma' \vdash e'{:}T') \wedge (x' \notin \mathrm{dom}(\Gamma')) \wedge (\Gamma = \Gamma', x'{:}T')$. Just substituting $\Gamma = \Gamma', x'{:}T'$ into the rule that we're looking at, for intuition:

  $$\text{(app)} \quad \frac{\Gamma', x'{:}T' \vdash e_1{:}T \to T' \qquad \Gamma', x'{:}T' \vdash e_2{:}T}{\Gamma', x'{:}T' \vdash e_1\ e_2{:}T'}$$

  We have to show $\Gamma' \vdash \{e'/x'\}(e_1\ e_2){:}T'$.

  By the definition of substitution $\{e'/x'\}(e_1\ e_2) = \{e'/x'\}e_1\ \{e'/x'\}e_2$.

  Hence, we have to show $\Gamma' \vdash \{e'/x'\}e_1\ \{e'/x'\}e_2{:}T'$.

  By the induction hypothesis (*) (checking ...) we have $\Gamma' \vdash \{e'/x'\}e_1{:}T \to T'$ and $\Gamma' \vdash \{e'/x'\}e_2{:}T$, so by (app) we have $\Gamma' \vdash \{e'/x'\}e_1\ \{e'/x'\}e_2{:}T'$ as required.

- Inductive case: type rule (fn):

$$\text{(fn)} \quad \frac{\Gamma, x{:}T \vdash e{:}T'}{\Gamma \vdash \textbf{fn } x{:}T \Rightarrow e : T \rightarrow T'}$$

...????....