Quantum Computing: Exercise Sheet 1

Anuj Dawar and Ben Travaglione

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Bits and Qubits

1. Which of the following are possible states of a qubit?

   (a) $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

   (b) $\frac{\sqrt{3}}{2}|1\rangle - \frac{1}{2}|0\rangle$

   (c) $0.7|0\rangle + 0.3|1\rangle$

   (d) $0.8|0\rangle + 0.6|1\rangle$

   (e) $\cos\theta|0\rangle + i\sin\theta|1\rangle$

   (f) $\cos^2\theta|0\rangle - \sin^2\theta|1\rangle$

   For each valid state among the above, give the probabilities of observing $|0\rangle$ and $|1\rangle$ when the system is measured in the standard computational basis.

   What are the probabilities of the two outcomes when the state is measured in the basis $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$?

2. A two-qubit system is in the following state

   $$\frac{1}{\sqrt{30}}(|00\rangle + 2i|01\rangle - 3|10\rangle - 4i|11\rangle)$$

   The first qubit is measured and observed to be 1. What is the state of the system after the measurement? What is the probability that a subsequent measurement of the second qubit will observe a 1?

Linear Algebra

3. Show that the operations of addition and scalar multiplication defined on slide 4 satisfy the axioms of a vector space given on slide 3.

4. **Uniqueness of dimension**. This exercise asks you to prove the assertion on slide 5, namely that the size of the basis is uniquely determined by the vector space V. Suppose that $|v_1\rangle, \ldots, |v_n\rangle$ is a basis for V. Let $|u_1\rangle, \ldots, |u_{n+1}\rangle$ be any collection of $n+1$ vectors. Show that they cannot all be linearly independent, i.e. one of them must be expressible as a linear combination of the others.

5. For each of the two example linear operators on slide 9, express them in the basis $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

6. Verify that the inner product on $\mathbb{C}^n$ defined on slide 11 satisfies the axioms on slide 10.

7. Express the two linear operators on slide 9 as a linear combination of outer products.

8. Find the eigenvalues and associated eigenvectors of the two linear operators on slide 9.

9. For each of the four matrices on slide 7 of Lecture 4, show that they are unitary.

10. If I is the 2-dimensional identity matrix and H is the Hadamard operator (Lecture 4, slide 7), give matrix representations of the operators $I \otimes H$ and $H \otimes I$. Work out, by multiplying these matrices with the corresponding vectors the eight vectors

$$(I \otimes H)|00\rangle, (I \otimes H)|01\rangle, (I \otimes H)|10\rangle, (I \otimes H)|11\rangle, \text{and}$$

$$(H \otimes I)|00\rangle, (H \otimes I)|01\rangle, (H \otimes I)|10\rangle, (H \otimes I)|11\rangle$$

Quantum Mechanics

11. (The exercise on slide 12) Prove that unitary evolution preserves normalisation.

12. (Exercises on slide 14) Prove that $XY = iZ$. Show that $X^2 = Y^2 = Z^2 = I$.

13. (Exercise on slide 28) Suppose a two-qubit system is in the state $0.8|00\rangle + 0.6|11\rangle$. A NOT gate is applied to the second qubit and a measurement performed in the computational basis. What are the probabalities of the possible measurement outcomes?

Quantum Circuits

14. The exercise on slide 6 asks you to descibe a unitary operation that implements the Boolean *And* operation. In fact, the unitary operation is not uniquely determined by the description given on that slide. Describe *two* different unitary operations that satisfy the definition of the Boolean And in the sense of slide 6.

15. Describe, in matrix form, the action of the controlled-H gate, where H is the Hadamard gate.

16. Show how the controlled-Not gate can be constucted from Hadamard gates and the controlled-Z. Demonstrate that the construction is correct by multiplying the corresponding matrices.

17. Write the Toffolli gate in matrix form.

18. The circuit on slide 13 is formed as the composition of 13 unitary operations, each of them obtained as the tensor product of one and two-qubit operations. Write each of these operators in matrix form and verify that their product is the same as your answer to Question 17.

19. **No-cloning theorem**. The no-cloning theorem states that a quantum state $|\psi\rangle$ cannot be duplicated. That is, there is no quantum operation that will map an arbitrary state $|\psi 0\rangle$ to $|\psi\psi\rangle$. Show this by considering any unitary map U that maps $|00\rangle$ to $|00\rangle$ and $|10\rangle$ to $|11\rangle$ and exhibiting a state $|\psi\rangle$ for which $U|\psi 0\rangle \neq |\psi\psi\rangle$.

20. **Deutsch-Josza Algorithm**. This exercise is aimed at proving the correctness of the circuit described on slide 16. Consider the operator U_f on slide 15 and show that:

    (a) if $|y\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$, then $U_f|xy\rangle = (-1)^{f(x)}|xy\rangle$;

    (b) if, in addition, $|x\rangle = H|0\rangle$, then $U_f|xy\rangle = ((-1)^{f(0)}|0\rangle + (-1)^{f(1)}|1\rangle)|y\rangle$;

    (c) the final output of the circuit is $(H \otimes I)U_f|xy\rangle$ where $|x\rangle$ and $|y\rangle$ are as above. Show that this is (up to global phase and normalisation) $|0y\rangle$ if f is constant and $|1y\rangle$ if f is balanced.