Slide 1

Semantics of Programming Languages

Peter Sewell

1B, 12 lectures

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This contains the notes and slides for lectures 1–8 and 10–12. Lecture 9 was a guest lecture by Prof. Greg Morrisett (Cornell University) on Typed Assembly Language; there is a separate handout with his slides.

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Syllabus

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No. of lectures: 12

Aims

The aim of this course is to introduce the structural, operational approach to programming language semantics. It will show how to specify the meaning of typical programming language constructs, in the context of language design, and how to reason formally about semantic properties of programs.

Lectures

- **Introduction.** Transition systems. The idea of structural operational semantics. Transition semantics of a simple imperative language. Language design options. [2 lectures]
- **Types.** Introduction to formal type systems. Typing for the simple imperative language. Statements of desirable properties. [1 lecture]
- **Induction.** Review of mathematical induction. Abstract syntax trees and structural induction. Rule-based inductive definitions and proofs. Proofs of type safety properties. [2 lectures]
- **Functions.** Call-by-name and call-by-value function application, semantics and typing. Local recursive definitions. [2 lectures]
- **Data.** Semantics and typing for products, sums, records, references. [1 lecture]
- **Subtyping.** Record subtyping and simple object encoding. [1 lecture]
- **Low-level Semantics.** Monomorphic typed assembly language. [1 lecture]
- **Semantic equivalence.** Semantic equivalence of phrases in a simple imperative language, including the congruence property. Examples of equivalence and non-equivalence. [1.5 lectures]
- **Concurrency.** Shared variable interleaving. [0.5 lectures]

Objectives

At the end of the course students should

- be familiar with rule-based presentations of the operational semantics and type systems for some simple imperative, functional and interactive program constructs
- be able to prove properties of an operational semantics using various forms of induction (mathematical, structural, and rule-based)
- be familiar with some operationally-based notions of semantic equivalence of program phrases and their basic properties

Recommended books

Hennessy, M. (1990). *The Semantics of Programming Languages.* Wiley.
Pierce, B. C. (2002) *Types and Programming Languages.* MIT Press
Winskel, G. (1993). *The Formal Semantics of Programming Languages.* MIT Press.

Books:

- Hennessy, M. (1990). *The Semantics of Programming Languages.* Wiley. Out of print, but available on the web at `http://www.cogs.susx.ac.uk/users/matthewh/semnotes.ps.gz`.

  Introduces many of the key topics of the course.

- Pierce, B. C. (2002) *Types and Programming Languages.* MIT Press.

  This is a graduate-level text, covering a great deal of material on programming language semantics. The first half (through to Chapter 15) is relevant to this course, and some of the later material relevant to the Part II Types course.

- Winskel, G. (1993). *The Formal Semantics of Programming Languages.* MIT Press.

  An introduction to both operational and denotational semantics; recommended for the Part II Denotational Semantics course.

Further reading:

- Plotkin, G. D.(1981). A structural approach to operational semantics. Technical Report DAIMI FN-19, Aarhus University.

  These notes first popularised the 'structural' approach to operational semantics—the approach emphasised in this course—but couched solely in terms of transition relations ('small-step' semantics), rather than evaluation relations ('big-step', 'natural', or 'relational' semantics). Although somewhat dated and hard to get hold of (the Computer Laboratory Library has a copy), they are still a mine of interesting examples.

- The two essays:
  Hoare, C. A. R.. Algebra and Models.
  Milner, R. Semantic Ideas in Computing.
  In: Wand, I. and R. Milner (Eds) (1996). *Computing Tomorrow.* CUP.

  Two accessible essays giving somewhat different perspectives on the semantics of computation and programming languages.

Tripos questions: This is a new course, substantially changed from the Semantics courses of previous years, so most old Tripos questions assume a different presentation (in particular, most use 'big-step' semantics). Two that are directly relevant are 1996 Paper 5 Question 12 and 1997 Paper 5 Question 12. Some more sample questions may be put up on the web page during the term.

Exercises and Implementations: Implementations of some of the languages are available on the course web page, accessible via `http://www.cl.cam.ac.uk/UoCCL/teaching/current.html`.

They are written in Moscow ML. This is installed on the Intel Lab machines. If you want to work with them on your own machine instead, there are Linux, Windows, and Mac versions of Moscow ML available at `http://www.dina.dk/~sestoft/mosml.html`.

The notes contain various exercises, some related to the implementations. Those marked ★ should be straightforward checks that you are grasping the material; I suggest you attempt all of these. Exercises marked ★★ may need a little more thought – both proofs and some implementation-related; you should do some of each. Exercises marked ★★★ may need material beyond the notes, and/or be quite time-consuming.

Feedback: Please do complete the on-line feedback form at the end of the course, and let me know during it if you discover errors in the notes or if the pace is too fast or slow. A list of corrections will be on the course web page.

Acknowledgements: These notes draw, with thanks, on earlier courses by Andrew Pitts, on Benjamin Pierce's book, and many other sources. Any errors are, of course, newly introduced by me.

Summary of Notation

Each section is roughly in the order that notation is introduced. The grammars of the languages are not included here, but are in the Collected Definitions of L1, L2 and L3 later in this document.

Finite Partial Functions

$\{a_1 \mapsto b_1, ..., a_n \mapsto b_n\}$	finite partial function mapping each a_i to b_i
$\mathrm{dom}(s)$	set of elements in the domain of s
$f + \{a \mapsto b\}$	the finite partial function f extended or overridden with a maps to b
$\Gamma, x{:}T$	the finite partial function Γ extended with $\{x \mapsto T\}$ – only used where x not in $\mathrm{dom}(\Gamma)$
Γ, Γ'	the finite partial function which is the union of Γ and Γ – only used where they have disjoint domains
$\{l_1 \mapsto n_1, ..., l_k \mapsto n_k\}$	an L1 or L2 store – the finite partial function mapping each l_i to n_i
$\{l_1 \mapsto v_1, ..., l_k \mapsto v_k\}$	an L3 store – the finite partial function mapping each l_i to v_i
$l_1{:}\mathsf{intref}, ..., l_k{:}\mathsf{intref}$	an L1 type environment – the finite partial function mapping each l_i to intref
$\ell{:}\mathsf{intref}, ..., x{:}T, ...$	an L2 type environment
$\ell{:}T_{loc}, ..., x{:}T, ...$	an L3 type environment
$\{e_1/x_1, .., e_k/x_k\}$	a substitution – the finite partial function $\{x_1 \mapsto e_1, ..., x_k \mapsto e_k\}$ mapping x_1 to e_1 etc.

Particular sets

$\mathbb{B} = \{\mathbf{true}, \mathbf{false}\}$	the set of booleans
$\mathbb{L} = \{l, l_1, l_2, ...\}$	the set of locations
$\mathbb{Z} = \{.., -1, 0, 1, ...\}$	the set of integers
$\mathbb{X} = \{\mathrm{x}, \mathrm{y}, ...\}$	the set of L2 and L3 variables
$\mathbb{LAB} = \{\mathrm{p}, \mathrm{q}, ...\}$	the set of record labels
$\mathbb{M} = \{\mathrm{m}, \mathrm{m}_0, \mathrm{m}_1, ...\}$	the set of mutex names
T	the set of all types (in whichever language)
$\mathrm{T}_{\mathrm{loc}}$	the set of all location types (in whichever language)
L_1	the set of all L1 expressions
$\mathrm{TypeEnv}$	the set of all L1 type environments, finite partial functions from $\mathbb{L}$ to $\mathbb{Z}$
$\mathrm{TypeEnv2}$	the set of all L2 type environments, the finite partial functions from $\mathbb{L} \cup \mathbb{X}$ to $\mathrm{T}_{\mathrm{loc}} \cup \mathrm{T}$ such that $\forall \ell \in \mathrm{dom}(\Gamma).\Gamma(\ell) \in \mathrm{T}_{\mathrm{loc}}$ and $\forall x \in \mathrm{dom}(\Gamma).\Gamma(x) \in \mathrm{T}$
$\mathbb{A}$	thread actions

Metavariables

$b \in \mathbb{B}$	boolean
$n \in \mathbb{Z}$	integer
$\ell \in \mathbb{L}$	location
op	binary operation
e, f	expression (of whichever language)
v	value (of whichever language)
s	store (of whichever language)
$T \in \mathrm{T}$	type (of whichever language)
$T_{loc} \in \mathrm{T}_{\mathrm{loc}}$	location type (of whichever language)
Γ	type environment (also, set of propositional assumptions)
i, k, y	natural numbers
c	configuration (or state), typically $\langle e, s \rangle$ with expression e and store s
Φ	formula
c	tree constructor
R	set of rules
(H, c)	a rule with hypotheses $H \subseteq A$ and conclusion $c \in A$ for some set A
S_R	a subset inductively defined by the set of rules R
$x \in \mathbb{X}$	variable
σ	substitution
$lab \in \mathbb{LAB}$	record label
E	evaluation context
C	arbitrary context
π	permutation of natural numbers
$m \in \mathbb{M}$	mutex name
M	state of all mutexes (a function $M{:}\mathbb{M} \longrightarrow \mathbb{B}$)
a	thread action, for $a \in \mathbb{A}$

Relations and auxiliary functions

$\langle e, s\rangle \longrightarrow \langle e', s'\rangle$	reduction (or transition) step
$\langle e, s\rangle \longrightarrow^* \langle e', s'\rangle$	reflexive transitive closure of $\longrightarrow$
$\langle e, s\rangle \longrightarrow^k \langle e', s'\rangle$	the k-fold composition of $\longrightarrow$
$\langle e, s\rangle \longrightarrow^\omega$	has an infinite reduction sequence (a unary predicate)
$\langle e, s\rangle \not\longrightarrow$	cannot reduce (a unary predicate)
$\Gamma \vdash e{:}T$	in type environment Γ, expression e has type T
$\mathrm{value}(e)$	e is a value
$\mathrm{fv}(e)$	the set of free variables of e
$\{e/x\}e'$	the expression resulting from substituting e for x in e'
$\sigma\ e$	the expression resulting from applying the substituting σ to e
$\langle e, s\rangle \Downarrow \langle v, s'\rangle$	big-step evaluation
$\Gamma \vdash s$	store s is well-typed with respect to type environment Γ
$T <: T'$	type T is a subtype of type T'
$e \simeq e'$	semantic equivalence (informal)
$e \simeq^T_\Gamma e'$	semantic equivalence at type T with respect to type environment Γ
$e \xrightarrow{a} e'$	single thread transition step, labelled with action a

Other

$\_$	hole in a context
$C[e]$	context C with e replacing the hole $\_$

Logic and Set Theory

$\Phi \wedge \Phi'$	and
$\Phi \vee \Phi'$	or
$\Phi \Rightarrow \Phi'$	implies
$\neg\Phi$	not
$\forall x.\Phi(x)$	for all
$\exists x.\Phi(x)$	exists
$a \in A$	element of
$\{a_1, ..., a_n\}$	the set with elements $a_1, ..., a_n$
$A_1 \cup A_2$	union
$A_1 \cap A_2$	intersection
$A_1 \subseteq A_2$	subset or equal

1 Introduction

In this course we will take a close look at *programming languages*. We will focus on how one can define precisely what a language is – how the programs of a language behave, or, more generally, what their *meaning*, or *semantics*, is.

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Semantics - What is it?

How to describe a programming language? Need to give:

- the *syntax* of programs; and
- their *semantics* (the meaning of programs, or how they behave).

Styles of description:

- the language is defined by whatever compiler Foo does
- natural language 'definitions'
- mathematically

Mathematical descriptions of syntax use formal grammars (eg BNF) – precise, concise, clear. In this course we'll see how to work with mathematical definitions of semantics/behaviour.

Slide 3

What do we use semantics for?

1. to understand a particular language - what you can depend on as a programmer; what you must provide as a compiler writer
2. as a tool for language design:
   (a) for expressing design choices, understanding language features and how they interact.
   (b) for proving properties of a language, eg type safety, decidability of type inference.
3. as a foundation for proving properties of particular programs

Here 1 is a complement to language implementation (cf. courses on Compiler Construction and Optimising Compilers). We need languages to be *both* clearly understandable, with precise definitions, *and* have good implementations; we don't want the definition of a language to be "whatever Compiler Foo does". Some optimising compilers use semantics explicitly, to know what optimisations are legal.

Point 2 is developed further in the Types course, which looks in particular at polymorphism.

Q: How many of you will do language design...? A: lots!

Point 3 is pursued in the Specification and Verification I and II courses. It's important especially for safety-critical systems, key parts of processors, etc.. Our focus in this course is on 1 and 2, though.

More broadly, while we will look mostly at semantics for conventional programming languages, similar techniques can be used for hardware description languages, verification of distributed algorithms, security protocols, and so on – all manner of subtle gizmos for which relying on informal intuition alone leads to bad systems. Some of these are explored in the Topics in Concurrency course.

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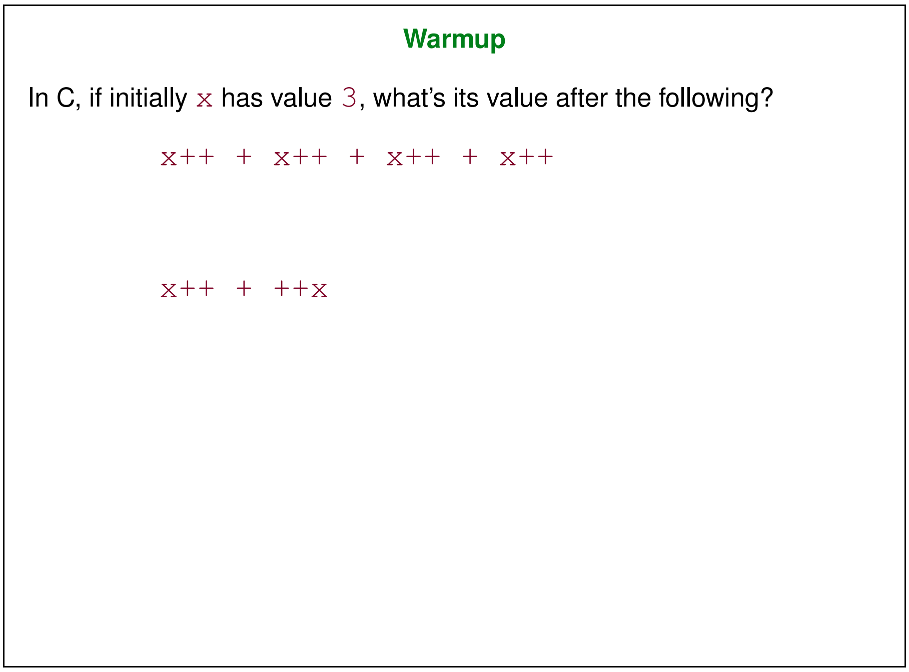

Warmup

In C, if initially `x` has value `3`, what's its value after the following?

```
x++ + x++ + x++ + x++
```

```
x++ + ++x
```

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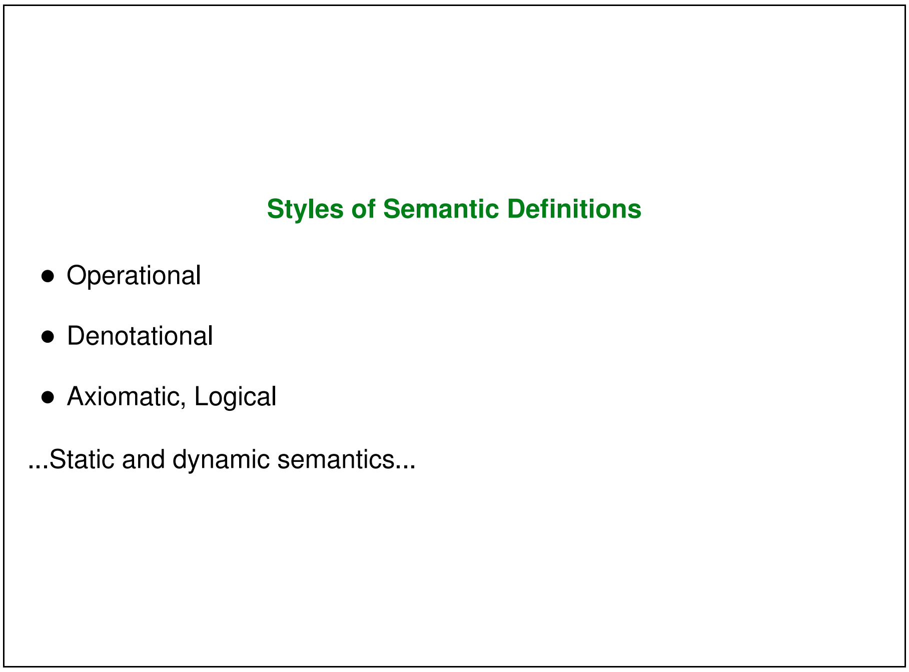

Styles of Semantic Definitions

- Operational
- Denotational
- Axiomatic, Logical

...Static and dynamic semantics...

Operational: define the meaning of a program in terms of the computation steps it takes in an idealised execution (this is *dynamic* semantics; in contrast the *static* semantics of a language describes its compile-time typechecking).

Denotational: define the meaning of a program as elements of some abstract mathematical structure, e.g. regarding programming-language functions as certain mathematical functions. cf. the Denotational Semantics course.

Axiomatic,Logical: define the meaning of a program indirectly, by giving the axioms of a logic of program properties. cf. Specification and Verification.

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What's this course?

Core

- operational semantics and typing for a tiny language
- technical tools (abstract syntax, inductive definitions, proof)
- design for functions, data and references

More advanced topics

- Subtyping
- Low-level Semantics (Typed Assembly Language)
- Semantic Equivalence
- Concurrency

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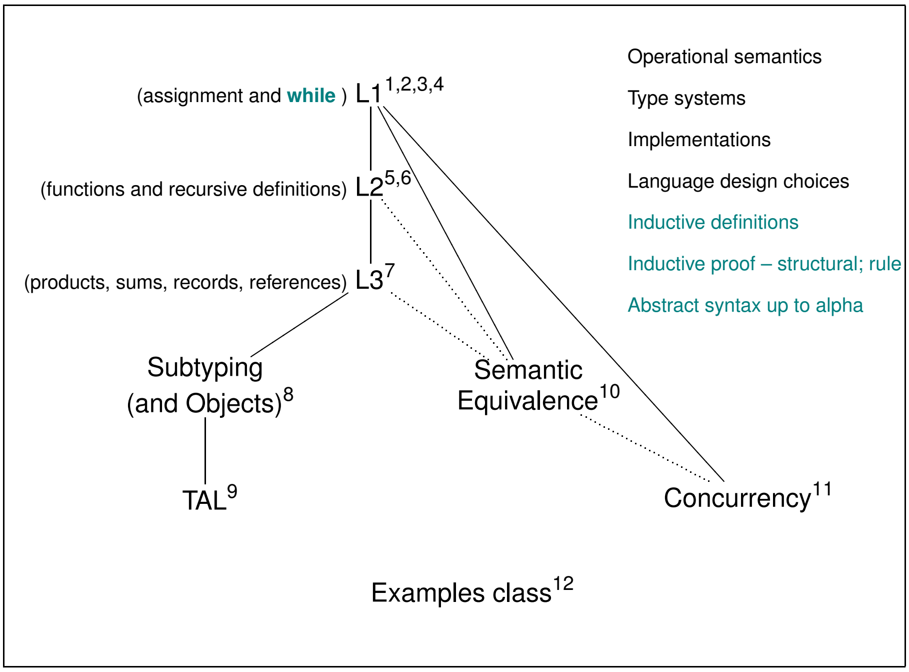

(assignment and **while**) $L1^{1,2,3,4}$

(functions and recursive definitions) $L2^{5,6}$

(products, sums, records, references) $L3^{7}$

Subtyping (and Objects)^{8}

TAL^{9}

Semantic Equivalence^{10}

Concurrency^{11}

Examples class^{12}

Operational semantics

Type systems

Implementations

Language design choices

Inductive definitions

Inductive proof – structural; rule

Abstract syntax up to alpha

In the core we will develop enough techniques to deal with the semantics of a non-trivial small language, showing some language-design pitfalls and alternatives along the way. It will end up with a decent fragment of ML. The second part will cover a selection of more advanced topics.

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Admin

- This is a new course, so:
  - ˘ please let me know of typos, and if it is too fast/too slow/too interesting/too dull (on-line feedback at the end)
  - ˘ not all previous Tripos questions are relevant (see the notes)
- Exercises in the notes.
- Implementations on web.
- Books

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‘Toy’ languages

Real programming languages are large, with many features and, often, with redundant constructs – things that can be expressed in the rest of the language.

When trying to understand some particular combination of features it’s usual to define a small ‘toy’ language with just what you’re interested in, then scale up later. Even small languages can involve delicate design choices.

2 A First Imperative Language: L1

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L1 (untyped)

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Our First Language: L1 – Syntax

Booleans $b \in \mathbb{B} = \{\textbf{true}, \textbf{false}\}$
Integers $n \in \mathbb{Z} = \{..., -1, 0, 1, ...\}$
Locations $\ell \in \mathbb{L} = \{l, l_0, l_1, l_2, ...\}$

Operations $op \ ::= + \mid \geq$

Expressions

$$\begin{array}{rcl} e & ::= & n \mid b \mid e_1 \ op \ e_2 \mid \textbf{if } e_1 \textbf{ then } e_2 \textbf{ else } e_3 \mid \\ & & \ell := e \mid !\ell \mid \\ & & \textbf{skip} \mid e_1; e_2 \mid \\ & & \textbf{while } e_1 \textbf{ do } e_2 \end{array}$$

Write expr for the set of all expressions.

Points to note:

- we'll return later to *exactly* what the set expr is when we talk about abstract syntax
- unbounded integers
- abstract locations – can't do pointer arithmetic on them
- untyped, so have nonsensical expressions like $3 + \textbf{true}$
- what kind of grammar is that?
- don't have expression/command distinction
- doesn't much matter what basic operators we have

- carefully distinguish metavariables $b, n, \ell, \ op\ , e$ etc. from program locations l etc..

2.1 L1: Operational Semantics

How to describe the behaviour of a program? Going to use various forms of automata:

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Transition systems

A *transition system* consists of

- a set Config, and
- a binary relation $\longrightarrow \subseteq \mathrm{Config} * \mathrm{Config}$.

The elements of Config are often called *configurations* or *states*. The relation $\longrightarrow$ is called the *transition* or *reduction* relation. We write $\longrightarrow$ infix, so $c \longrightarrow c'$ should be read as 'state c can make a transition to state c''.

To compare with RLFA: a transition system is like an NFA$^{\varepsilon}$ with an empty alphabet (so only ε transitions) except (a) it can have infinitely many states, and (b) we don't specify a start state or accepting states. Sometimes one adds labels (e.g. to represent IO) but mostly we'll just look at the values of terminated states, those that cannot do any transitions.

Some handy auxilary definitions:

- $\longrightarrow^*$ is the reflexive transitive closure of $\longrightarrow$, so $c \longrightarrow^* c'$ iff there exist $k \geq 0$ and $c_0, .., c_k$ such that $c = c_0 \longrightarrow c_1 ... \longrightarrow c_k = c'$.
- $\not\longrightarrow$ is a unary predicate (a subset of Config) defined by $c \not\longrightarrow$ iff $\neg\exists c'.c \longrightarrow c'$.
- The transition relation is *deterministic* if for all states c there is at most one c' such that $c \longrightarrow c'$, ie if $\forall c.\forall c', c''.(c \longrightarrow c' \wedge c \longrightarrow c'') \implies c' = c''$.

Instantiating that definition for L1:

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L1 Semantics (1 of 4) – Configurations

Say *stores* s are finite partial functions from $\mathbb{L}$ to $\mathbb{Z}$. For example:

$$\{l_1 \mapsto 7,\ l_3 \mapsto 23\}$$

Take configurations to be pairs $\langle e, s\rangle$ of an expression e and a store s, so our transition relation will have the form

$$\langle e, s\rangle \longrightarrow \langle e', s'\rangle$$

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Transitions are single computation steps, for example will have:

$$\begin{array}{lll} & \langle l := 2 + !l, & \{l \mapsto 3\}\rangle \\ \longrightarrow & \langle l := 2 + 3, & \{l \mapsto 3\}\rangle \\ \longrightarrow & \langle l := 5, & \{l \mapsto 3\}\rangle \\ \longrightarrow & \langle \textbf{skip}, & \{l \mapsto 5\}\rangle \\ \not\longrightarrow & & \end{array}$$

want to keep on until we get to a *value* v, an expression in

$$\mathbb{V} = \mathbb{B} \cup \mathbb{Z} \cup \{\textbf{skip}\}.$$

Say $\langle e, s\rangle$ is *stuck* if e is not a value and $\langle e, s\rangle \not\longrightarrow$. For example $2 + \textbf{true}$ will be stuck.

Equivalently: Say *values* v are expressions from the grammar $v ::= b \mid n \mid \textbf{skip}$.

What is a finite partial function f from a set A to a set B? It's a set containing a finite number $n \geq 0$ of pairs $\{(a_1, b_1), ..., (a_n, b_n)\}$, often written $\{a_1 \mapsto b_1, ..., a_n \mapsto b_n\}$, for which

- $\forall i \in \{1, .., n\}.a_i \in A$ (the domain is a subset of$A$)
- $\forall i \in \{1, .., n\}.b_i \in B$ (the range is a subset of B)
- $\forall i \in \{1, .., n\}, j \in \{1, .., n\}.i \neq j \Rightarrow a_i \neq a_j$ (f is functional, i.e. each element of A is mapped to at most one element of B)

Notation: for a partial function s, write $\mathrm{dom}(s)$ for the set of elements in the domain of s (things that s maps to something) and $\mathrm{ran}(s)$ for the set of elements in the range of s (things that something is mapped to by s). For example, for the s above we have $\mathrm{dom}(s) = \{l_1, l_3\}$ and $\mathrm{ran}(s) = \{7, 23\}$.

Note that a finite partial function can be *empty*, if $n = 0$ above.

Here using elements of expression syntax in configurations ... cf abstract machines...

Write store for the set of all stores.

Now define the behaviour for each construct of L1 by giving some rules that (together) define a transition relation $\longrightarrow$.

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L1 Semantics (2 of 4) – Rules (basic operations)

(op $+$) $\langle n_1 + n_2, s\rangle \longrightarrow \langle n, s\rangle$ if $n = n_1 + n_2$

(op $\geq$) $\langle n_1 \geq n_2, s\rangle \longrightarrow \langle b, s\rangle$ if $b = (n_1 \geq n_2)$

(op1) $$\frac{\langle e_1, s\rangle \longrightarrow \langle e_1', s'\rangle}{\langle e_1 \ op \ e_2, s\rangle \longrightarrow \langle e_1' \ op \ e_2, s'\rangle}$$

(op2) $$\frac{\langle e_2, s\rangle \longrightarrow \langle e_2', s'\rangle}{\langle v \ op \ e_2, s\rangle \longrightarrow \langle v \ op \ e_2', s'\rangle}$$

How to read these? The rule (op $+$) says that for any instantiation of the metavariables n, n_1 and n_2 (i.e. any choice of three integers), that satisfies the sidecondition, there is a transition from the instantiated configuration on the left to the one on the right.

We use a variable naming convention: n can *only* be instantiated by integers, not by arbitrary expressions, cabbages, or what-have-you.

The rule (op1) says that for any instantiation of e_1, e_1', e_2, s, s' (i.e. any three expressions and two stores), *if* a transition of the form above the line can be deduced *then* we can deduce the transition below the line. We'll be more precise about this later.

Observe that – as you would expect – none of these rules introduce a change in the store part of configurations.

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Example

If we want to find the possible sequences of transitions of $\langle (2+3)+(6+7), \emptyset \rangle$ ... look for derivations of transitions. (you might think the answer should be 18 - but we want to know what *this definition* says should happen)

$$\text{(op1)}\ \dfrac{\text{(op }+\text{)}\ \dfrac{}{\langle 2+3, \emptyset \rangle \longrightarrow \langle 5, \emptyset \rangle}}{\langle (2+3)+(6+7), \emptyset \rangle \longrightarrow \langle 5+(6+7), \emptyset \rangle}$$

$$\text{(op2)}\ \dfrac{\text{(op }+\text{)}\ \dfrac{}{\langle 6+7, \emptyset \rangle \longrightarrow \langle 13, \emptyset \rangle}}{\langle 5+(6+7), \emptyset \rangle \longrightarrow \langle 5+13, \emptyset \rangle}$$

$$\text{(op }+\text{)}\ \dfrac{}{\langle 5+13, \emptyset \rangle \longrightarrow \langle 18, \emptyset \rangle}$$

First transition: using (op1) with $e_1 = 2+3$, $e_1' = 5$, $e_2 = 6+7$, $op = +$, $s = \emptyset$, $s' = \emptyset$, and using (op $+$) with $n_1 = 2$, $n_2 = 3$, $s = \emptyset$. Note couldn't begin with (op2) as $e_1 = 2+3$ is not a value, and couldn't use (op $+$) directly on $(2+3)+(6+7)$ as $2+3$ and $6+7$ are not numbers from $\mathbb{Z}$ – just expressions which might eventually evaluate to numbers (recall, by convention the n in the rules ranges over $\mathbb{Z}$ only).

Second transition: using (op2) with $e_1 = 5$, $e_2 = 6+7$, $e_2' = 13$, $op = +$, $s = \emptyset$, $s' = \emptyset$, and using (op $+$) with $n_1 = 6$, $n_2 = 7$, $s = \emptyset$. Note that to use (op2) we needed that $e_1 = 5$ is a value. We couldn't use (op1) as $e_1 = 5$ does not have any transitions itself.

Third transition: using (op $+$) with $n_1 = 5$, $n_2 = 13$, $s = \emptyset$.

For each transition we do something like proof search in natural deduction: starting with a state (at the bottom left), look for a rule and an instantiation of the metavariables in that rule that makes the left-hand-side of its conclusion match that state.

Warning: in general, there might be more than one rule and one instantiation that does this.

If there isn't a derivation concluding in $\langle e, s \rangle \longrightarrow \langle e', s' \rangle$ then there isn't such a transition.

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L1 Semantics (3 of 4) – store and sequencing

(deref) $\langle !\ell, s\rangle \longrightarrow \langle n, s\rangle$ if $\ell \in \mathrm{dom}(s)$ and $s(\ell) = n$

(assign1) $\langle \ell := n, s\rangle \longrightarrow \langle \textbf{skip}, s + \{\ell \mapsto n\}\rangle$ if $\ell \in \mathrm{dom}(s)$

(assign2) $$\frac{\langle e, s\rangle \longrightarrow \langle e', s'\rangle}{\langle \ell := e, s\rangle \longrightarrow \langle \ell := e', s'\rangle}$$

(seq1) $\langle \textbf{skip}; e_2, s\rangle \longrightarrow \langle e_2, s\rangle$

(seq2) $$\frac{\langle e_1, s\rangle \longrightarrow \langle e_1', s'\rangle}{\langle e_1; e_2, s\rangle \longrightarrow \langle e_1'; e_2, s'\rangle}$$

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Example

$$\begin{array}{lll}
\langle l := 3; !l, \{l \mapsto 0\}\rangle & \longrightarrow & \langle \textbf{skip}; !l, \{l \mapsto 3\}\rangle \\
 & \longrightarrow & \langle !l, \{l \mapsto 3\}\rangle \\
 & \longrightarrow & \langle 3, \{l \mapsto 3\}\rangle
\end{array}$$

$$\langle l := 3; l := !l, \{l \mapsto 0\}\rangle \longrightarrow \ ?$$

$$\langle 15 + !l, \emptyset\rangle \longrightarrow \ ?$$

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L1 Semantics (4 of 4) – The rest (conditionals and while)

(if1) $\langle \textbf{if true then } e_2 \textbf{ else } e_3, s\rangle \longrightarrow \langle e_2, s\rangle$

(if2) $\langle \textbf{if false then } e_2 \textbf{ else } e_3, s\rangle \longrightarrow \langle e_3, s\rangle$

(if3) $$\frac{\langle e_1, s\rangle \longrightarrow \langle e_1', s'\rangle}{\langle \textbf{if } e_1 \textbf{ then } e_2 \textbf{ else } e_3, s\rangle \longrightarrow \langle \textbf{if } e_1' \textbf{ then } e_2 \textbf{ else } e_3, s'\rangle}$$

(while)

$\langle \textbf{while } e_1 \textbf{ do } e_2, s\rangle \longrightarrow \langle \textbf{if } e_1 \textbf{ then } (e_2; \textbf{while } e_1 \textbf{ do } e_2) \textbf{ else skip}, s\rangle$

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Example

If

$e = (l_2 := 0; \textbf{while } !l_1 \geq 1 \textbf{ do } (l_2 :=!l_2+!l_1; l_1 :=!l_1 + -1))$

$s = \{l_1 \mapsto 3, l_2 \mapsto 0\}$

then

$\langle e, s\rangle \longrightarrow^* ?$

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Determinacy

Theorem 1 (L1 Determinacy) *If* $\langle e, s\rangle \longrightarrow \langle e_1, s_1\rangle$ *and* $\langle e, s\rangle \longrightarrow \langle e_2, s_2\rangle$ *then* $\langle e_1, s_1\rangle = \langle e_2, s_2\rangle$*.*

Proof - see later

Note that top-level universal quantifiers are usually left out – the theorem really says "For all e, s, e_1, s_1, e_2, s_2, if $\langle e, s\rangle \longrightarrow \langle e_1, s_1\rangle$ and $\langle e, s\rangle \longrightarrow \langle e_2, s_2\rangle$ then $\langle e_1, s_1\rangle = \langle e_2, s_2\rangle$".

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L1 Implementation

Will implement an interpreter for L1, following the definition. Use mosml (Moscow ML) as the implementation language, as datatypes and pattern matching are good for this kind of thing.

First, must pick representations for locations, stores, and expressions:

```
type loc = string

type store = (loc * int) list
```

We've chosen to represent locations as strings, rather arbitrarily (really, so they pretty-print trivially). A lower-level implementation would use machine pointers.

In the semantics, a store is a finite partial function from locations to integers. In the implementation, we represent a store as a list of `loc*int` pairs containing, for each ℓ in the domain of the store, exactly one element of the form `(l,n)`. The order of the list will not be important. This is not a very efficient implementation, but it is simple.

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```
datatype oper = Plus | GTEQ

datatype expr =
           Integer of int
         | Boolean of bool
         | Op of expr * oper * expr
         | If of expr * expr * expr
         | Assign of loc * expr
         | Deref of loc
         | Skip
         | Seq of expr * expr
         | While of expr * expr
```

The expression and operation datatypes have essentially the same form as the abstract grammar. Note, though, that it does not exactly match the semantics, as that allowed arbitrary integers whereas here we use the bounded mosml integers – so not every term of the abstract syntax is representable as an element of type `expr`, and the interpreter will fail with an overflow exception if `+` overflows.

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Store operations

Define auxiliary operations

```
lookup : store*loc        -> int option
update : store*(loc*int)  -> store option
```

which both return `NONE` if given a location that is not in the domain of the store. Recall that a value of type `T option` is either `NONE` or `SOME v` for a value `v` of `T`.

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The single-step function

Now define the single-step function

```
reduce : expr*store -> (expr*store) option
```

which takes a configuration `(e,s)` and returns either

`NONE`, if $\langle e, s \rangle \not\longrightarrow$,

or `SOME (e',s')`, if it has a transition $\langle e, s \rangle \longrightarrow \langle e', s' \rangle$.

Note that if the semantics didn't define a deterministic transition system we'd have to be more elaborate.

(you might think it would be better ML style to use exceptions instead of these options; that would be fine).

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(op $+$), (op $\geq$)

```
fun reduce (Integer n,s) = NONE
  | reduce (Boolean b,s) = NONE
  | reduce (Op (e1,opr,e2),s) =
    (case (e1,opr,e2) of
         (Integer n1, Plus, Integer n2) =>
           SOME(Integer (n1+n2), s)
       | (Integer n1, GTEQ, Integer n2) =>
           SOME(Boolean (n1 >= n2), s)
       | (e1,opr,e2) =>
           ...
```

Contrast this code with the semantic rules given earlier.

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(op1), (op2)

```
...
if (is_value e1) then
    case reduce (e2,s) of
        SOME (e2',s') =>
          SOME (Op(e1,opr,e2'),s')
      | NONE => NONE
else
    case reduce (e1,s) of
        SOME (e1',s') =>
          SOME(Op(e1',opr,e2),s')
      | NONE => NONE )
```

Note that the code depends on global properties of the semantics, including the fact that it defines a deterministic transition system, so the comments indicating that particular lines of code implement particular semantic rules are not the whole story.

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(assign1), (assign2)

```
| reduce (Assign (l,e),s) =
  (case e of
       Integer n =>
         (case update (s,(l,n)) of
             SOME s' => SOME(Skip, s')
           | NONE => NONE)
     | _ =>
         (case reduce (e,s) of
             SOME (e',s') =>
               SOME(Assign (l,e'), s')
           | NONE => NONE  ) )
```

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The many-step evaluation function

Now define the many-step evaluation function

```
evaluate: expr*store -> (expr*store) option
```

which takes a configuration `(e,s)` and returns the `(e',s')` such that $\langle e, s\rangle \longrightarrow^* \langle e', s'\rangle \not\longrightarrow$, if there is such, or does not return.

```
fun evaluate (e,s) =
  case reduce (e,s) of
    NONE => (e,s)
  | SOME (e',s') => evaluate (e',s')
```

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Demo

The full interpreter code is available on the web, in the file `l1.ml`, together with a pretty-printer and the type-checker we will come to soon. You should make it go...

```
(* 2002-11-08 -- Time-stamp: <2003-04-07 14:18:57 pes20>    -*-SML-*- *)
(* Peter Sewell                                                       *)

(*
   This file contains an interpreter, pretty-printer and type-checker
   for the language L1.

   To make it go, copy it into a working directory, ensure Moscow ML
   is available, and type
```

```
        mosml -P full l1.ml

    That will give you a MoscowML top level in which these definitions
    are present.  You can then type

       doit ();

    to show the reduction sequence of < l1:=3;!l1 , l1=0  >, and

       doit2 ();

    to run the type-checker on the same simple example; you can try
    other examples analogously.  This file doesn't have a parser for
    l1, so you'll have to enter the abstract syntax directly, eg

       prettyreduce (Seq( Assign ("l1",Integer 3), Deref "l1"), [("l1",0)]);

    This has been tested with Moscow ML version 2.00 (June 2000), but
    should work with any other implementation of Standard ML.
  *)

(* *********************)
(* the abstract syntax *)
(* *********************)

type loc = string

datatype oper = Plus | GTEQ

datatype expr =
         Integer of int
       | Boolean of bool
       | Op of expr * oper * expr
       | If of expr * expr * expr
       | Assign of loc * expr
       | Deref of loc
       | Skip
       | Seq of expr * expr
       | While of expr * expr

  (* note that this does not exactly match the semantics, as that
  allowed arbitrary integers whereas here we use the bounded mosml
  integers -- so not every term of the abstract syntax is
  representable as an element of type expr, and the interpreter will
  fail with an overflow exception if + overflows.

  We've chosen a particular set of locations, the strings.*)

(* **********************************)
(* an interpreter for the semantics *)
(* **********************************)

fun is_value (Integer n) = true
  | is_value (Boolean b) = true
  | is_value (Skip) = true
  | is_value _ = false
  (* In the semantics, a store is a finite partial function from
  locations to integers.  In the implementation, we represent a store
  as a list of loc*int pairs containing, for each l in the domain of
  the store, exactly one element of the form (l,n).  The operations

    lookup : store * loc         -> int option
    update : store * (loc * int) -> store option

  both return NONE if given a location that is not in the domain of
  the store.

  This is not a very efficient implementation, but it is simple. *)


type store = (loc * int) list

fun lookup ( [], l ) = NONE
  | lookup ( (l',n')::pairs, l) =
    if l=l' then SOME n' else lookup (pairs,l)

fun update'  front [] (l,n) = NONE
 |  update'  front ((l',n')::pairs) (l,n) =
    if l=l' then
        SOME(front @ ((l,n)::pairs) )
    else
        update' ((l',n')::front) pairs (l,n)

fun update (s, (l,n)) = update' [] s (l,n)


  (* now define the single-step function

     reduce :  expr * store -> (expr * store) option

  which takes a configuration (e,s) and returns either NONE, if it has
  no transitions, or SOME (e',s'), if it has a transition (e,s) -->
  (e',s').

  Note that the code depends on global properties of the semantics,
  including the fact that it defines a deterministic transition
  system, so the comments indicating that particular lines of code
  implement particular semantic rules are not the whole story.

  *)


fun reduce (Integer n,s) = NONE
  | reduce (Boolean b,s) = NONE
  | reduce (Op (e1,opr,e2),s) =
    (case (e1,opr,e2) of
         (Integer n1, Plus, Integer n2) => SOME(Integer (n1+n2), s)   (*op + *)
       | (Integer n1, GTEQ, Integer n2) => SOME(Boolean (n1 >= n2), s)(*op >=*)
       | (e1,opr,e2) =>
         if (is_value e1) then
             case reduce (e2,s) of
                 SOME (e2',s') => SOME (Op(e1,opr,e2'),s')     (* (op2) *)
```

```
               | NONE => NONE
         else
             case reduce (e1,s) of
                 SOME (e1',s') => SOME(Op(e1',opr,e2),s')      (* (op1) *)
               | NONE => NONE )
  | reduce (Deref l,s) =
    (case lookup  (s,l) of
          SOME n => SOME(Integer n,s)                          (* (deref) *)
        | NONE => NONE )
  | reduce (Assign (l,e),s) =
    (case e of
         Integer n => (case update (s,(l,n)) of
                           SOME s' => SOME(Skip, s')          (* (assign1) *)
                         | NONE => NONE)
       | _ => (case reduce (e,s) of
                   SOME (e',s') => SOME(Assign (l,e'), s')    (* (assign2) *)
                 | NONE => NONE  ) )
  | reduce (While (e1,e2),s) = SOME( If(e1,Seq(e2,While(e1,e2)),Skip),s) (* (while) *)
  | reduce (Skip,s) = NONE
  | reduce (Seq (e1,e2),s) =
    case e1 of
        Skip => SOME(e2,s)                                     (* (seq1) *)
      | _ => ( case reduce (e1,s) of
                 SOME (e1',s') => SOME(Seq (e1',e2), s')       (* (seq2) *)
               | NONE => NONE )


  (* now define the many-step evaluation function

     evaluate :  expr * store -> (expr * store) option

  which takes a configuration (e,s) and returns the unique (e',s')
  such that   (e,s) -->* (e',s') -/->.  *)


fun evaluate (e,s) = case reduce (e,s) of
                         NONE => (e,s)
                       | SOME (e',s') => evaluate (e',s')

[...]
```

L1 is a simple language, but it nonetheless involves several language design choices.

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Language design 1. Order of evaluation

For $(e_1 \; op \; e_2)$, the rules above say e_1 should be fully reduced, to a value, before we start reducing e_2. For example:

$$\langle (l := 1; 0) + (l := 2; 0), \{l \mapsto 0\} \rangle \longrightarrow^5 \langle 0, \{l \rightarrow \boxed{2}\} \rangle$$

For right-to-left evaluation, replace (op1) and (op2) by

$$\text{(op1b)} \quad \frac{\langle e_2, s \rangle \longrightarrow \langle e_2', s' \rangle}{\langle e_1 \; op \; e_2, s \rangle \longrightarrow \langle e_1 \; op \; e_2', s' \rangle}$$

$$\text{(op2b)} \quad \frac{\langle e_1, s \rangle \longrightarrow \langle e_1', s' \rangle}{\langle e_1 \; op \; v, s \rangle \longrightarrow \langle e_1' \; op \; v, s' \rangle}$$

In this language (call it L1b)

$$\langle (l := 1; 0) + (l := 2; 0), \{l \mapsto 0\} \rangle \longrightarrow^5 \langle 0, \{l \rightarrow \boxed{1}\} \rangle$$

For programmers whose first language has left-to-right reading order, left-to-right evaluation is arguably more intuitive than right-to-left. Nonetheless, some languages are right-to-left for efficiency reasons (e.g. OCaml bytecode).

It is important to have the *same* order for all operations, otherwise we certainly have a counter-intuitive language.

One could also *underspecify*, taking both (op1) and (op1b) rules. That language doesn't have the Determinacy property.

Sometimes ordering really is not always guaranteed, say for two writes $l := 1; l := 2$. In L1 it is defined, but if we were talking about a setting with a cache (either processors, or disk block writes, or something) we might have to do something additional to force ordering. Similarly if you have concurrency $l := 1 \mid l := 2$. Work on redesigning the Java Memory Model by Doug Lea and Bill Pugh, which involves this kind of question, can be found at `http://www.cs.umd.edu/~pugh/java/memoryModel/`.

One could also underspecify in a language definition but require each implementation to use a consistent order, or require each implementation to use a consistent order for each operator occurrence in the program source code. A great encouragement to the bugs...

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Language design 2. Assignment results

Recall

(assign1) $\langle \ell := n, s\rangle \longrightarrow \langle \textbf{skip}, s + \{\ell \mapsto n\}\rangle$ if $\ell \in \mathrm{dom}(s)$

(seq1) $\langle \textbf{skip}; e_2, s\rangle \longrightarrow \langle e_2, s\rangle$

So

$$\begin{array}{rcl} \langle l := 1; l := 2, \{l \mapsto 0\}\rangle & \longrightarrow & \langle \textbf{skip}; l := 2, \{l \mapsto 1\}\rangle \\ & \longrightarrow^* & \langle \textbf{skip}, \{l \mapsto 2\}\rangle \end{array}$$

We've chosen $\ell := v$ to result in skip, and $e_1; e_2$ to only progress if $e_1 = \textbf{skip}$, not for any value. Instead could have this:

(assign1') $\langle \ell := v, s\rangle \longrightarrow \langle v, s + (\ell \mapsto v)\rangle$ if $\ell \in \mathrm{dom}(s)$

(seq1') $\langle v; e_2, s\rangle \longrightarrow \langle e_2, s\rangle$

Matter of taste?

Another possiblity: return the *old* value, e.g. in ANSI C signal handler installation `signal(n,h)`.

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Language design 3. Store initialisation

Recall that

(deref) $\langle !\ell, s\rangle \longrightarrow \langle n, s\rangle$ if $\ell \in \mathrm{dom}(s)$ and $s(\ell) = n$

(assign1) $\langle \ell := n, s\rangle \longrightarrow \langle \textbf{skip}, s + \{\ell \mapsto n\}\rangle$ if $\ell \in \mathrm{dom}(s)$

both require $\ell \in \mathrm{dom}(s)$, otherwise the expressions are stuck.

Instead, could

1. implicitly initialise *all* locations to 0, or
2. allow assignment to an $\ell \notin \mathrm{dom}(s)$ to initialise that ℓ.

These would both be bad design decisions, liable to lead to ghastly bugs, with locations initialised on some code path but not others. 1 would be particularly awkward in a richer language where values other than integers can be stored, where there may not be any sensible value to default-initialise to.

Adding typing will rule out stuckness at compile-time.

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Language design 4. Storable values

Recall stores s are finite partial functions from $\mathbb{L}$ to $\mathbb{Z}$, with rules:

(deref) $\langle !\ell, s\rangle \longrightarrow \langle n, s\rangle$ if $\ell \in \mathrm{dom}(s)$ and $s(\ell) = n$

(assign1) $\langle \ell := n, s\rangle \longrightarrow \langle \textbf{skip}, s + \{\ell \mapsto n\}\rangle$ if $\ell \in \mathrm{dom}(s)$

(assign2) $$\frac{\langle e, s\rangle \longrightarrow \langle e', s'\rangle}{\langle \ell := e, s\rangle \longrightarrow \langle \ell := e', s'\rangle}$$

Can store only integers. $\langle l := \textbf{true}, s\rangle$ is stuck.

This is annoying – unmotivated *irregularity* – why not allow storage of any value? of locations? of expressions???

Also, store is global....leading to ghastly programming in big code. Will revisit later.

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Language design 5. Operators and basic values

Booleans are really not integers (pace C)

How many operators? Obviously want more than just $+$ and $\geq$. But this is semantically dull - in a full language would add in many, in standard libraries.

(beware, it's not completely dull - eg floating point specs! Even the L1 impl and semantics aren't in step.).

Exercise: fix the implementation to match the semantics.

Exercise: fix the semantics to match the implementation.

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Expressiveness

Is L1 expressive enough to write interesting programs?

- yes: it's Turing-powerful (try coding an arbitrary register machine in L1).
- no: there's no support for gadgets like functions, objects, lists, trees, modules,.....

Is L1 *too* expressive? (ie, can we write too many programs in it)

- yes: we'd like to forbid programs like $3 + \textbf{false}$ as early as possible, not wait for a runtime error (which might occur only on some execution paths). We'll do so with a *type system*.

2.2 L1: Typing

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L1 Typing

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Type Systems

used for

- preventing certain kinds of errors
- structuring programs
- guiding language design

Also: (1) optimising compilers can make great use of type information; (2) types can be used to enforce security properties, from simple absence of buffer overflows to (in research languages) sophisticated information-flow policies; and (3) there are connections with logic, which we'll return to later.

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Run-time errors

Trapped errors. Cause execution to halt immediately. (E.g. jumping to an illegal address, raising a top-level exception, etc.) Innocuous?

Untrapped errors. May go unnoticed for a while and later cause arbitrary behaviour. (E.g. accessing data past the end of an array, security loopholes in Java abstract machines, etc.) Insidious!

Given a precise definition of what constitutes an untrapped run-time error, then a language is *safe* if all its syntactically legal programs cannot cause such errors.

Usually, safety is desirable. Moreover, we'd like as few trapped errors as possible.

We cannot expect to exclude *all* trapped errors, eg arith overflows, or out-of-memory errors, but certainly want to exclude all untrapped errors.

So, how to do so? Can use runtime checks and compile-time checks – want compile-time where possible.

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Formal type systems

Divide programs into the good and the bad...

We will define a ternary relation $\Gamma \vdash e{:}T$, read as 'expression e has type T, under assumptions Γ on the types of locations that may occur in e'.

For example (according to the definition coming up):

$$\begin{array}{lcllll}
\{\} & \vdash & \textbf{if true then } 2 \textbf{ else } 3+4 & : & \mathsf{int} & \\
l_1{:}\mathsf{intref} & \vdash & \textbf{if } !l_1 \geq 3 \textbf{ then } !l_1 \textbf{ else } 3 & : & \mathsf{int} & \\
\{\} & \not\vdash & 3 + \textbf{false} & : & T & \text{for any } T \\
\{\} & \not\vdash & \textbf{if true then } 3 \textbf{ else false} & : & \mathsf{int} &
\end{array}$$

Note that the last is excluded despite the fact that when you execute the program you will always get an int – type systems define *approximations* to the behaviour of programs, often quite crude – and this has to be so, as we generally would like them to be decidable.

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Types

Types of expressions:

$$T \quad ::= \quad \mathsf{int} \mid \mathsf{bool} \mid \mathsf{unit}$$

Types of locations:

$$T_{loc} \quad ::= \quad \mathsf{intref}$$

Write T and $\mathrm{T}_{\mathrm{loc}}$ for the sets of all terms of these grammars.

Let Γ range over $\mathrm{TypeEnv}$, the finite partial functions from locations $\mathbb{L}$ to $\mathrm{T}_{\mathrm{loc}}$. Notation: write a Γ as $l_1{:}\mathsf{intref}, ..., l_k{:}\mathsf{intref}$ instead of $\{l_1 \mapsto \mathsf{intref}, ..., l_k \mapsto \mathsf{intref}\}$.

- concretely, $\mathrm{T} = \{\mathsf{int}, \mathsf{bool}, \mathsf{unit}\}$ and $\mathrm{T}_{\mathrm{loc}} = \{\mathsf{intref}\}$.
- in this (very small!) language, there is only one type in $\mathrm{T}_{\mathrm{loc}}$, so a Γ is (up to isomorphism) just a set of locations. Later, $\mathrm{T}_{\mathrm{loc}}$ will be more interesting...
- our semantics only let you store integers, so we have stratified types into T and $\mathrm{T}_{\mathrm{loc}}$. If you wanted to store other values, you'd say

$$\begin{array}{lcl}
T & ::= & \mathsf{int} \mid \mathsf{bool} \mid \mathsf{unit} \\
T_{loc} & ::= & T\ \mathsf{ref}
\end{array}$$

If you wanted to be able to manipulate references as first-class objects, the typing would be

$$\begin{array}{lll} T & ::= & \mathsf{int} \mid \mathsf{bool} \mid \mathsf{unit} \mid T\ \mathsf{ref} \\ T_{loc} & ::= & T\ \mathsf{ref} \end{array}$$

and there would be consequent changes (what exactly?) to the syntax and the semantics. This is our first sight of an important theme: type-system-directed language design.

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Defining the type judgement $\boxed{\Gamma \vdash e{:}T}$ (1 of 3)

(int) $\Gamma \vdash n{:}\mathsf{int}$ for $n \in \mathbb{Z}$

(bool) $\Gamma \vdash b{:}\mathsf{bool}$ for $b \in \{\textbf{true}, \textbf{false}\}$

$$\text{(op }+\text{)}\ \frac{\begin{array}{c}\Gamma \vdash e_1{:}\mathsf{int} \\ \Gamma \vdash e_2{:}\mathsf{int}\end{array}}{\Gamma \vdash e_1 + e_2{:}\mathsf{int}} \qquad \text{(op }\geq\text{)}\ \frac{\begin{array}{c}\Gamma \vdash e_1{:}\mathsf{int} \\ \Gamma \vdash e_2{:}\mathsf{int}\end{array}}{\Gamma \vdash e_1 \geq e_2{:}\mathsf{bool}}$$

$$\text{(if)}\ \frac{\begin{array}{c}\Gamma \vdash e_1{:}\mathsf{bool} \\ \Gamma \vdash e_2{:}T \\ \Gamma \vdash e_3{:}T\end{array}}{\Gamma \vdash \textbf{if } e_1 \textbf{ then } e_2 \textbf{ else } e_3{:}T}$$

NB in (if) both the same T - but any

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Example

To show $\{\} \vdash \textbf{if true then } 2 \textbf{ else } 3 + 4{:}\mathsf{int}$ we can give a type derivation like this:

$$\text{(if)}\ \frac{\text{(bool)}\ \dfrac{}{\{\} \vdash \textbf{true}{:}\mathsf{bool}} \qquad \text{(int)}\ \dfrac{}{\{\} \vdash 2{:}\mathsf{int}} \qquad \nabla}{\{\} \vdash \textbf{if true then } 2 \textbf{ else } 3 + 4{:}\mathsf{int}}$$

where ∇ is

$$\text{(op }+\text{)}\ \frac{\text{(int)}\ \dfrac{}{\{\} \vdash 3{:}\mathsf{int}} \qquad \text{(int)}\ \dfrac{}{\{\} \vdash 4{:}\mathsf{int}}}{\{\} \vdash 3 + 4{:}\mathsf{int}}$$

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Defining the type judgement $\boxed{\Gamma \vdash e{:}T}$ (2 of 3)

(assign) $$\frac{\Gamma(\ell) = \mathsf{intref} \qquad \Gamma \vdash e{:}\mathsf{int}}{\Gamma \vdash \ell := e{:}\mathsf{unit}}$$

(deref) $$\frac{\Gamma(\ell) = \mathsf{intref}}{\Gamma \vdash !\ell{:}\mathsf{int}}$$

Here the $\Gamma(\ell) = \mathsf{intref}$ just means $\ell \in \mathrm{dom}(\Gamma)$.

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Defining the type judgement $\boxed{\Gamma \vdash e{:}T}$ (3 of 3)

(skip) $\Gamma \vdash \textbf{skip}{:}\mathsf{unit}$

(seq) $$\frac{\Gamma \vdash e_1{:}\mathsf{unit} \qquad \Gamma \vdash e_2{:}T}{\Gamma \vdash e_1; e_2{:}T}$$

(while) $$\frac{\Gamma \vdash e_1{:}\mathsf{bool} \qquad \Gamma \vdash e_2{:}\mathsf{unit}}{\Gamma \vdash \textbf{while } e_1 \textbf{ do } e_2{:}\mathsf{unit}}$$

Note that the typing rules are *syntax-directed* – for each clause of the abstract syntax for expressions there is exactly one rule with a conclusion of that form.

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Properties

Theorem 2 (Progress) *If* $\Gamma \vdash e{:}T$ *and* $dom(\Gamma) \subseteq dom(s)$ *then either* e *is a value or there exist* e', s' *such that* $\langle e, s \rangle \longrightarrow \langle e', s' \rangle$.

Theorem 3 (Type Preservation) *If* $\Gamma \vdash e{:}T$ *and* $dom(\Gamma) \subseteq dom(s)$ *and* $\langle e, s \rangle \longrightarrow \langle e', s' \rangle$ *then* $\Gamma \vdash e'{:}T$ *and* $dom(\Gamma) \subseteq dom(s')$.

From these two we have that well-typed programs don't get stuck:

Theorem 4 (Safety) *If* $\Gamma \vdash e{:}T$, $dom(\Gamma) \subseteq dom(s)$, *and* $\langle e, s \rangle \longrightarrow^* \langle e', s' \rangle$ *then either* e' *is a value or there exist* e'', s'' *such that* $\langle e', s' \rangle \longrightarrow \langle e'', s'' \rangle$.

(we'll discuss how to *prove* these results soon)

Semantic style: one could make an explicit definition of what configurations are runtime errors. Here, instead, those configurations are just stuck.

For L1 we don't need to type the range of the store, as by definition all stored things are integers.

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Type checking, typeability, and type inference

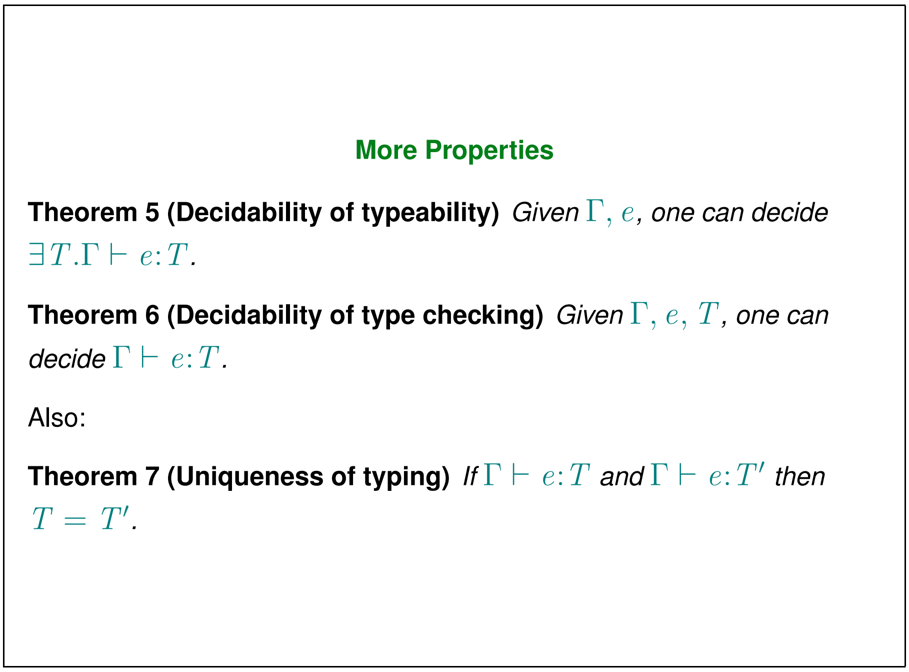
Type checking problem for a type system: given Γ, e, T, is $\Gamma \vdash e{:}T$ derivable?

Typeability problem: given Γ and e, find T such that $\Gamma \vdash e{:}T$ is derivable, or show there is none.

Second problem is usually harder than the first. Solving it usually results in a type inference algorithm: computing a type T for a phrase e, given type environment Γ (or failing, if there is none).

For this type system, though, both are easy.

Slide 48



More Properties

Theorem 5 (Decidability of typeability) *Given Γ, e, one can decide $\exists T.\Gamma \vdash e{:}T$.*

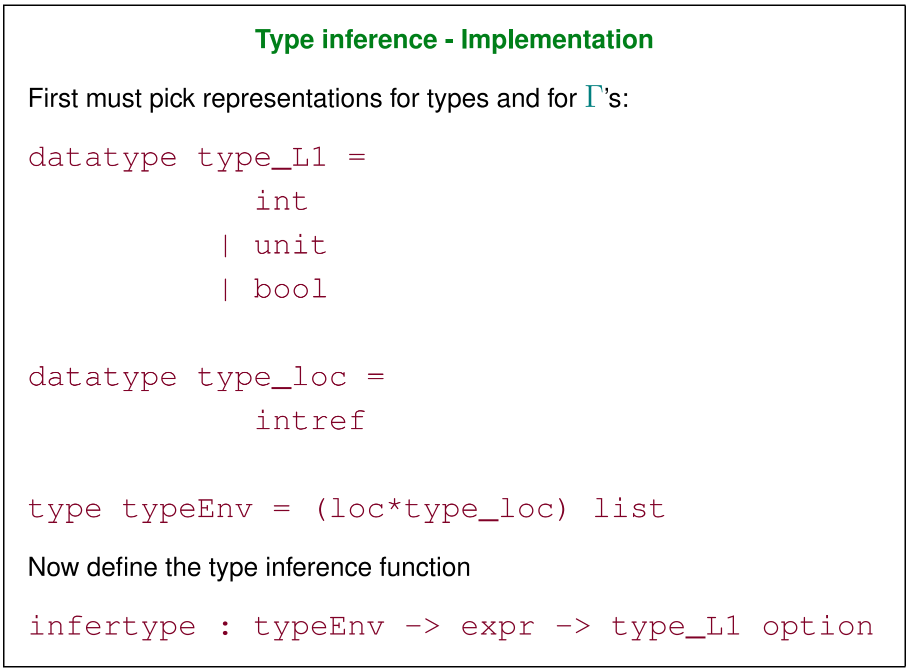
Theorem 6 (Decidability of type checking) *Given Γ, e, T, one can decide $\Gamma \vdash e{:}T$.*

Also:

Theorem 7 (Uniqueness of typing) *If $\Gamma \vdash e{:}T$ and $\Gamma \vdash e{:}T'$ then $T = T'$.*

The file `l1.ml` contains also an implementation of a type inference algorithm for L1 – take a look.

Slide 49



Type inference - Implementation

First must pick representations for types and for Γ's:

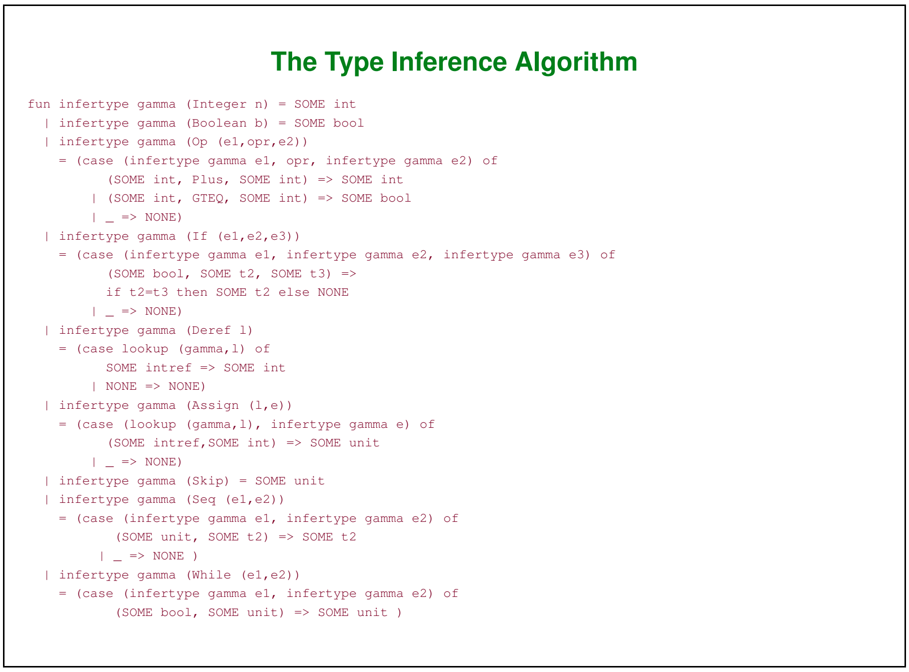
```
datatype type_L1 =
           int
         | unit
         | bool

datatype type_loc =
           intref

type typeEnv = (loc*type_loc) list
```

Now define the type inference function

```
infertype : typeEnv -> expr -> type_L1 option
```

In the semantics, type environments Γ are partial functions from locations to the singleton set $\{\mathsf{intref}\}$. Here, just as we did for stores, we represent them as a list of `loc*type_loc` pairs containing, for each ℓ in the domain of the type environment, exactly one element of the form `(l,intref)`.

Slide 50

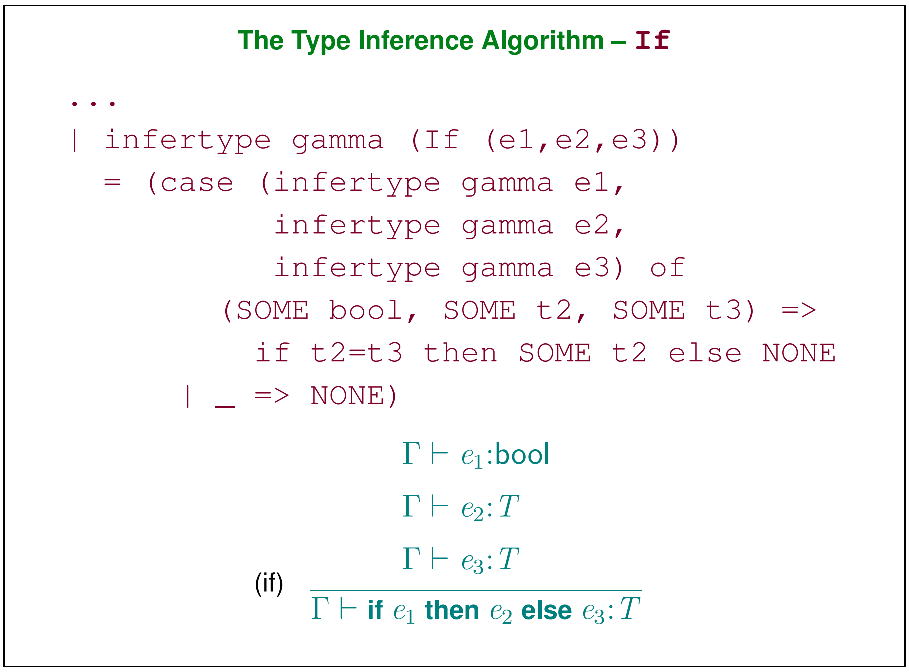


The Type Inference Algorithm

```
fun infertype gamma (Integer n) = SOME int
  | infertype gamma (Boolean b) = SOME bool
  | infertype gamma (Op (e1,opr,e2))
    = (case (infertype gamma e1, opr, infertype gamma e2) of
          (SOME int, Plus, SOME int) => SOME int
        | (SOME int, GTEQ, SOME int) => SOME bool
        | _ => NONE)
  | infertype gamma (If (e1,e2,e3))
    = (case (infertype gamma e1, infertype gamma e2, infertype gamma e3) of
          (SOME bool, SOME t2, SOME t3) =>
          if t2=t3 then SOME t2 else NONE
        | _ => NONE)
  | infertype gamma (Deref l)
    = (case lookup (gamma,l) of
          SOME intref => SOME int
        | NONE => NONE)
  | infertype gamma (Assign (l,e))
    = (case (lookup (gamma,l), infertype gamma e) of
          (SOME intref,SOME int) => SOME unit
        | _ => NONE)
  | infertype gamma (Skip) = SOME unit
  | infertype gamma (Seq (e1,e2))
    = (case (infertype gamma e1, infertype gamma e2) of
           (SOME unit, SOME t2) => SOME t2
         | _ => NONE )
  | infertype gamma (While (e1,e2))
    = (case (infertype gamma e1, infertype gamma e2) of
           (SOME bool, SOME unit) => SOME unit )
```

ahem.

Slide 51



The Type Inference Algorithm – If

```
...
| infertype gamma (If (e1,e2,e3))
  = (case (infertype gamma e1,
           infertype gamma e2,
           infertype gamma e3) of
         (SOME bool, SOME t2, SOME t3) =>
          if t2=t3 then SOME t2 else NONE
      | _ => NONE)
```

(if)
$$\frac{\Gamma \vdash e_1{:}\mathsf{bool} \qquad \Gamma \vdash e_2{:}T \qquad \Gamma \vdash e_3{:}T}{\Gamma \vdash \textbf{if } e_1 \textbf{ then } e_2 \textbf{ else } e_3{:}T}$$

Slide 52

The Type Inference Algorithm – Deref

```
...
| infertype gamma (Deref l)
  = (case lookup (gamma,l) of
         SOME intref => SOME int
       | NONE => NONE)
...
```

(deref) $\dfrac{\Gamma(\ell) = \mathsf{intref}}{\Gamma \vdash !\ell{:}\mathsf{int}}$

Again, the code depends on a uniqueness property (Theorem 7), without which we would have to have infertype return a `type_L1` list of all the possible types.

Slide 53

Demo

Q: Why didn't we build the type system into the syntax? (original FORTRAN, BASIC etc. had typing built into variable names, with e.g. those beginning with `I` or `J` storing integers). A: You quickly go beyond context-free grammars.

Slide 54

Executing L1 in mosml

L1 is essentially a fragment of mosml - given a typable L1 expression e and an initial store s, e can be executed in mosml by wrapping it

```
let val skip = ()
    and l1 = ref n1
    and l2 = ref n2
    ..
    and lk = ref nk
in
  e
end;
```

where s is the store $\{l_1 \mapsto n_1, ..., l_k \mapsto n_k\}$ and all locations that occur in e are contained in $\{l_1, ..., l_k\}$.

(watch out for ~1 and -1)

Slide 55

Why *Not* Types?

- *"I can't write the code I want in this type system."*

  (the Pascal complaint) usually false for a modern typed language
- *"It's too tiresome to get the types right throughout development."*

  (the untyped-scripting-language complaint)
- *"Type annotations are too verbose."*

  type inference means you only have to write them where it's useful
- *"Type error messages are incomprehensible."*

  hmm. Sadly, sometimes true.
- *"I really can't write the code I want."*

  Garbage collection? Marshalling? Multi-stage computation?

2.3 L1: Collected Definition

Syntax

Booleans $b \in \mathbb{B} = \{\textbf{true}, \textbf{false}\}$
Integers $n \in \mathbb{Z} = \{..., -1, 0, 1, ...\}$
Locations $\ell \in \mathbb{L} = \{l, l_0, l_1, l_2, ...\}$

Operations $op ::= + \mid \geq$

Expressions

$$\begin{array}{rcl} e & ::= & n \mid b \mid e_1 \; op \; e_2 \mid \textbf{if } e_1 \textbf{ then } e_2 \textbf{ else } e_3 \mid \\ & & \ell := e \mid !\ell \mid \\ & & \textbf{skip} \mid e_1; e_2 \mid \\ & & \textbf{while } e_1 \textbf{ do } e_2 \end{array}$$

Operational Semantics

Say *stores* s are finite partial functions from $\mathbb{L}$ to $\mathbb{Z}$. Say *values* v are expressions from the grammar $v ::= b \mid n \mid \textbf{skip}$.

(op $+$) $\quad \langle n_1 + n_2, s\rangle \longrightarrow \langle n, s\rangle \quad$ if $n = n_1 + n_2$

(op $\geq$) $\quad \langle n_1 \geq n_2, s\rangle \longrightarrow \langle b, s\rangle \quad$ if $b = (n_1 \geq n_2)$

(op1) $\quad \dfrac{\langle e_1, s\rangle \longrightarrow \langle e_1', s'\rangle}{\langle e_1 \; op \; e_2, s\rangle \longrightarrow \langle e_1' \; op \; e_2, s'\rangle}$

(op2) $\quad \dfrac{\langle e_2, s\rangle \longrightarrow \langle e_2', s'\rangle}{\langle v \; op \; e_2, s\rangle \longrightarrow \langle v \; op \; e_2', s'\rangle}$

(deref) $\quad \langle !\ell, s\rangle \longrightarrow \langle n, s\rangle \quad$ if $\ell \in \mathrm{dom}(s)$ and $s(\ell) = n$

(assign1) $\quad \langle \ell := n, s\rangle \longrightarrow \langle \textbf{skip}, s + \{\ell \mapsto n\}\rangle \quad$ if $\ell \in \mathrm{dom}(s)$

(assign2) $\quad \dfrac{\langle e, s\rangle \longrightarrow \langle e', s'\rangle}{\langle \ell := e, s\rangle \longrightarrow \langle \ell := e', s'\rangle}$

(seq1) $\quad \langle \textbf{skip}; e_2, s\rangle \longrightarrow \langle e_2, s\rangle$

(seq2) $\quad \dfrac{\langle e_1, s\rangle \longrightarrow \langle e_1', s'\rangle}{\langle e_1; e_2, s\rangle \longrightarrow \langle e_1'; e_2, s'\rangle}$

(if1) $\quad \langle \textbf{if true then } e_2 \textbf{ else } e_3, s\rangle \longrightarrow \langle e_2, s\rangle$

(if2) $\quad \langle \textbf{if false then } e_2 \textbf{ else } e_3, s\rangle \longrightarrow \langle e_3, s\rangle$

(if3) $\quad \dfrac{\langle e_1, s\rangle \longrightarrow \langle e_1', s'\rangle}{\langle \textbf{if } e_1 \textbf{ then } e_2 \textbf{ else } e_3, s\rangle \longrightarrow \langle \textbf{if } e_1' \textbf{ then } e_2 \textbf{ else } e_3, s'\rangle}$

(while)
$\langle \textbf{while } e_1 \textbf{ do } e_2, s\rangle \longrightarrow \langle \textbf{if } e_1 \textbf{ then } (e_2; \textbf{while } e_1 \textbf{ do } e_2) \textbf{ else skip}, s\rangle$

Typing

Types of expressions:

$$T \quad ::= \quad \mathsf{int} \mid \mathsf{bool} \mid \mathsf{unit}$$

Types of locations:

$$T_{loc} \quad ::= \quad \mathsf{intref}$$

Write T and $\mathrm{T}_{\mathrm{loc}}$ for the sets of all terms of these grammars.

Let Γ range over $\mathrm{TypeEnv}$, the finite partial functions from locations $\mathbb{L}$ to $\mathrm{T}_{\mathrm{loc}}$.

$$\text{(int)} \quad \Gamma \vdash n{:}\mathsf{int} \quad \text{for } n \in \mathbb{Z}$$

$$\text{(bool)} \quad \Gamma \vdash b{:}\mathsf{bool} \quad \text{for } b \in \{\textbf{true}, \textbf{false}\}$$

$$\text{(op }+\text{)} \quad \frac{\begin{array}{c}\Gamma \vdash e_1{:}\mathsf{int}\\ \Gamma \vdash e_2{:}\mathsf{int}\end{array}}{\Gamma \vdash e_1 + e_2{:}\mathsf{int}} \qquad \text{(op }\geq\text{)} \quad \frac{\begin{array}{c}\Gamma \vdash e_1{:}\mathsf{int}\\ \Gamma \vdash e_2{:}\mathsf{int}\end{array}}{\Gamma \vdash e_1 \geq e_2{:}\mathsf{bool}}$$

$$\text{(if)} \quad \frac{\begin{array}{c}\Gamma \vdash e_1{:}\mathsf{bool}\\ \Gamma \vdash e_2{:}T\\ \Gamma \vdash e_3{:}T\end{array}}{\Gamma \vdash \textbf{if } e_1 \textbf{ then } e_2 \textbf{ else } e_3{:}T}$$

$$\text{(assign)} \quad \frac{\begin{array}{c}\Gamma(\ell) = \mathsf{intref}\\ \Gamma \vdash e{:}\mathsf{int}\end{array}}{\Gamma \vdash \ell := e{:}\mathsf{unit}}$$

$$\text{(deref)} \quad \frac{\Gamma(\ell) = \mathsf{intref}}{\Gamma \vdash !\ell{:}\mathsf{int}}$$

$$\text{(skip)} \quad \Gamma \vdash \textbf{skip}{:}\mathsf{unit}$$

$$\text{(seq)} \quad \frac{\begin{array}{c}\Gamma \vdash e_1{:}\mathsf{unit}\\ \Gamma \vdash e_2{:}T\end{array}}{\Gamma \vdash e_1; e_2{:}T}$$

$$\text{(while)} \quad \frac{\begin{array}{c}\Gamma \vdash e_1{:}\mathsf{bool}\\ \Gamma \vdash e_2{:}\mathsf{unit}\end{array}}{\Gamma \vdash \textbf{while } e_1 \textbf{ do } e_2{:}\mathsf{unit}}$$

2.4 Exercises

Exercise 1 ★*Write a program to compute the factorial of the integer initially in location* l_1. *Take care to ensure that your program really is an expression in L1.*

Exercise 2 ★*Give full derivations of all the reduction steps of* $\langle (l_0 := 7); (l_1 := (!l_0 + 2)), \{l_0 \mapsto 0, l_1 \mapsto 0\}\rangle$.

Exercise 3 ★*Give full derivations of the first four reduction steps of the* $\langle e, s\rangle$ *on Slide 20.*

Exercise 4 ★*Adapt the implementation code to correspond to the two rules (op1b) and (op2b) on Slide 31. Give some test cases that distinguish between the original and the new semantics.*

Exercise 5 ★*Adapt the implementation code to correspond to the two rules (assign1') and (seq1') on Slide 32. Give some test cases that distinguish between the original and the new semantics.*

Exercise 6 ★★*Fix the L1 implementation to match the semantics, taking care with the representation of integers.*

Exercise 7 ★★*Fix the L1 semantics to match the implementation, taking care with the representation of integers.*

Exercise 8 ★*Give a type derivation for* $(l_0 := 7); (l_1 := (!l_0 + 2))$ *with* $\Gamma = l_0{:}\mathsf{intref}, l_1{:}\mathsf{intref}$.

Exercise 9 ★*Give a type derivation for the* e *on Slide 20 with* $\Gamma = l_1{:}\mathsf{intref}, l_2{:}\mathsf{intref}, l_3{:}\mathsf{intref}$.

Exercise 10 ★*Does Type Preservation hold for the variant language with rules (assign1') and (seq1') on Slide 32? If not, give an example, and show how the type rules could be adjusted to make it true.*

Exercise 11 ★*Adapt the type inference implementation to match your revised type system from Exercise* **??**

Exercise 12 ★*Check whether mosml, the L1 implementation and the L1 agree on the order of evaluation for operators and sequencing.*

Exercise 13 ★ *(just for fun) Adapt the implementation to output derivation trees, in ASCII, (or to show where proof search gets stuck) for* $\longrightarrow$ *or* $\vdash$.

3 Induction

Slide 56

Induction

Slide 57

We’ve stated several ‘theorems’, but how do we know they are true?

Intuition is often wrong – we need *proof*.

Use proof process also for strengthening our intuition about subtle language features, and for debugging definitions – it helps you examine all the various cases.

Most of our definitions are inductive – so to prove things about them, we need the corresponding *induction principles*.

Slide 58

Principle of Mathematical Induction

For any property $\Phi(x)$ of natural numbers $x \in \mathbb{N} = \{0, 1, 2, ...\}$, to prove

$\forall x \in \mathbb{N}.\Phi(x)$

it's enough to prove

$\Phi(0)$ and $\forall x \in \mathbb{N}.\Phi(x) \Rightarrow \Phi(x+1)$.

i.e.

$$\boxed{\big(\Phi(0) \wedge (\forall x \in \mathbb{N}.\Phi(x) \Rightarrow \Phi(x+1))\big) \Rightarrow \forall x \in \mathbb{N}.\Phi(x)}$$

Slide 59

$$\boxed{\big(\Phi(0) \wedge (\forall x \in \mathbb{N}.\Phi(x) \Rightarrow \Phi(x+1))\big) \Rightarrow \forall x \in \mathbb{N}.\Phi(x)}$$

For example, to prove

Theorem 8 $1 + 2 + ... + x = 1/2 * x * (x + 1)$

use mathematical induction for

$\Phi(x) = (1 + 2 + ... + x = 1/2 * x * (x + 1))$

There's a model proof in the notes, (annotated to say what's going on), as an example of good style. Writing a clear proof structure like this becomes essential when things get more complex – you have to *use* the formalism to help you get things right. Emulate it! (but without the annotations!)

NB, the natural numbers include 0 around here.

Theorem 8 $1 + 2 + ... + x = 1/2 * x * (x + 1)$.

Proof We prove $\forall x.\Phi(x)$, where *(state Φ explicitly)*

$$\Phi(x) \stackrel{\text{def}}{=} (1 + 2 + ... + x = 1/2 * x * (x + 1))$$

by mathematical induction. *(state the induction principle you're using)*

(Now show each conjunct of the premise of the induction principle)

Base case: *(conjunct $\Phi(0)$)*

$\Phi(0)$ is $\overset{\textit{(instantiate } \Phi\textit{)}}{(1 + ... + 0 = 1/2 * 0 * (0 + 1))}$, which holds as both sides are equal to 0.

Inductive step: *(conjunct* $\forall x \in \mathbb{N}.\Phi(x) \Rightarrow \Phi(x+1)$ *)*

Consider an arbitrary $k \in \mathbb{N}$ *(it's a universal ($\forall$), so consider an arbitrary one).*

Suppose $\Phi(k)$ *(to show the implication $\Phi(k) \Rightarrow \Phi(k+1)$, assume the premise and try to show the conclusion).*

We have to show $\Phi(k+1)$, i.e. *(state what we have to show explicitly)*

$$(1 + 2 + ... + (k+1)) = 1/2 * (k+1) * ((k+1)+1)$$

Now, the left hand side is

$$\begin{aligned}(1 + 2 + ... + (k+1)) &= (1 + 2 + ... + k) + (k+1) && \text{(rearranging)}\\ &= (1/2 * k * (k+1)) + (k+1) && \text{(using } \Phi(k)\text{)}\end{aligned}$$

(say where you use the 'induction hypothesis' assumption $\Phi(k)$ made above)

and the right hand side is

$$\begin{aligned}1/2 * (k+1) * ((k+1)+1) &= 1/2 * (k * (k+1) + (k+1) * 1 + 1 * k + 1) && \text{(rearranging)}\\ &= 1/2 * k * (k+1) + 1/2 * ((k+1) + k + 1) && \text{(rearranging)}\\ &= 1/2 * k * (k+1) + (k+1) && \text{(rearranging)}\end{aligned}$$

which is equal to the LHS.

□

Slide 60

Complete Induction

There is an equivalent principle, sometimes more convenient:

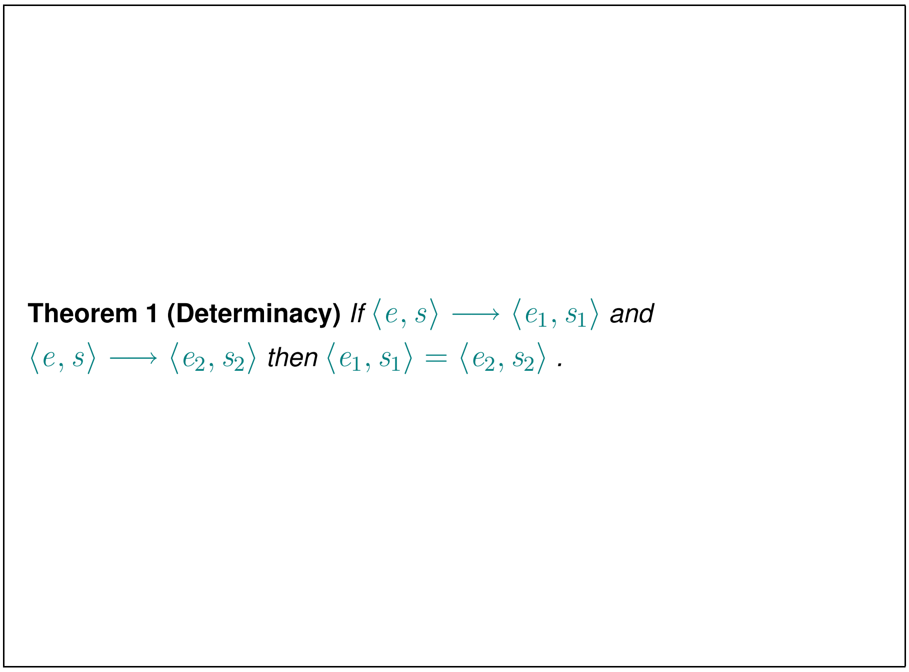
For any property $\Phi(k)$ of natural numbers $k \in \mathbb{N} = \{0, 1, 2, ...\}$, to prove

$\forall k \in \mathbb{N}.\Phi(k)$

it's enough to prove

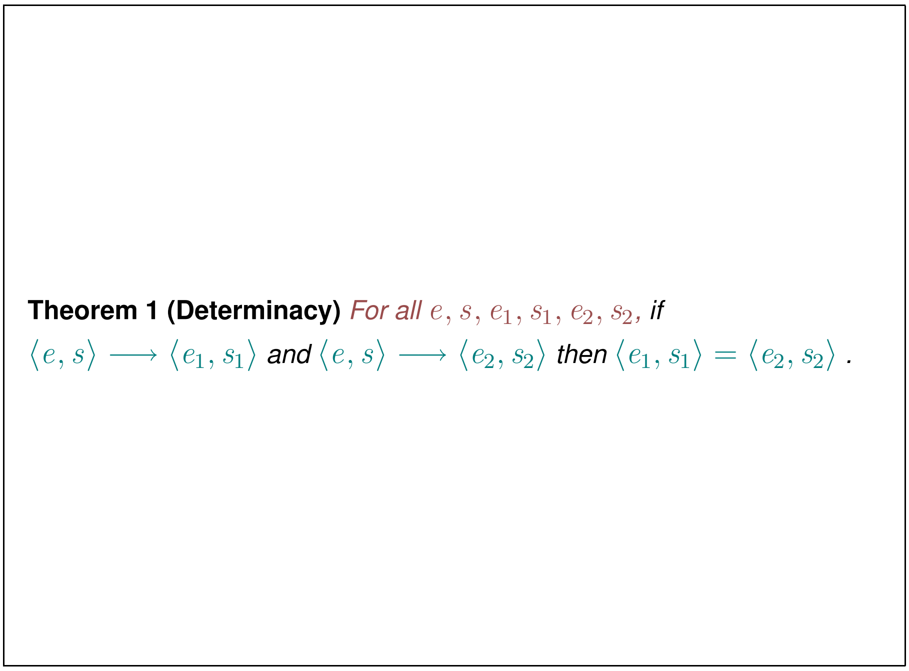
$\forall k \in \mathbb{N}.(\forall y \in \mathbb{N}.y < k \Rightarrow \Phi(y)) \Rightarrow \Phi(k)$.

Slide 61



Theorem 1 (Determinacy) *If* $\langle e, s\rangle \longrightarrow \langle e_1, s_1\rangle$ *and* $\langle e, s\rangle \longrightarrow \langle e_2, s_2\rangle$ *then* $\langle e_1, s_1\rangle = \langle e_2, s_2\rangle$.

Slide 62



Theorem 1 (Determinacy) *For all* e, s, e_1, s_1, e_2, s_2, *if* $\langle e, s\rangle \longrightarrow \langle e_1, s_1\rangle$ *and* $\langle e, s\rangle \longrightarrow \langle e_2, s_2\rangle$ *then* $\langle e_1, s_1\rangle = \langle e_2, s_2\rangle$.

3.1 Abstract Syntax and Structural Induction

Slide 63

Abstract Syntax

How to prove things for all expressions? First, have to pay attention to what an expression *is*.

Recall we said:

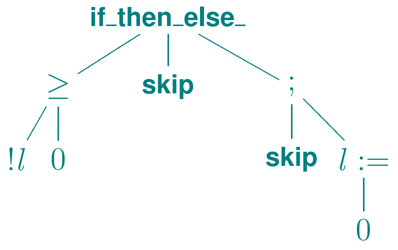
$$
\begin{aligned}
e \quad ::= \quad & n \mid b \mid e \; op \; e \mid \textbf{if } e \textbf{ then } e \textbf{ else } e \mid \\
& \ell := e \mid !\ell \mid \\
& \textbf{skip} \mid e; e \mid \\
& \textbf{while } e \textbf{ do } e
\end{aligned}
$$

defining a set of expressions.

Slide 64

Q: Is an expression, e.g. $\textbf{if } !l \geq 0 \textbf{ then skip else } (\textbf{skip}; l := 0)$:

1. a list of characters `` [`i', `f', `_', `!', `l', ..] ``;
2. a list of tokens `[ IF, DEREF, LOC "l", GTEQ, ..]`; or
3. an abstract syntax tree?



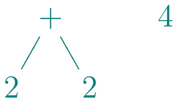
All those are (sometimes) useful ways of looking at them (for lexing and parsing you start with (1) and (2)), but for semantics we don't want to be distracted by concrete syntax – it's easiest to work with abstract syntax trees, which for this grammar are finite trees, with ordered children, labelled as follows:

- leaves (nullary nodes) labelled by $\mathbb{B} \cup \mathbb{Z} \cup (\{!\} * \mathbb{L}) \cup \{\textbf{skip}\} = \{\textbf{true}, \textbf{false}, \textbf{skip}\} \cup \{..., -1, 0, 1, ...\} \cup \{!l, !l_1, !l_2, ...\}$.
- unary nodes labelled by $\{l :=, l_1 :=, l_2 :=, ...\}$
- binary nodes labelled by $\{+, \geq, :=, ; , \textbf{while\_do\_}\}$
- ternary nodes labelled by $\{\textbf{if\_then\_else\_}\}$

Abstract grammar *suggests* a concrete syntax – we write expressions as strings just for convenience, using parentheses to disambiguate where required and infix/mixfix notation, but really mean trees.

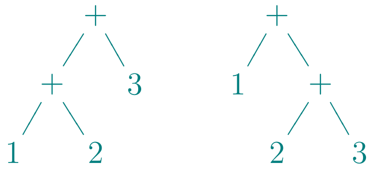
Slide 65

A: an abstract syntax tree. Hence: $2 + 2 \neq 4$



$1 + 2 + 3$ – ambiguous

$(1 + 2) + 3 \neq 1 + (2 + 3)$



Parentheses are only used for disambiguation – they are not part of the grammar. $1 + 2 = (1 + 2) = ((1 + 2)) = (((((1)))) + ((2)))$

Slide 66

Principle of Structural Induction (for abstract syntax)

For any property $\Phi(e)$ of expressions e, to prove

$\forall e \in L_1.\Phi(e)$

it's enough to prove for each tree constructor c (taking $k \geq 0$ arguments) that if Φ holds for the subtrees $e_1, .., e_k$ then Φ holds for the tree $c(e_1, .., e_k)$. i.e.

$$\boxed{\big(\forall c.\forall e_1, .., e_k.(\Phi(e_1) \wedge \ldots \wedge \Phi(e_k)) \Rightarrow \Phi(c(e_1, .., e_k))\big) \Rightarrow \forall e.\Phi(e)}$$

where the tree constructors (or node labels) c are n, **true**, **false**, $!l$, **skip**, $l :=$, **while_do_**, **if_then_else_**, etc.

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In particular, for L1: to show $\forall e \in L_1 . \Phi(e)$ it's enough to show:

nullary: $\Phi(\textbf{skip})$

$\forall b \in \{\textbf{true}, \textbf{false}\} . \Phi(b)$

$\forall n \in \mathbb{Z} . \Phi(n)$

$\forall \ell \in \mathbb{L} . \Phi(!\ell)$

unary: $\forall \ell \in \mathbb{L} . \forall e . \Phi(e) \Rightarrow \Phi(\ell := e)$

binary: $\forall\ op\ . \forall e_1, e_2 . (\Phi(e_1) \wedge \Phi(e_2)) \Rightarrow \Phi(e_1\ op\ e_2)$

$\forall e_1, e_2 . (\Phi(e_1) \wedge \Phi(e_2)) \Rightarrow \Phi(e_1 ; e_2)$

$\forall e_1, e_2 . (\Phi(e_1) \wedge \Phi(e_2)) \Rightarrow \Phi(\textbf{while}\ e_1\ \textbf{do}\ e_2)$

ternary: $\forall e_1, e_2, e_3 . (\Phi(e_1) \wedge \Phi(e_2) \wedge \Phi(e_3)) \Rightarrow \Phi(\textbf{if}\ e_1\ \textbf{then}\ e_2\ \textbf{else}\ e_3)$

(See how this comes directly from the grammar)

(compare with the Principal of Mathematical Induction for natural numbers, which we could view as abstract syntax trees of the grammar $n ::= \textbf{zero} \mid \textbf{succ}\ (n)$)

(in the other direction, instead of using structural induction, could do complete induction on the *size* of expressions)

Slide 68

Proving Determinacy

Theorem 1 (Determinacy) *If* $\langle e, s \rangle \longrightarrow \langle e_1, s_1 \rangle$ *and* $\langle e, s \rangle \longrightarrow \langle e_2, s_2 \rangle$ *then* $\langle e_1, s_1 \rangle = \langle e_2, s_2 \rangle$.

Take

$$\begin{aligned}\Phi(e) \overset{\text{def}}{=} {} & \forall s, e', s', e'', s'' . \\ & (\langle e, s \rangle \longrightarrow \langle e', s' \rangle \wedge \langle e, s \rangle \longrightarrow \langle e'', s'' \rangle) \\ & \Rightarrow \langle e', s' \rangle = \langle e'', s'' \rangle\end{aligned}$$

We show $\forall e \in L_1 . \Phi(e)$ by structural induction.

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$$\Phi(e) \stackrel{\text{def}}{=} \forall s, e', s', e'', s''. \; (\langle e, s\rangle \longrightarrow \langle e', s'\rangle \wedge \langle e, s\rangle \longrightarrow \langle e'', s''\rangle) \Rightarrow \langle e', s'\rangle = \langle e'', s''\rangle$$

nullary: $\Phi(\textbf{skip})$

$\forall b \in \{\textbf{true}, \textbf{false}\}.\Phi(b)$

$\forall n \in \mathbb{Z}.\Phi(n)$

$\forall \ell \in \mathbb{L}.\Phi(!\ell)$

unary: $\forall \ell \in \mathbb{L}.\forall e.\Phi(e) \Rightarrow \Phi(\ell := e)$

binary: $\forall \; op \; .\forall e_1, e_2.(\Phi(e_1) \wedge \Phi(e_2)) \Rightarrow \Phi(e_1 \; op \; e_2)$

$\forall e_1, e_2.(\Phi(e_1) \wedge \Phi(e_2)) \Rightarrow \Phi(e_1; e_2)$

$\forall e_1, e_2.(\Phi(e_1) \wedge \Phi(e_2)) \Rightarrow \Phi(\textbf{while } e_1 \textbf{ do } e_2)$

ternary: $\forall e_1, e_2, e_3.(\Phi(e_1) \wedge \Phi(e_2) \wedge \Phi(e_3)) \Rightarrow \Phi(\textbf{if } e_1 \textbf{ then } e_2 \textbf{ else } e_3)$

Slide 70

(op $+$) $\langle n_1 + n_2, s\rangle \longrightarrow \langle n, s\rangle$ if $n = n_1 + n_2$

(op $\geq$) $\langle n_1 \geq n_2, s\rangle \longrightarrow \langle b, s\rangle$ if $b = (n_1 \geq n_2)$

(op1) $\dfrac{\langle e_1, s\rangle \longrightarrow \langle e_1', s'\rangle}{\langle e_1 \; op \; e_2, s\rangle \longrightarrow \langle e_1' \; op \; e_2, s'\rangle}$

(op2) $\dfrac{\langle e_2, s\rangle \longrightarrow \langle e_2', s'\rangle}{\langle v \; op \; e_2, s\rangle \longrightarrow \langle v \; op \; e_2', s'\rangle}$

(deref) $\langle !\ell, s\rangle \longrightarrow \langle n, s\rangle$ if $\ell \in \mathrm{dom}(s)$ and $s(\ell) = n$

(assign1) $\langle \ell := n, s\rangle \longrightarrow \langle \textbf{skip}, s + \{\ell \mapsto n\}\rangle$ if $\ell \in \mathrm{dom}(s)$

(assign2) $\dfrac{\langle e, s\rangle \longrightarrow \langle e', s'\rangle}{\langle \ell := e, s\rangle \longrightarrow \langle \ell := e', s'\rangle}$

(seq1) $\langle \textbf{skip}; e_2, s\rangle \longrightarrow \langle e_2, s\rangle$

(seq2) $\dfrac{\langle e_1, s\rangle \longrightarrow \langle e_1', s'\rangle}{\langle e_1; e_2, s\rangle \longrightarrow \langle e_1'; e_2, s'\rangle}$

(if1) $\langle \textbf{if true then } e_2 \textbf{ else } e_3, s\rangle \longrightarrow \langle e_2, s\rangle$

(if2) $\langle \textbf{if false then } e_2 \textbf{ else } e_3, s\rangle \longrightarrow \langle e_3, s\rangle$

(if3) $\dfrac{\langle e_1, s\rangle \longrightarrow \langle e_1', s'\rangle}{\langle \textbf{if } e_1 \textbf{ then } e_2 \textbf{ else } e_3, s\rangle \longrightarrow \langle \textbf{if } e_1' \textbf{ then } e_2 \textbf{ else } e_3, s'\rangle}$

(while)
$\langle \textbf{while } e_1 \textbf{ do } e_2, s\rangle \longrightarrow \langle \textbf{if } e_1 \textbf{ then } (e_2; \textbf{while } e_1 \textbf{ do } e_2) \textbf{ else skip}, s\rangle$

Slide 71

$$\Phi(e) \overset{\text{def}}{=} \forall s, e', s', e'', s''. \\ (\langle e, s\rangle \longrightarrow \langle e', s'\rangle \wedge \langle e, s\rangle \longrightarrow \langle e'', s''\rangle) \\ \Rightarrow \langle e', s'\rangle = \langle e'', s''\rangle$$

(assign1) $\langle \ell := n, s\rangle \longrightarrow \langle \textbf{skip}, s + \{\ell \mapsto n\}\rangle$ if $\ell \in \mathrm{dom}(s)$

(assign2) $$\frac{\langle e, s\rangle \longrightarrow \langle e', s'\rangle}{\langle \ell := e, s\rangle \longrightarrow \langle \ell := e', s'\rangle}$$

Slide 72

Values don't reduce

It's handy to have this lemma:

Lemma 1 *For all $e \in L_1$, if e is a value then* $\forall s.\neg\exists e', s'.\langle e, s\rangle \longrightarrow \langle e', s'\rangle$.

Proof By defn e is a value if it is of one of the forms $n, b,$ **skip**. By examination of the rules on slides ..., there is no rule with conclusion of the form $\langle e, s\rangle \longrightarrow \langle e', s'\rangle$ for e one of $n, b,$ **skip**. □

Slide 73

All the other cases are in the notes.

Theorem 1 (Determinacy) *If* $\langle e, s\rangle \longrightarrow \langle e_1, s_1\rangle$ *and* $\langle e, s\rangle \longrightarrow \langle e_2, s_2\rangle$ *then* $\langle e_1, s_1\rangle = \langle e_2, s_2\rangle$.

Proof Take

$$\Phi(e) \stackrel{\text{def}}{=} \forall s, e', s', e'', s''.(\langle e, s\rangle \longrightarrow \langle e', s'\rangle \wedge \langle e, s\rangle \longrightarrow \langle e'', s''\rangle) \Rightarrow \langle e', s'\rangle = \langle e'', s''\rangle$$

We show $\forall e \in L_1.\Phi(e)$ by structural induction.

Cases skip, b, n**.** For e of these forms there are no rules with a conclusion of the form $\langle e, ...\rangle \longrightarrow \langle .., ..\rangle$ so the left hand side of the implication cannot hold, so the implication is true.

Case $!\ell$**.** Take arbitrary s, e', s', e'', s'' such that $\langle !\ell, s\rangle \longrightarrow \langle e', s'\rangle \wedge \langle !\ell, s\rangle \longrightarrow \langle e'', s''\rangle$.

The only rule which could be applicable is (deref), in which case, for those transitions to be instances of the rule we must have

$$\begin{array}{ll} \ell \in \mathrm{dom}(s) & \ell \in \mathrm{dom}(s) \\ e' = s(\ell) & e'' = s(\ell) \\ s' = s & s'' = s \end{array}$$

so $e' = e''$ and $s' = s''$.

Case $\ell := e$**.** Suppose $\Phi(e)$ (then we have to show $\Phi(\ell := e)$).

Take arbitrary s, e', s', e'', s'' such that $\langle \ell := e, s\rangle \longrightarrow \langle e', s'\rangle \wedge \langle \ell := e, s\rangle \longrightarrow \langle e'', s''\rangle$.

It's handy to have this lemma:

Lemma 2 *For all* $e \in L_1$*, if* e *is a value then* $\forall s.\neg\exists e', s'.\langle e, s\rangle \longrightarrow \langle e', s'\rangle$.

Proof By defn e is a value if it is of one of the forms $n, b,$ **skip**. By examination of the rules on slides ..., there is no rule with conclusion of the form $\langle e, s\rangle \longrightarrow \langle e', s'\rangle$ for e one of $n, b,$ **skip**. □

The only rules which could be applicable, for each of the two transitions, are (assign1) and (assign2).

case $\langle \ell := e, s\rangle \longrightarrow \langle e', s'\rangle$ is an instance of (assign1). Then for some n we have $e = n$ and $\ell \in \mathrm{dom}(s)$ and $e' =$ **skip** and $s' = s + \{\ell \mapsto n\}$.

case $\langle \ell := n, s\rangle \longrightarrow \langle e'', s''\rangle$ is an instance of (assign1) (note we are using the fact that $e = n$ here). Then $e'' =$ **skip** and $s'' = s + (\ell \mapsto n)$ so $\langle e', s'\rangle = \langle e'', s''\rangle$ as required.

case $\langle \ell := e, s\rangle \longrightarrow \langle e'', s''\rangle$ is an instance of (assign2). Then $\langle n, s\rangle \longrightarrow \langle e'', s''\rangle$, which contradicts the lemma, so this case cannot arise.

case $\langle \ell := e, s\rangle \longrightarrow \langle e', s'\rangle$ is an instance of (assign2). Then for some e_1' we have $\langle e, s\rangle \longrightarrow \langle e_1', s'\rangle$ (*) and $e' = (\ell := e_1')$.

case $\langle \ell := e, s\rangle \longrightarrow \langle e'', s''\rangle$ is an instance of (assign1). Then for some n we have $e = n$, which contradicts the lemma, so this case cannot arise.

case $\langle \ell := e, s\rangle \longrightarrow \langle e'', s''\rangle$ is an instance of (assign2). Then for some e_1'' we have $\langle e, s\rangle \longrightarrow \langle e_1'', s''\rangle$(**) and $e'' = (\ell := e_1'')$. Now, by the induction hypothesis $\Phi(e)$, (*) and (**) we have $\langle e_1', s'\rangle = \langle e_1'', s''\rangle$, so $\langle e', s'\rangle = \langle \ell := e_1', s'\rangle = \langle \ell := e_1'', s''\rangle = \langle e'', s''\rangle$ as required.

Case $e_1\ op\ e_2$. Suppose $\Phi(e_1)$ and $\Phi(e_2)$.

Take arbitrary s, e', s', e'', s'' such that $\langle e_1\ op\ e_2, s\rangle \longrightarrow \langle e', s'\rangle \wedge \langle e_1\ op\ e_2, s\rangle \longrightarrow \langle e'', s''\rangle$.

By examining the expressions in the left-hand-sides of the conclusions of the rules, and using the lemma above, the only possibilities are those below (you should check why this is so for yourself).

case $op = +$ and $\langle e_1 + e_2, s\rangle \longrightarrow \langle e', s'\rangle$ is an instance of (op+) and $\langle e_1 + e_2, s\rangle \longrightarrow \langle e'', s''\rangle$ is an instance of (op+).

Then for some n_1, n_2 we have $e_1 = n_1$, $e_2 = n_2$, $e' = n_3 = e''$ for $n_3 = n_1 + n_2$, and $s' = s = s''$.

case $op = \geq$ and $\langle e_1 \geq e_2, s\rangle \longrightarrow \langle e', s'\rangle$ is an instance of (op$\geq$) and $\langle e_1 \geq e_2, s\rangle \longrightarrow \langle e'', s''\rangle$ is an instance of (op$\geq$).

Then for some n_1, n_2 we have $e_1 = n_1$, $e_2 = n_2$, $e' = b = e''$ for $b = (n_1 \geq n_2)$, and $s' = s = s''$.

case $\langle e_1\ op\ e_2, s\rangle \longrightarrow \langle e', s'\rangle$ is an instance of (op1) and $\langle e_1\ op\ e_2, s\rangle \longrightarrow \langle e'', s''\rangle$ is an instance of (op1).

Then for some e_1' and e_1'' we have $\langle e_1, s\rangle \longrightarrow \langle e_1', s'\rangle$ (*), $\langle e_1, s\rangle \longrightarrow \langle e_1'', s''\rangle$ (**), $e' = e_1'\ op\ e_2$, and $e'' = e_1''\ op\ e_2$. Now, by the induction hypothesis $\Phi(e_1)$, (*) and (**) we have $\langle e_1', s'\rangle = \langle e_1'', s''\rangle$, so $\langle e', s'\rangle = \langle e_1'\ op\ e_2, s'\rangle = \langle e_1''\ op\ e_2, s''\rangle = \langle e'', s''\rangle$ as required.

case $\langle e_1\ op\ e_2, s\rangle \longrightarrow \langle e', s'\rangle$ is an instance of (op2) and $\langle e_1\ op\ e_2, s\rangle \longrightarrow \langle e'', s''\rangle$ is an instance of (op2).

Similar, save that we use the induction hypothesis $\Phi(e_2)$.

Case $e_1; e_2$. Suppose $\Phi(e_1)$ and $\Phi(e_2)$.

Take arbitrary s, e', s', e'', s'' such that $\langle e_1; e_2, s\rangle \longrightarrow \langle e', s'\rangle \wedge \langle e_1; e_2, s\rangle \longrightarrow \langle e'', s''\rangle$.

By examining the expressions in the left-hand-sides of the conclusions of the rules, and using the lemma above, the only possibilities are those below.

case $e_1 = \mathbf{skip}$ and both transitions are instances of (seq1).

Then $\langle e', s'\rangle = \langle e_2, s\rangle = \langle e'', s''\rangle$.

case e_1 is not a value and both transitions are instances of (seq2). Then for some e_1' and e_1'' we have $\langle e_1, s\rangle \longrightarrow \langle e_1', s'\rangle$ (*), $\langle e_1, s\rangle \longrightarrow \langle e_1'', s''\rangle$ (**), $e' = e_1'; e_2$, and $e'' = e_1''; e_2$

Then by the induction hypothesis $\Phi(e_1)$ we have $\langle e_1', s'\rangle = \langle e_1'', s''\rangle$, so $\langle e', s'\rangle = \langle e_1'; e_2, s'\rangle = \langle e_1''; e_2, s''\rangle = \langle e'', s''\rangle$ as required.

Case $\mathbf{while}\ e_1\ \mathbf{do}\ e_2$. Suppose $\Phi(e_1)$ and $\Phi(e_2)$.

Take arbitrary s, e', s', e'', s'' such that $\langle \mathbf{while}\ e_1\ \mathbf{do}\ e_2, s\rangle \longrightarrow \langle e', s'\rangle \wedge \langle \mathbf{while}\ e_1\ \mathbf{do}\ e_2, s\rangle \longrightarrow \langle e'', s''\rangle$.

By examining the expressions in the left-hand-sides of the conclusions of the rules both must be instances of (while), so $\langle e', s'\rangle = \langle \mathbf{if}\ e_1\ \mathbf{then}\ (e_2; \mathbf{while}\ e_1\ \mathbf{do}\ e_2)\ \mathbf{else}\ \mathbf{skip}, s\rangle = \langle e'', s''\rangle$.

Case $\mathbf{if}\ e_1\ \mathbf{then}\ e_2\ \mathbf{else}\ e_3$. Suppose $\Phi(e_1)$, $\Phi(e_2)$ and $\Phi(e_3)$.

Take arbitrary s, e', s', e'', s'' such that $\langle \mathbf{if}\ e_1\ \mathbf{then}\ e_2\ \mathbf{else}\ e_3, s\rangle \longrightarrow \langle e', s'\rangle \wedge \langle \mathbf{if}\ e_1\ \mathbf{then}\ e_2\ \mathbf{else}\ e_3, s\rangle \longrightarrow \langle e'', s''\rangle$.

By examining the expressions in the left-hand-sides of the conclusions of the rules, and using the lemma above, the only possibilities are those below.

case $e_1 =$ **true** and both transitions are instances of (if1).

case $e_1 =$ **false** and both transitions are instances of (if2).

case e_1 is not a value and both transitions are instances of (if3).

The first two cases are immediate; the last uses $\Phi(e_1)$.

□

(check we've done all the cases!)

(note that the level of written detail can vary, as here – if you and the reader agree – but you must do all the steps in your head. If in any doubt, write it down, as an aid to thought...!)

Slide 74

Interlude

Why do we spend all this effort proving obvious facts?

1. so you can see (and explain) *why* they are obvious
2. sometimes the obvious facts are false...
3. sometimes the obvious facts are not obvious at all
4. sometimes a proof contains or suggests an algorithm that you need – eg, proofs that type inference is decidable (for fancier type systems)

3.2 Inductive Definitions and Rule Induction

Slide 75

Theorem 2 (Progress) *If* $\Gamma \vdash e{:}T$ *and* $\mathit{dom}(\Gamma) \subseteq \mathit{dom}(s)$ *then either* e *is a value or there exist* e', s' *such that* $\langle e, s\rangle \longrightarrow \langle e', s'\rangle$*.*

Slide 76

Inductive Definitions

We defined the transition relation $\langle e, s\rangle \longrightarrow \langle e', s'\rangle$ and the typing relation $\Gamma \vdash e{:}T$ by giving some rules, eg

$$(\text{op }+)\quad \langle n_1 + n_2, s\rangle \longrightarrow \langle n, s\rangle \quad \text{if } n = n_1 + n_2$$

$$(\text{op1})\quad \frac{\langle e_1, s\rangle \longrightarrow \langle e_1', s'\rangle}{\langle e_1\ op\ e_2, s\rangle \longrightarrow \langle e_1'\ op\ e_2, s'\rangle}$$

What did we actually mean?

- for each rule we can make the set of concrete rule instances, taking all values of n_1, n_2, s, n (satisfying the side condition) for (op +) and all values of e_1, e_2, s, e_1', s' for (op1).

$$(\text{op}+)\quad \frac{}{\langle 2+2, \{\}\rangle \longrightarrow \langle 4, \{\}\rangle}\ , \qquad (\text{op }+)\quad \frac{}{\langle 2+3, \{\}\rangle \longrightarrow \langle 5, \{\}\rangle}\ , \ldots$$

$$(\text{op1})\quad \frac{\langle 2+2, \{\}\rangle \longrightarrow \langle 4, \{\}\rangle}{\langle (2+2)+3, \{\}\rangle \longrightarrow \langle 4+3, \{\}\rangle}\ , \qquad (\text{op1})\quad \frac{\langle 2+2, \{\}\rangle \longrightarrow \langle \mathbf{false}, \{\}\rangle}{\langle (2+2)+3, \{\}\rangle \longrightarrow \langle \mathbf{false}+3, \{\}\rangle}$$

  (note the premise of the second isn't going to be derivable - but that's still an instance of this rule)

- a *derivation* of a transition is a finite tree with nodes labelled by $\langle e, s\rangle \longrightarrow \langle e', s'\rangle$ such that each step *is* a concrete rule instance.

$$\dfrac{\dfrac{\dfrac{}{\langle 2+2, \{\}\rangle \longrightarrow \langle 4, \{\}\rangle}\ (\mathrm{op}+)}{\langle (2+2)+3, \{\}\rangle \longrightarrow \langle 4+3, \{\}\rangle}\ (\mathrm{op}1)}{\langle (2+2)+3 \geq 5, \{\}\rangle \longrightarrow \langle 4+3 \geq 5, \{\}\rangle}\ (\mathrm{op}1)$$

- we say $\langle e, s\rangle \longrightarrow \langle e', s'\rangle$ if there is a derivation with that as the root node.

More formally:

Slide 77

Inductive Definitions, Take 2

To make an *inductive definition* of a particular subset of a set A, take a set R of some concrete rules, each of which is a pair (H, c) where H is a finite subset of A – the hypotheses – and c is an element of A – the conclusion.

Consider finite trees labelled by elements of A for which every step is in R, eg

$$\dfrac{\dfrac{}{a_1} \qquad \dfrac{\dfrac{}{a_3}}{a_2}}{a_0}$$

where $(\{\}, a_1)$, $(\{\}, a_3)$, $(\{a_3\}, a_2)$, and $(\{a_1, a_2\}, a_0)$ all elements of R.

The subset S_R of A *inductively defined* by the rules R is the set of $a \in A$ such that there is such a proof with root node labelled by a.

Slide 78

For the definition of the transition relation:

- Start with $A = \mathrm{expr} * \mathrm{store} * \mathrm{expr} * \mathrm{store}$
- We define $\longrightarrow \subseteq A$
  (write infix, e.g.$\langle e, s\rangle \longrightarrow \langle e', s'\rangle$ instead of $(e, s, e', s') \in \longrightarrow$).
- The rules R are the concrete rule instances of the rules on slides 15,17, and 19.

For the definition of the typing relation:

- Start with $A = \mathrm{TypeEnv} * \mathrm{expr} * \mathrm{types}$.
- We define $\vdash \subseteq A$
  (write mixfix, e.g.$\Gamma \vdash e{:}T$ instead of $(\Gamma, e, T) \in \vdash$).
- The rules are the concrete rule instances of the rules on slides 42, 44, 45.

Even more formally (optionally):

Inductive Definitions, take 3

Given rules R, define $F_R{:}PA \to PA$ by

$$F_R(S) = \{c \mid \exists H.(H,c) \in R \wedge H \subseteq S\}$$

($F_R(S)$ is the set of all things you can derive in exactly one step from things in S)

$$\begin{array}{lcl} S_R^0 & = & \{\} \\ S_R^{k+1} & = & F_R(S_R^k) \\ S_R^\omega & = & \bigcap_{k\in\mathbb{N}} {S_R}^k \end{array}$$

Theorem 9 $S_R = S_R^\omega$.

Say a subset $S \subseteq A$ is *closed under rules* R if $\forall (H,c) \in R.(H \subseteq S) \Rightarrow c \in S$, ie, if $F_R(S) \subseteq S$.

Theorem 10 $S_R = \bigcap\{S \mid S \subseteq A \wedge F_R(S) \subseteq S\}$

'the subset S_R of A inductively defined by R is the smallest set closed under the rules R'

(intersection of all of them, so smaller than any of them)

Now, to prove something about an inductively-defined set...

Slide 79

Principle of Rule Induction

For any property $\Phi(a)$ of elements a of A, and any set of rules R which inductively define the set S_R, to prove

$$\forall a \in S_R.\Phi(a)$$

it's enough to prove that $\{a \mid \Phi(a)\}$ is closed under the rules, ie for each rule

$$\frac{\{h_1,..,h_k\}}{c}$$

if $\Phi(h_1) \wedge ... \wedge \Phi(h_k)$ then $\Phi(c)$.

$$\boxed{\big(\forall (H,c) \in R.(\forall h \in H.\Phi(h)) \Rightarrow \Phi(c)\big) \Rightarrow \forall a \in S_R.\Phi(a)}$$

(Optionally) To see why rule induction is sound, using the Take 3 definition above: Saying $\{a \mid \Phi(a)\}$ closed under the rules means exactly $F_R(\{a \mid \Phi(a)\}) \subseteq \{a \mid \Phi(a)\}$ so by Theorem 10 we have $S_R \subseteq \{a \mid \Phi(a)\}$ i.e.$\forall a \in S_R.a \in \{a' \mid \Phi(a')\}$ i.e.$\forall a \in S_R.\Phi(a)$.

Slide 80

Principle of rule induction (a slight variant)

For any property $\Phi(a)$ of elements a of A, and any set of rules R which inductively define the set S_R, to prove

$$\forall a \in S_R . \Phi(a)$$

it's enough to prove that

for each rule

$$\frac{\{h_1, .., h_k\}}{c}$$

if $\Phi(h_1) \wedge ... \wedge \Phi(h_k) \wedge h_1 \in S_R \wedge .. \wedge h_k \in S_R$ then $\Phi(c)$.

Slide 81

Theorem 2 (Progress) *If $\Gamma \vdash e{:}T$ and dom$(\Gamma) \subseteq$ dom(s) then either e is a value or there exist e', s' such that $\langle e, s \rangle \longrightarrow \langle e', s' \rangle$.*

Proof Take

$$\begin{aligned}\Phi(\Gamma, e, T) \overset{\text{def}}{=} \forall s.\ & \text{dom}(\Gamma) \subseteq \text{dom}(s) \Rightarrow \\ & \text{value}(e) \vee (\exists e', s'.\langle e, s \rangle \longrightarrow \langle e', s' \rangle)\end{aligned}$$

We show that for all Γ, e, T, if $\Gamma \vdash e{:}T$ then $\Phi(\Gamma, e, T)$, by rule induction on the definition of $\vdash$.

Slide 82

$$\boxed{\begin{aligned}\Phi(\Gamma, e, T) \overset{\text{def}}{=} \forall s.\ &\mathsf{dom}(\Gamma) \subseteq \mathsf{dom}(s) \Rightarrow \\ &\mathsf{value}(e) \vee (\exists e', s'.\langle e, s\rangle \longrightarrow \langle e', s'\rangle)\end{aligned}}$$

Case (op$+$). Recall the rule

$$(\mathsf{op}\ +)\quad \frac{\begin{array}{c}\Gamma \vdash e_1{:}\mathsf{int}\\ \Gamma \vdash e_2{:}\mathsf{int}\end{array}}{\Gamma \vdash e_1 + e_2{:}\mathsf{int}}$$

Suppose $\Phi(\Gamma, e_1, \mathsf{int})$, $\Phi(\Gamma, e_2, \mathsf{int})$, $\Gamma \vdash e_1{:}\mathsf{int}$, and $\Gamma \vdash e_2{:}\mathsf{int}$.

We have to show $\Phi(\Gamma, e_1 + e_2, \mathsf{int})$.

Consider an arbitrary s. Assume $\mathsf{dom}(\Gamma) \subseteq \mathsf{dom}(s)$.

Now $e_1 + e_2$ is not a value, so we have to show

$\exists \langle e', s'\rangle.\langle e_1 + e_2, s\rangle \longrightarrow \langle e', s'\rangle$.

Slide 83

Using $\Phi(\Gamma, e_1, \mathsf{int})$ and $\Phi(\Gamma, e_2, \mathsf{int})$ we have:

case e_1 reduces. Then $e_1 + e_2$ does, using (op1).

case e_1 is a value but e_2 reduces. Then $e_1 + e_2$ does, using (op2).

case Both e_1 and e_2 are values. Want to use:

$$\boxed{(\mathsf{op}\ +)\quad \langle n_1 + n_2, s\rangle \longrightarrow \langle n, s\rangle \quad \text{if } n = n_1 + n_2}$$

Lemma 3 *for all* Γ, e, T, *if* $\Gamma \vdash e{:}T$, e *is a value and* $T = \mathsf{int}$ *then for some* $n \in \mathbb{Z}$ *we have* $e = n$.

We assumed (the variant rule induction principle) that $\Gamma \vdash e_1{:}\mathsf{int}$ and $\Gamma \vdash e_2{:}\mathsf{int}$, so using this Lemma have $e_1 = n_1$ and $e_2 = n_2$.

Then $e_1 + e_2$ reduces, using rule (op$+$).

Theorem 2 (Progress) *If* $\Gamma \vdash e{:}T$ *and* $dom(\Gamma) \subseteq dom(s)$ *then either* e *is a value or there exist* e', s' *such that* $\langle e, s\rangle \longrightarrow \langle e', s'\rangle$.

Proof Take

$$\Phi(\Gamma, e, T) \overset{\text{def}}{=} \forall s.\mathrm{dom}(\Gamma) \subseteq \mathrm{dom}(s) \Rightarrow \mathrm{value}(e) \vee (\exists e', s'.\langle e, s\rangle \longrightarrow \langle e', s'\rangle)$$

We show that for all Γ, e, T, if $\Gamma \vdash e{:}T$ then $\Phi(\Gamma, e, T)$, by rule induction on the definition of $\vdash$.

Case (int). Recall the rule scheme

$$(\mathrm{int})\quad \Gamma \vdash n{:}\mathsf{int} \quad \text{for } n \in \mathbb{Z}$$

It has no premises, so we have to show that for all instances Γ, e, T of the conclusion we have $\Phi(\Gamma, e, T)$.

For any such instance, there must be an $n \in \mathbb{Z}$ for which $e = n$.

Now Φ is of the form $\forall s.\mathrm{dom}(\Gamma) \subseteq \mathrm{dom}(s) \Rightarrow ...$, so consider an arbitrary s and assume $\mathrm{dom}(\Gamma) \subseteq \mathrm{dom}(s)$.

We have to show $\mathrm{value}(e) \vee (\exists e', s'.\langle e, s\rangle \longrightarrow \langle e', s'\rangle)$. But the first disjunct is true as integers are values (according to the definition on Slide 14).

Case (bool) similar.

Case (op$+$). Recall the rule

$$(\text{op } +)\quad \frac{\begin{array}{c}\Gamma \vdash e_1{:}\mathsf{int}\\ \Gamma \vdash e_2{:}\mathsf{int}\end{array}}{\Gamma \vdash e_1 + e_2{:}\mathsf{int}}$$

We have to show that for all Γ, e_1, e_2, if $\Phi(\Gamma, e_1, \mathsf{int})$ and $\Phi(\Gamma, e_2, \mathsf{int})$ then $\Phi(\Gamma, e_1 + e_2, \mathsf{int})$.

Suppose $\Phi(\Gamma, e_1, \mathsf{int})$ (*), $\Phi(\Gamma, e_2, \mathsf{int})$ (**), $\Gamma \vdash e_1{:}\mathsf{int}$ (***), and $\Gamma \vdash e_2{:}\mathsf{int}$ (****) (note that we're using the variant form of rule induction here).

Consider an arbitrary s. Assume $\mathrm{dom}(\Gamma) \subseteq \mathrm{dom}(s)$.

We have to show $\mathrm{value}(e_1 + e_2) \vee (\exists e', s'.\langle e_1 + e_2, s\rangle \longrightarrow \langle e', s'\rangle)$.

Now the first disjunct is false ($e_1 + e_2$ is not a value), so we have to show the second, i.e.$\exists\langle e', s'\rangle.\langle e_1 + e_2, s\rangle \longrightarrow \langle e', s'\rangle$.

By (*) one of the following holds.

case $\exists e_1', s'.\langle e_1, s\rangle \longrightarrow \langle e_1', s'\rangle$.

Then by (op1) we have $\langle e_1 + e_2, s\rangle \longrightarrow \langle e_1' + e_2, s\rangle$, so we are done.

case e_1 is a value. By (**) one of the following holds.

case $\exists e_2', s'.\langle e_2, s\rangle \longrightarrow \langle e_2', s'\rangle$.

Then by (op3) $\langle e_1 + e_2, s\rangle \longrightarrow \langle e_1 + e_2', s\rangle$, so we are done.

case e_2 is a value.

(Now want to use (op$+$), but need to know that e_1 and e_2 are really integers.)

Lemma 4 *for all* Γ, e, T, *if* $\Gamma \vdash e{:}T$, e *is a value and* $T = \mathsf{int}$ *then for some* $n \in \mathbb{Z}$ *we have* $e = n$.

Proof By rule induction. Take $\Phi'(\Gamma, e, T) = ((\mathrm{value}(e) \wedge T = \mathsf{int}) \Rightarrow \exists n \in \mathbb{Z}.e = n)$.

Case (int). ok

Case (bool),(skip). In instances of these rules the conclusion is a value but the type is not int, so ok.

Case otherwise. In instances of all other rules the conclusion is not a value, so ok.

(a rather trivial use of rule induction – we never needed to use the induction hypothesis, just to do case analysis of the last rule that might have been used in a derivation of $\Gamma \vdash e{:}T$). □

Using the Lemma, (***) and (****) there exist $n_1 \in \mathbb{Z}$ and $n_2 \in \mathbb{Z}$ such that $e_1 = n_1$ and $e_2 = n_2$. Then by (op+) $\langle e_1 + e_2, s\rangle \longrightarrow \langle n, s\rangle$ where $n = n_1 + n_2$, so we are done.

Case (op $\geq$). Similar to (op $+$).

Case (if). Recall the rule

$$\text{(if)}\ \frac{\Gamma \vdash e_1{:}\mathsf{bool} \quad \Gamma \vdash e_2{:}T \quad \Gamma \vdash e_3{:}T}{\Gamma \vdash \textbf{if}\ e_1\ \textbf{then}\ e_2\ \textbf{else}\ e_3{:}T}$$

Suppose $\Phi(\Gamma, e_1, \mathsf{bool})$ (*1), $\Phi(\Gamma, e_2, T)$ (*2), $\Phi(\Gamma, e_3, T)$ (*3), $\Gamma \vdash e_1{:}\mathsf{bool}$ (*4), $\Gamma \vdash e_2{:}T$ (*5) and $\Gamma \vdash e_3{:}T$ (*6).

Consider an arbitrary s. Assume $\mathrm{dom}(\Gamma) \subseteq \mathrm{dom}(s)$. Write e for $\textbf{if}\ e_1\ \textbf{then}\ e_2\ \textbf{else}\ e_3$.

This e is not a value, so we have to show $\langle e, s\rangle$ has a transition.

case $\exists e_1', s'.\langle e_1, s\rangle \longrightarrow \langle e_1', s'\rangle$.

Then by (if3) $\langle e, s\rangle \longrightarrow \langle \textbf{if}\ e_1'\ \textbf{then}\ e_2\ \textbf{else}\ e_3, s\rangle$, so we are done.

case e_1 is a value.

(Now want to use (if1) or (if2), but need to know that $e_1 \in \{\textbf{true}, \textbf{false}\}$. Realise should have proved a stronger Lemma above).

Lemma 5 *For all* Γ, e, T. *if* $\Gamma \vdash e{:}T$ *and* e *is a value, then* $T = \mathsf{int} \Rightarrow \exists n \in \mathbb{Z}.e = n$, $T = \mathsf{bool} \Rightarrow \exists b \in \{\textbf{true}, \textbf{false}\}.e = b$, *and* $T = \mathsf{unit} \Rightarrow e = \textbf{skip}$.

Proof By rule induction – details omitted. □

Using the Lemma and (*4) we have $\exists b \in \{\textbf{true}, \textbf{false}\}.e_1 = b$.

case $b = \textbf{true}$. Use (if1).

case $b = \textbf{false}$. Use (if2).

Case (deref). Recall the rule

$$\text{(deref)}\ \frac{\Gamma(\ell) = \mathsf{intref}}{\Gamma \vdash !\ell{:}\mathsf{int}}$$

(This is a leaf – it has no $\Gamma \vdash e{:}T$ premises - so no Φs to assume).

Consider an arbitrary s with $\mathrm{dom}(\Gamma) \subseteq \mathrm{dom}(s)$.

By the condition $\Gamma(\ell) = \mathsf{intref}$ we have $\ell \in \mathrm{dom}(\Gamma)$, so $\ell \in \mathrm{dom}(s)$, so there is some n with $s(\ell) = n$, so there is an instance of (deref) $\langle !\ell, s\rangle \longrightarrow \langle n, s\rangle$.

Cases (assign), (skip), (seq), (while). Left as an exercise.

□

Theorem 3 (Type Preservation) *If* $\Gamma \vdash e{:}T$ *and* $\mathit{dom}(\Gamma) \subseteq \mathit{dom}(s)$ *and* $\langle e, s\rangle \longrightarrow \langle e', s'\rangle$ *then* $\Gamma \vdash e'{:}T$ *and* $\mathit{dom}(\Gamma) \subseteq \mathit{dom}(s')$.

Proof First show the second part, using the following lemma.

Lemma 6 *If* $\langle e, s\rangle \longrightarrow \langle e', s'\rangle$ *then* $\mathit{dom}(s') = \mathit{dom}(s)$.

Proof Rule induction on derivations of $\langle e, s\rangle \longrightarrow \langle e', s'\rangle$. Take $\Phi(e, s, e', s') = (\mathrm{dom}(s) = \mathrm{dom}(s'))$.

All rules are immediate uses of the induction hypothesis except (assign1), for which we note that if $\ell \in \mathrm{dom}(s)$ then $\mathrm{dom}(s + (\ell \mapsto n)) = \mathrm{dom}(s)$. □

Now prove the first part, ie If $\Gamma \vdash e{:}T$ and $\mathrm{dom}(\Gamma) \supseteq \mathrm{dom}(s)$ and $\langle e, s\rangle \longrightarrow \langle e', s'\rangle$ then $\Gamma \vdash e'{:}T$.

Prove by rule induction on derivations of $\langle e, s\rangle \longrightarrow \langle e', s'\rangle$.

Take $\Phi(e, s, e', s') = \forall \Gamma, T.(\Gamma \vdash e{:}T \wedge \mathrm{dom}(\Gamma) \subseteq \mathrm{dom}(s)) \Rightarrow \Gamma \vdash e'{:}T$.

Case (op+). Recall

$$(\text{op} +) \quad \langle n_1 + n_2, s\rangle \longrightarrow \langle n, s\rangle \quad \text{if } n = n_1 + n_2$$

Take arbitrary Γ, T. Suppose $\Gamma \vdash n_1 + n_2{:}T$ (*) and $\mathrm{dom}(\Gamma) \subseteq \mathrm{dom}(s)$. The last rule in the derivation of (*) must have been (op+), so must have $T = \mathsf{int}$. Then can use (int) to derive $\Gamma \vdash n{:}T$.

Case (op $\geq$). Similar.

Case (op1). Recall

$$(\text{op1}) \quad \frac{\langle e_1, s\rangle \longrightarrow \langle e_1', s'\rangle}{\langle e_1 \ op \ e_2, s\rangle \longrightarrow \langle e_1' \ op \ e_2, s'\rangle}$$

Suppose $\Phi(e_1, s, e_1', s')$ (*) and $\langle e_1, s\rangle \longrightarrow \langle e_1', s'\rangle$. Have to show $\Phi(e_1 \ op \ e_2, s, e_1' \ op \ e_2, s')$. Take arbitrary Γ, T. Suppose $\Gamma \vdash e_1 \ op \ e_2{:}T$ and $\mathrm{dom}(\Gamma) \subseteq \mathrm{dom}(\Gamma)$ (**).

case $op \ = +$. The last rule in the derivation of $\Gamma \vdash e_1 + e_2{:}T$ must have been (op+), so must have $T = \mathsf{int}$, $\Gamma \vdash e_1{:}\mathsf{int}$ (***) and $\Gamma \vdash e_2{:}\mathsf{int}$ (****). By the induction hypothesis (*), (**), and (***) we have $\Gamma \vdash e_1'{:}\mathsf{int}$. By the (op+) rule $\Gamma \vdash e_1' + e_2{:}T$.

case $op \ =\geq$. Similar.

Case s (op2) (deref), (assign1), (assign2), (seq1), (seq2), (if1), (if2), (if3), (while). Left as exercises.

□

Theorem 4 (Safety) *If $\Gamma \vdash e{:}T$, $dom(\Gamma) \subseteq dom(s)$, and $\langle e, s\rangle \longrightarrow^* \langle e', s'\rangle$ then either e' is a value or there exist e'', s'' such that $\langle e', s'\rangle \longrightarrow \langle e'', s''\rangle$.*

Proof Hint: induction along $\longrightarrow^*$ using the previous results. □

Theorem 7 (Uniqueness of typing) *If $\Gamma \vdash e{:}T$ and $\Gamma \vdash e{:}T'$ then $T = T'$.* The proof is left as Exercise 18.

Theorem 5 (Decidability of typeability) *Given Γ, e, one can decide $\exists T.\Gamma \vdash e{:}T$.*

Theorem 6 (Decidability of type checking) *Given Γ, e, T, one can decide $\Gamma \vdash e{:}T$.*

Proof The implementation gives a type inference algorithm, which, *if correct*, and together with Uniqueness, implies both of these results. □

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Structural Induction and Rule Induction – Recap

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When is a proof a proof?

What's a proof?

Formal: a derivation in formal logic (e.g. a big natural deduction proof tree). Often far too verbose to deal with by hand (but can *machine-check* such things).

Informal but rigorous: an argument to persuade the reader that, if pushed, you could write a fully formal proof (the usual mathematical notion, e.g. those we just did). Have to learn by practice to see when they are rigorous.

Bogus: neither of the above.

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Summarising Proof Techniques

Determinacy	structural induction for e
Progress	rule induction for $\Gamma \vdash e{:}T$
Type Preservation	rule induction for $\langle e, s \rangle \longrightarrow \langle e', s' \rangle$
Safety	mathematical induction on $\longrightarrow^k$
Uniqueness of typing	...
Decidability of typability	exhibiting an algorithm
Decidability of checking	corollary of other results

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Proving Determinacy

Theorem 1 (Determinacy) *If* $\langle e, s \rangle \longrightarrow \langle e_1, s_1 \rangle$ *and* $\langle e, s \rangle \longrightarrow \langle e_2, s_2 \rangle$ *then* $\langle e_1, s_1 \rangle = \langle e_2, s_2 \rangle$.

Take

$$\begin{aligned} \Phi(e) \quad \overset{\text{def}}{=} \quad & \forall s, e', s', e'', s''. \\ & (\langle e, s \rangle \longrightarrow \langle e', s' \rangle \wedge \langle e, s \rangle \longrightarrow \langle e'', s'' \rangle) \\ & \Rightarrow \langle e', s' \rangle = \langle e'', s'' \rangle \end{aligned}$$

We show $\forall e \in L_1 . \Phi(e)$ by structural induction.

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Principle of Structural Induction: to prove $\Phi(e)$ for all expressions e of L1, it's enough to prove for each tree constructor c that if Φ holds for the subtrees $e_1, .., e_k$ then Φ holds for the tree $c(e_1, .., e_k)$.

Instantating to the L1 grammar, have to show:

nullary:	$\Phi(\textbf{skip})$
	$\forall b \in \{\textbf{true}, \textbf{false}\}.\Phi(b)$
	$\forall n \in \mathbb{Z}.\Phi(n)$
	$\forall \ell \in \mathbb{L}.\Phi(!\ell)$
unary:	$\forall \ell \in \mathbb{L}.\forall e.\Phi(e) \Rightarrow \Phi(\ell := e)$
binary:	$\forall\ op\ .\forall e_1, e_2.(\Phi(e_1) \wedge \Phi(e_2)) \Rightarrow \Phi(e_1\ op\ e_2)$
	$\forall e_1, e_2.(\Phi(e_1) \wedge \Phi(e_2)) \Rightarrow \Phi(e_1; e_2)$
	$\forall e_1, e_2.(\Phi(e_1) \wedge \Phi(e_2)) \Rightarrow \Phi(\textbf{while}\ e_1\ \textbf{do}\ e_2)$
ternary:	$\forall e_1, e_2, e_3.(\Phi(e_1) \wedge \Phi(e_2) \wedge \Phi(e_3)) \Rightarrow \Phi(\textbf{if}\ e_1\ \textbf{then}\ e_2\ \textbf{else}\ e_3)$

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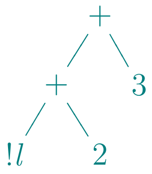
Having proved those 9 things, consider an example $(!l + 2) + 3$. To see why $\Phi((!l + 2) + 3)$ holds:



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Proving Progress

Theorem 2 (Progress) *If* $\Gamma \vdash e{:}T$ *and* $\mathit{dom}(\Gamma) \subseteq \mathit{dom}(s)$ *then either* e *is a value or there exist* e', s' *such that* $\langle e, s\rangle \longrightarrow \langle e', s'\rangle$.

Proof Take

$$\Phi(\Gamma, e, T) \overset{\text{def}}{=} \forall s.\ \mathsf{dom}(\Gamma) \subseteq \mathsf{dom}(s) \Rightarrow \\ \mathsf{value}(e) \vee (\exists e', s'.\langle e, s\rangle \longrightarrow \langle e', s'\rangle)$$

We show that for all Γ, e, T, if $\Gamma \vdash e{:}T$ then $\Phi(\Gamma, e, T)$, by rule induction on the definition of $\vdash$.

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Principle of Rule Induction (variant form): to prove $\Phi(a)$ for all a in the set S_R (defined by rules R), it's enough to prove that for each rule

$$\frac{\{h_1, .., h_k\}}{c}$$

if $\Phi(h_1) \wedge \ldots \wedge \Phi(h_k) \wedge h_1 \in S_R \wedge .. \wedge h_k \in S_R$ then $\Phi(c)$.
Instantiating to the L1 typing rules, have to show:

(int) $\forall \Gamma, n.\Phi(\Gamma, n, \mathsf{int})$

(deref) $\forall \Gamma, \ell.\Gamma(\ell) = \mathsf{intref} \Rightarrow \Phi(\Gamma, !\ell, \mathsf{int})$

(op $+$) $\forall \Gamma, e_1, e_2.(\Phi(\Gamma, e_1, \mathsf{int}) \wedge \Phi(\Gamma, e_2, \mathsf{int}) \wedge \Gamma \vdash e_1{:}\mathsf{int} \wedge \Gamma \vdash e_2{:}\mathsf{int})$
$\Rightarrow \Phi(\Gamma, e_1 + e_2, \mathsf{int})$

(seq) $\forall \Gamma, e_1, e_2, T.(\Phi(\Gamma, e_1, \mathsf{unit}) \wedge \Phi(\Gamma, e_2, T) \wedge \Gamma \vdash e_1{:}\mathsf{unit} \wedge \Gamma \vdash e_2{:}T)$
$\Rightarrow \Phi(\Gamma, e_1; e_2, T)$

etc.

(can also instantiate to the L1 reduction rules...)

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Having proved those 10 things, consider an example $\Gamma \vdash (!l + 2) + 3{:}\mathsf{int}$. To see why $\Phi(\Gamma,\ (!l+2)+3,\ \mathsf{int})$ holds:

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash !l{:}\mathsf{int}}\ \text{(deref)} \qquad \dfrac{}{\Gamma \vdash 2{:}\mathsf{int}}\ \text{(int)}}{\Gamma \vdash (!l+2){:}\mathsf{int}}\ \text{(op }+\text{)} \qquad \dfrac{}{\Gamma \vdash 3{:}\mathsf{int}}\ \text{(int)}}{\Gamma \vdash (!l+2)+3{:}\mathsf{int}}\ \text{(op }+\text{)}$$

Which of these induction principles to use is a matter of convenience – you want to use an induction principle that matches the definitions you're working with. That's not a hard constraint, though – for completeness, observe the following:

Mathematical induction over $\mathbb{N}$ is equivalent to complete induction over $\mathbb{N}$.

Mathematical induction over $\mathbb{N}$ is essentially the same as structural induction over $n ::= \textbf{zero} \mid \textbf{succ}\ (n)$.

Instead of using structural induction (for an arbitrary grammar), you could use complete induction on the *size* of terms.

Instead of using structural induction, you could use rule induction: supposing some fixed set of tree node labels (e.g. all the character strings), take A to be the set of all trees with those labels, and consider each clause of your grammar (e.g.$e ::= \ldots \mid e + e$) to be a rule

$$\frac{e \quad e}{e + e}$$

3.3 Exercises

Exercise 14 ★*Without looking at the proof in the notes, do the cases of the proof of Theorem 1 (Determinacy) for e_1 op e_2, $e_1; e_2$,* **while** e_1 **do** e_2, *and* **if** e_1 **then** e_2 **else** e_3.

Exercise 15 ★*Try proving Determinacy for the language with nondeterministic order of evaluation for e_1 op e_2 (ie with both (op1) and (op1b) rules), which is* not *determinate. Explain where exactly the proof can't be carried through.*

Exercise 16 ★*Complete the proof of Theorem 2 (Progress).*

Exercise 17 ★★*Complete the proof of Theorem 3 (Type Preservation).*

Exercise 18 ★★*Prove Theorem 7 (Uniqueness of Typing).*

4 Functions

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Functions – L2

Most languages have some kind of function, method, procedure, or what have you – some way of abstracting a piece of code on a formal parameter, so that you can use the code multiple times with different arguments, without having to duplicate the code in the source. The next two lectures explore the design space, adding functions to L1.

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Functions – Syntax

Going to add expressions like these

$$(\mathbf{fn}\ \mathrm{x}{:}\mathsf{int} \Rightarrow \mathrm{x} + 7)$$

$$(\mathbf{fn}\ \mathrm{x}{:}\mathsf{int} \rightarrow \mathsf{int} \Rightarrow (\mathbf{fn}\ \mathrm{y}{:}\mathsf{int} \Rightarrow \mathrm{x}\ \mathrm{y}))$$

So, add to the syntax of L1:

Variables $x \in \mathbb{X}$ for a set $\mathbb{X} = \{\mathrm{x}, \mathrm{y}, \mathrm{z}, ...\}$

Expressions

$$e \quad ::= \quad ... \mid \mathbf{fn}\ x{:}T \Rightarrow e \mid e_1\ e_2 \mid x$$

Types

$$\begin{array}{lll} T & ::= & \mathsf{int} \mid \mathsf{bool} \mid \mathsf{unit} \mid T_1 \rightarrow T_2 \\ T_{loc} & ::= & \mathsf{intref} \end{array}$$

Note the difference between metavariables x, y, z and non-meta (i.e., L2) variables $\mathrm{x}, \mathrm{y}, \mathrm{z}$. In the notes they are distinguished by font; in handwriting one just have to keep track in your head – not often a problem.

These are *anonymous* functions only.

They look like lambda terms ($\mathbf{fn}\ \mathrm{x}{:}\mathsf{int} \Rightarrow \mathrm{x}$ could be written $\lambda\mathrm{x}{:}\mathsf{int}.\mathrm{x}$). But, (a) we're adding them to a rich language, not working with pure lambda, and (b) we're going to explore several options for how they should behave.

Concrete syntax: by convention, application associates to the left, so $e_1\ e_2\ e_3$ denotes $(e_1\ e_2)\ e_3$, and type arrows associate to the right, so $T_1 \to T_2 \to T_3$ denotes $T_1 \to (T_2 \to T_3)$. A **fn** extends to the right as far as parentheses permit, so **fn** x:unit $\Rightarrow \mathrm{x};\mathrm{x}$ denotes **fn** x:unit $\Rightarrow (\mathrm{x};\mathrm{x})$, not (**fn** x:unit $\Rightarrow \mathrm{x});\mathrm{x}$.

4.1 Abstract Syntax up to Alpha Conversion; Substitution

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Alpha conversion

In expressions **fn** $x{:}T \Rightarrow e$ the x is a *binder*.

- inside e, any x's (that aren't themselves binders and are not inside another **fn** $x{:}T' \Rightarrow ...$) mean the same thing – the formal parameter of this function.
- outside this **fn** $x{:}T \Rightarrow e$, it doesn't matter which variable we used for the formal parameter – in fact, we shouldn't be able to tell. For example, **fn** x:int $\Rightarrow \mathrm{x} + 2$ should be the same as **fn** y:int $\Rightarrow \mathrm{y} + 2$. cf $\int_0^1 x + x^2 dx = \int_0^1 y + y^2 dy$

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Alpha conversion – free and bound occurrences

In a bit more detail (but still informally):

Say an occurrence of x in an expression e is *free* if it is not inside any (**fn** $x{:}T \Rightarrow ...$). For example:

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$\mathrm{x} + \mathrm{y}$

fn x:int $\Rightarrow \mathrm{x} + 2$

fn x:int $\Rightarrow \mathrm{x} + \mathrm{z}$

if y **then** $2 + \mathrm{x}$ **else** ((**fn** x:int $\Rightarrow \mathrm{x} + 2)\mathrm{z}$)

All the other occurrences of x are *bound* by the closest enclosing **fn** $x{:}T \Rightarrow ...$.

Note that in **fn** x:int $\Rightarrow 2$ the x is not an occurrence... likewise, in **fn** x:int $\Rightarrow \mathrm{x} + 2$ the left x is not an occurrence; the right x is an occurrence that is bound by the left x.

Sometimes handy to draw in the binding:

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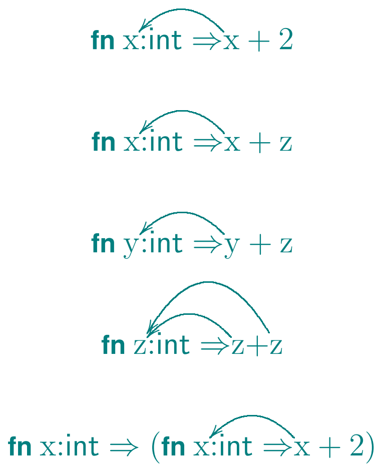
Alpha conversion – Binding examples



$\textbf{fn}\ \mathrm{x}{:}\mathtt{int} \Rightarrow \mathrm{x} + 2$

$\textbf{fn}\ \mathrm{x}{:}\mathtt{int} \Rightarrow \mathrm{x} + \mathrm{z}$

$\textbf{fn}\ \mathrm{y}{:}\mathtt{int} \Rightarrow \mathrm{y} + \mathrm{z}$

$\textbf{fn}\ \mathrm{z}{:}\mathtt{int} \Rightarrow \mathrm{z}{+}\mathrm{z}$

$\textbf{fn}\ \mathrm{x}{:}\mathtt{int} \Rightarrow (\textbf{fn}\ \mathrm{x}{:}\mathtt{int} \Rightarrow \mathrm{x} + 2)$

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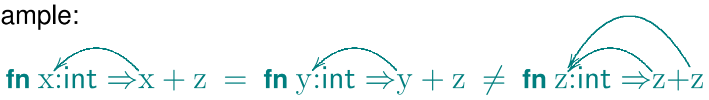
Alpha Conversion – The Convention

Convention: we will allow ourselves to *any time at all, in any expression* $\ldots(\textbf{fn}\ x{:}T \Rightarrow e)\ldots$, replace the binding x and all occurrences of x that are bound by that binder, by any other variable – so long as that doesn't change the binding graph.

For example:



$\textbf{fn}\ \mathrm{x}{:}\mathtt{int} \Rightarrow \mathrm{x} + \mathrm{z} \ = \ \textbf{fn}\ \mathrm{y}{:}\mathtt{int} \Rightarrow \mathrm{y} + \mathrm{z} \ \neq \ \textbf{fn}\ \mathrm{z}{:}\mathtt{int} \Rightarrow \mathrm{z}{+}\mathrm{z}$

This is called 'working up to alpha conversion'. It amounts to regarding the syntax not as abstract syntax trees, but as abstract syntax trees with pointers...

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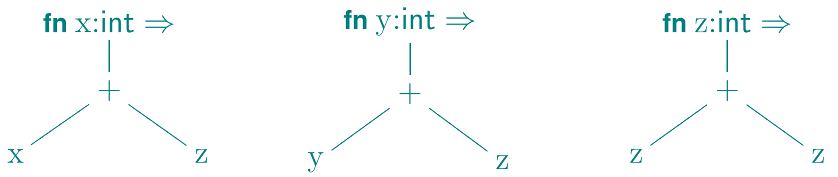
Abstract Syntax *up to Alpha Conversion*

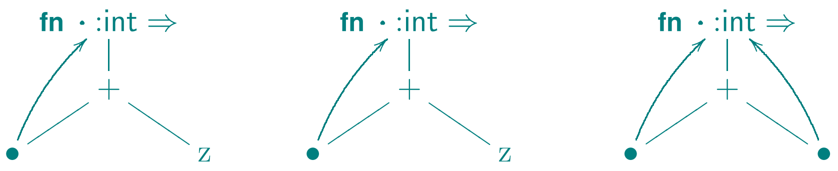
fn x:int $\Rightarrow \mathrm{x}+\mathrm{z} \quad = \quad$ **fn** y:int $\Rightarrow \mathrm{y}+\mathrm{z} \quad \neq \quad$ **fn** z:int $\Rightarrow \mathrm{z}+\mathrm{z}$

Start with naive abstract syntax trees:



add pointers (from each x node to the closest enclosing **fn** $x{:}T \Rightarrow$ node);

remove names of binders and the occurrences they bind



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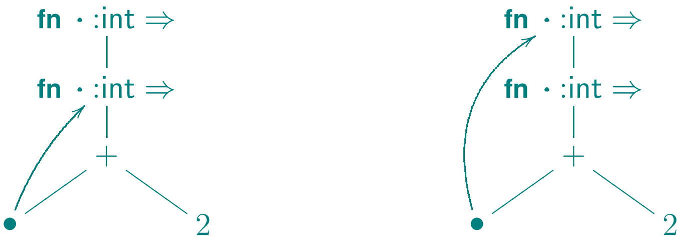
fn x:int $\Rightarrow ($**fn** x:int $\Rightarrow \mathrm{x}+2)$

$=$ **fn** y:int $\Rightarrow ($**fn** z:int $\Rightarrow \mathrm{z}+2) \quad \neq \quad$ **fn** z:int $\Rightarrow ($**fn** y:int $\Rightarrow \mathrm{z}+2)$



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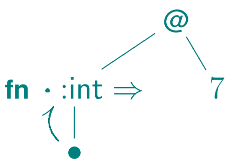
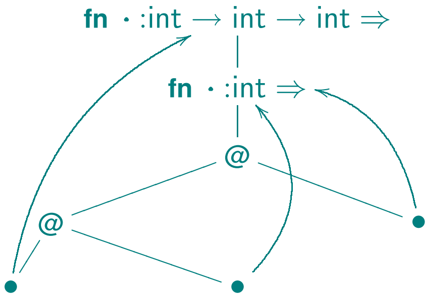
$(\textbf{fn}\ \mathrm{x}{:}\mathsf{int} \Rightarrow \mathrm{x})\ 7 \qquad \textbf{fn}\ \mathrm{z}{:}\mathsf{int} \to \mathsf{int} \to \mathsf{int} \Rightarrow (\textbf{fn}\ \mathrm{y}{:}\mathsf{int} \Rightarrow \mathrm{z}\ \mathrm{y}\ \mathrm{y})$





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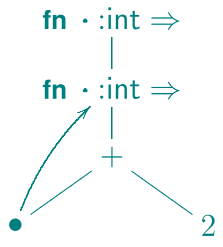
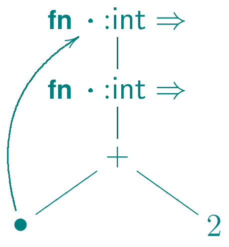
De Bruijn Indices

Our implementation will use those pointers – known as *De Bruijn Indices*. Each occurrence of a bound variable is represented by the number of $\textbf{fn}\ \cdot : T \Rightarrow$ nodes you have to count out to to get to its binder.

$$\textbf{fn}\ \cdot{:}\mathsf{int} \Rightarrow (\textbf{fn}\ \cdot{:}\mathsf{int} \Rightarrow \mathrm{v}_0 + 2) \neq \textbf{fn}\ \cdot{:}\mathsf{int} \Rightarrow (\textbf{fn}\ \cdot{:}\mathsf{int} \Rightarrow \mathrm{v}_1 + 2)$$





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Free Variables

Say the *free variables* of an expression e are the set of variables x for which there is an occurence of x free in e.

$$\begin{array}{lcl}
\mathsf{fv}(x) & = & \{x\} \\
\mathsf{fv}(e_1\ op\ e_2) & = & \mathsf{fv}(e_1) \cup \mathsf{fv}(e_2) \\
\mathsf{fv}(\mathbf{fn}\ x{:}T \Rightarrow e) & = & \mathsf{fv}(e) - \{x\}
\end{array}$$

Say e is *closed* if $\mathsf{fv}(e) = \{\}$.

If E is a set of expressions, write $\mathsf{fv}(E)$ for $\bigcup_{e \in E} \mathsf{fv}(e)$.

(note this definition is alpha-invariant - all our definitions should be)

For example

$$\begin{array}{lcl}
\mathrm{fv}(\mathrm{x} + \mathrm{y}) & = & \{\mathrm{x}, \mathrm{y}\} \\
\mathrm{fv}(\mathbf{fn}\ \mathrm{x}{:}\mathsf{int} \Rightarrow \mathrm{x} + \mathrm{y}) & = & \{\mathrm{y}\} \\
\mathrm{fv}(\mathrm{x} + (\mathbf{fn}\ \mathrm{x}{:}\mathsf{int} \Rightarrow \mathrm{x} + \mathrm{y})7) & = & \{\mathrm{x}, \mathrm{y}\}
\end{array}$$

Full definition of $\mathrm{fv}(e)$:

$$\begin{array}{lcl}
\mathrm{fv}(x) & = & \{x\} \\
\mathrm{fv}(\mathbf{fn}\ x{:}T \Rightarrow e) & = & \mathrm{fv}(e) - \{x\} \\
\mathrm{fv}(e_1\, e_2) & = & \mathrm{fv}(e_1) \cup \mathrm{fv}(e_2) \\
\mathrm{fv}(n) & = & \{\} \\
\mathrm{fv}(e_1\ op\ e_2) & = & \mathrm{fv}(e_1) \cup \mathrm{fv}(e_2) \\
\mathrm{fv}(\mathbf{if}\ e_1\ \mathbf{then}\ e_2\ \mathbf{else}\ e_3) & = & \mathrm{fv}(e_1) \cup \mathrm{fv}(e_2) \cup \mathrm{fv}(e_3) \\
\mathrm{fv}(b) & = & \{\} \\
\mathrm{fv}(\mathbf{skip}) & = & \{\} \\
\mathrm{fv}(\ell := e) & = & \mathrm{fv}(e) \\
\mathrm{fv}(!\ell) & = & \{\} \\
\mathrm{fv}(e_1; e_2) & = & \mathrm{fv}(e_1) \cup \mathrm{fv}(e_2) \\
\mathrm{fv}(\mathbf{while}\ e_1\ \mathbf{do}\ e_2) & = & \mathrm{fv}(e_1) \cup \mathrm{fv}(e_2)
\end{array}$$

(for an example of a definition that is *not* alpha-invariant, consider

$$\begin{array}{lcl}
\mathrm{bv}(x) & = & \{\} \\
\mathrm{bv}(\mathbf{fn}\ x{:}T \Rightarrow e) & = & \{x\} \cup \mathrm{bv}(e) \\
\mathrm{bv}(e_1\, e_2) & = & \mathrm{bv}(e_1) \cup \mathrm{bv}(e_2) \\
\ldots & &
\end{array}$$

This is fine for concrete terms, but we're working up to alpha conversion, so $(\mathbf{fn}\ \mathrm{x}{:}\mathsf{int} \Rightarrow 2) = (\mathbf{fn}\ \mathrm{y}{:}\mathsf{int} \Rightarrow 2)$ but $\mathrm{bv}(\mathbf{fn}\ \mathrm{x}{:}\mathsf{int} \Rightarrow 2) = \{\mathrm{x}\} \neq \{\mathrm{y}\} = \mathrm{bv}(\mathbf{fn}\ \mathrm{y}{:}\mathsf{int} \Rightarrow 2)$. Argh! Can't be having that. Can see from looking back at the abstract syntax trees up to alpha conversion that they just don't have this information in, anyway.)

The semantics for functions will involve substituting actual parameters for formal parameters. That's a bit delicate in a world with binding...

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Substitution – Examples

The semantics for functions will involve *substituting* actual parameters for formal parameters.

Write $\{e/x\}e'$ for the result of substituting e for all *free* occurrences of x in e'. For example

$$\begin{array}{lcl}
\{3/\mathrm{x}\}(\mathrm{x} \geq \mathrm{x}) & = & (3 \geq 3) \\
\{3/\mathrm{x}\}((\textbf{fn}\ \mathrm{x}{:}\mathsf{int} \Rightarrow \mathrm{x} + \mathrm{y})\mathrm{x}) & = & (\textbf{fn}\ \mathrm{x}{:}\mathsf{int} \Rightarrow \mathrm{x} + \mathrm{y})3 \\
\{\mathrm{y} + 2/\mathrm{x}\}(\textbf{fn}\ \mathrm{y}{:}\mathsf{int} \Rightarrow \mathrm{x} + \mathrm{y}) & = & \textbf{fn}\ \mathrm{z}{:}\mathsf{int} \Rightarrow (\mathrm{y} + 2) + \mathrm{z}
\end{array}$$

Note that substitution is a meta-operation – it's *not* part of the L2 expression grammar.

The notation used for substitution varies – people write $\{3/x\}e$, or $[3/x]e$, or $e[3/x]$, or $\{x \leftarrow 3\}e$, or...

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Substitution – Definition

Defining that:

$$\begin{array}{lcll}
\{e/z\}x & = & e & \text{if } x = z \\
 & = & x & \text{otherwise} \\
\{e/z\}(\textbf{fn}\ x{:}T \Rightarrow e_1) & = & \textbf{fn}\ x{:}T \Rightarrow (\{e/z\}e_1) & \text{if } x \neq z \text{ (*)} \\
 & & & \text{and } x \notin \mathsf{fv}(e) \text{ (*)} \\
\{e/z\}(e_1\, e_2) & = & (\{e/z\}e_1)(\{e/z\}e_2) & \\
\ldots & & &
\end{array}$$

if (*) is not true, we first have to pick an alpha-variant of $\textbf{fn}\ x{:}T \Rightarrow e_1$ to make it so (always can)

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Substitution – Example Again

$$\begin{aligned}
& \{y + 2/x\}(\textbf{fn}\ y{:}\mathrm{int} \Rightarrow x + y) \\
= \ & \{y + 2/x\}(\textbf{fn}\ y'{:}\mathrm{int} \Rightarrow x + y') \text{ renaming} \\
= \ & \textbf{fn}\ y'{:}\mathrm{int} \Rightarrow \{y + 2/x\}(x + y') \text{ as } y' \neq x \text{ and } y' \notin \mathrm{fv}(y + 2) \\
= \ & \textbf{fn}\ y'{:}\mathrm{int} \Rightarrow \{y + 2/x\}x + \{y + 2/x\}y' \\
= \ & \textbf{fn}\ y'{:}\mathrm{int} \Rightarrow (y + 2) + y'
\end{aligned}$$

(could have chosen any other z instead of y', except y or x)

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Substitution – Simultaneous

Generalising to simultaneous substitution: Say a *substitution* σ is a finite partial function from variables to expressions.

Notation: write a σ as $\{e_1/x_1, .., e_k/x_k\}$ instead of $\{x_1 \mapsto e_1, ..., x_k \mapsto e_k\}$ (for the function mapping x_1 to e_1 etc.)

Define $\sigma\, e$ in the notes.

Write $\mathrm{dom}(\sigma)$ for the set of variables in the domain of σ; $\mathrm{ran}(\sigma)$ for the set of expressions in the range of σ, ie

$$\begin{aligned}
\mathrm{dom}(\{e_1/x_1, .., e_k/x_k\}) &= \{x_1, .., x_k\} \\
\mathrm{ran}(\{e_1/x_1, .., e_k/x_k\}) &= \{e_1, .., e_k\}
\end{aligned}$$

Define the application of a substitution to a term by:

$$\begin{array}{lcll}
\sigma x & = & \sigma(x) & \text{if } x \in \mathrm{dom}(\sigma) \\
 & = & x & \text{otherwise} \\
\sigma(\textbf{fn } x{:}T \Rightarrow e) & = & \textbf{fn } x{:}T \Rightarrow (\sigma\, e) & \text{if } x \notin \mathrm{dom}(\sigma) \text{ and } x \notin \mathrm{fv}(\mathrm{ran}(\sigma)) \text{ (*)} \\
\sigma(e_1\, e_2) & = & (\sigma\, e_1)(\sigma\, e_2) & \\
\sigma n & = & n & \\
\sigma(e_1 \; op \; e_2) & = & \sigma(e_1) \; op \; \sigma(e_2) & \\
\sigma(\textbf{if } e_1 \textbf{ then } e_2 \textbf{ else } e_3) & = & \textbf{if } \sigma(e_1) \textbf{ then } \sigma(e_2) \textbf{ else } \sigma(e_3) & \\
\sigma(b) & = & b & \\
\sigma(\textbf{skip}) & = & \textbf{skip} & \\
\sigma(\ell := e) & = & \ell := \sigma(e) & \\
\sigma(!\ell) & = & !\ell & \\
\sigma(e_1; e_2) & = & \sigma(e_1); \sigma(e_2) & \\
\sigma(\textbf{while } e_1 \textbf{ do } e_2) & = & \textbf{while } \sigma(e_1) \textbf{ do } \sigma(e_2) &
\end{array}$$

4.2 Function Typing

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Typing functions (1)

Before, Γ gave the types of store locations; it ranged over $\mathrm{TypeEnv}$ which was the set of all finite partial functions from locations $\mathbb{L}$ to $\mathrm{T}_{\mathrm{loc}}$.

Now, it must also give assumptions on the types of variables: e.g. $l_1{:}\mathsf{intref}, \mathrm{x}{:}\mathsf{int}, \mathrm{y}{:}\mathsf{bool} \rightarrow \mathsf{int}$.

Take $\Gamma \in \mathrm{TypeEnv2}$, the finite partial functions from $\mathbb{L} \cup \mathbb{X}$ to $\mathrm{T}_{\mathrm{loc}} \cup \mathrm{T}$ such that

$\forall \ell \in \mathsf{dom}(\Gamma).\Gamma(\ell) \in \mathrm{T}_{\mathrm{loc}}$

$\forall x \in \mathsf{dom}(\Gamma).\Gamma(x) \in \mathrm{T}$

Notation: if $x \notin \mathsf{dom}(\Gamma)$, write $\Gamma, x{:}T$ for the partial function which maps x to T but otherwise is like Γ.

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Typing functions (2)

(var) $\Gamma \vdash x{:}T$ if $\Gamma(x) = T$

(fn) $$\dfrac{\Gamma, x{:}T \vdash e{:}T'}{\Gamma \vdash \textbf{fn}\ x{:}T \Rightarrow e : T \to T'}$$

(app) $$\dfrac{\Gamma \vdash e_1{:}T \to T' \qquad \Gamma \vdash e_2{:}T}{\Gamma \vdash e_1\ e_2{:}T'}$$

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Typing functions – Example

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathsf{x{:}int} \vdash \mathsf{x{:}int}}\,(\text{var}) \quad \dfrac{}{\mathsf{x{:}int} \vdash \mathsf{2{:}int}}\,(\text{int})}{\mathsf{x{:}int} \vdash \mathsf{x} + \mathsf{2{:}int}}\,(\text{op}+)}{\{\} \vdash (\textbf{fn}\ \mathsf{x{:}int} \Rightarrow \mathsf{x} + 2){:}\mathsf{int} \to \mathsf{int}}\,(\text{fn}) \qquad \dfrac{}{\{\} \vdash \mathsf{2{:}int}}\,(\text{int})}{\{\} \vdash (\textbf{fn}\ \mathsf{x{:}int} \Rightarrow \mathsf{x} + 2)\ \mathsf{2{:}int}}\,(\text{app})$$

- The syntax is explicitly typed, so don't need to 'guess' a T in the **fn** rule.
- Note that variables of these types are quite different from locations – you can't assign to variables; you can't abstract on locations. For example, $(\textbf{fn}\ l{:}\mathsf{intref} \Rightarrow !l)$ is not in the syntax.
- Note that sometimes you need alpha convention, e.g. to type

  $\textbf{fn}\ \mathsf{x{:}int} \Rightarrow \mathsf{x} + (\textbf{fn}\ \mathsf{x{:}bool} \Rightarrow \textbf{if}\ \mathsf{x}\ \textbf{then}\ 3\ \textbf{else}\ 4)\textbf{true}$

  It's a good idea to start out with all binders different from each other and from all free variables. It would be a bad idea to prohibit variable shadowing like this in source programs.
- In ML you have *parametrically polymorphic* functions, but we won't talk about them here – that's in Part II Types.

- Note that these functions are not recursive (as you can see in the syntax: there's no way in the body of $\textbf{fn}\ x{:}T \Rightarrow e$ to refer to the function as a whole).
- With our notational convention for $\Gamma, x{:}T$, we could rewrite the (var) rule as $\Gamma, x{:}T \vdash x{:}T$. By the convention, x is not in the domain of Γ, and $\Gamma + \{x \mapsto T\}$ is a perfectly good partial function.

Another example:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{l{:}\mathsf{intref}, \mathrm{x}{:}\mathsf{unit} \vdash 1{:}\mathsf{int}}\,\text{(int)}}{l{:}\mathsf{intref}, \mathrm{x}{:}\mathsf{unit} \vdash (l := 1){:}\mathsf{unit}}\,\text{(assign)} \quad \dfrac{}{l{:}\mathsf{intref}, \mathrm{x}{:}\mathsf{unit} \vdash \mathrm{x}{:}\mathsf{unit}}\,\text{(var)}}{l{:}\mathsf{intref}, \mathrm{x}{:}\mathsf{unit} \vdash (l := 1); \mathrm{x}{:}\mathsf{unit}}\,\text{(seq)}}{l{:}\mathsf{intref} \vdash (\textbf{fn}\ \mathrm{x}{:}\mathsf{unit} \Rightarrow (l := 1); \mathrm{x}){:}\mathsf{unit} \rightarrow \mathsf{unit}}\,\text{(fn)} \quad \dfrac{\dfrac{}{l{:}\mathsf{intref} \vdash 2{:}\mathsf{int}}\,\text{(int)}}{l{:}\mathsf{intref} \vdash (l := 2){:}\mathsf{unit}}\,\text{(assign)}}{l{:}\mathsf{intref} \vdash (\textbf{fn}\ \mathrm{x}{:}\mathsf{unit} \Rightarrow (l := 1); \mathrm{x})\ (l := 2){:}\mathsf{unit}}\,\text{(app)}$$

4.3 Function Behaviour

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Function Behaviour

Consider the expression

$e = (\textbf{fn}\ \mathrm{x}{:}\mathsf{unit} \Rightarrow (l := 1); \mathrm{x})\ (l := 2)$

then

$\langle e, \{l \mapsto 0\}\rangle \longrightarrow^{*} \langle \textbf{skip}, \{l \mapsto ???\}\rangle$

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Function Behaviour. Choice 1: Call-by-value

Informally: reduce left-hand-side of application to a **fn**-term; reduce argument to a value; then replace all occurrences of the formal parameter in the **fn**-term by that value.

$e = (\textbf{fn}\ \mathrm{x}{:}\mathsf{unit} \Rightarrow (l := 1); \mathrm{x})(l := 2)$

$$
\begin{array}{lll}
\langle e, \{l = 0\}\rangle & \longrightarrow & \langle(\textbf{fn}\ \mathrm{x}{:}\mathsf{unit} \Rightarrow (l := 1); \mathrm{x})\textbf{skip}, \{l = 2\}\rangle \\
 & \longrightarrow & \langle((l := 1); \textbf{skip}) \qquad\qquad , \{l = 2\}\rangle \\
 & \longrightarrow & \langle(\textbf{skip}; \textbf{skip}) \qquad\qquad , \{l = 1\}\rangle \\
 & \longrightarrow & \langle \textbf{skip} \qquad\qquad , \{l = 1\}\rangle
\end{array}
$$

This is most common design choice - ML, Java,...

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L2 Call-by-value

Values $v ::= b \mid n \mid \textbf{skip} \mid \textbf{fn}\ x{:}T \Rightarrow e$

(app1) $$\frac{\langle e_1, s\rangle \longrightarrow \langle e_1', s'\rangle}{\langle e_1\ e_2, s\rangle \longrightarrow \langle e_1'\ e_2, s'\rangle}$$

(app2) $$\frac{\langle e_2, s\rangle \longrightarrow \langle e_2', s'\rangle}{\langle v\ e_2, s\rangle \longrightarrow \langle v\ e_2', s'\rangle}$$

(fn) $$\langle(\textbf{fn}\ x{:}T \Rightarrow e)\ v, s\rangle \longrightarrow \langle\{v/x\}e, s\rangle$$

- This is a *strict* semantics – fully evaluating the argument to function before doing the application.
- Could (perversely) evaluate $e_1 e_2$ right-to-left instead. Better design is to match order for operators etc..
- The syntax has explicit types and the semantics involves syntax, so types appear in semantics – but they are not used in any interesting way, so an implementation could erase them before execution.
- The rules for these constructs, and those in the next couple of lectures, don't touch the store, but we need to include it to get the sequencing of side-effects right. In a *pure* functional language, would have configurations just be expressions.
- Relating to implementation: a naive implementation of these rules would have to traverse e and copy v as many times as there are free occurrences of x in e. Real implementations use environments – but there's lots of machinery there that we don't need for a definition.

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L2 Call-by-value – reduction examples

$$\begin{array}{ll}
 & \langle(\textbf{fn}\ \mathrm{x{:}int} \Rightarrow \textbf{fn}\ \mathrm{y{:}int} \Rightarrow \mathrm{x}+\mathrm{y})\ (3+4)\ 5\ , s\rangle \\
= & \langle\big((\textbf{fn}\ \mathrm{x{:}int} \Rightarrow \textbf{fn}\ \mathrm{y{:}int} \Rightarrow \mathrm{x}+\mathrm{y})\ (3+4)\big)\ 5\ , s\rangle \\
\longrightarrow & \langle\big((\textbf{fn}\ \mathrm{x{:}int} \Rightarrow \textbf{fn}\ \mathrm{y{:}int} \Rightarrow \mathrm{x}+\mathrm{y})\ 7\big)\ 5\ , s\rangle \\
\longrightarrow & \langle\big(\{7/\mathrm{x}\}(\textbf{fn}\ \mathrm{y{:}int} \Rightarrow \mathrm{x}+\mathrm{y})\big)\ 5\ , s\rangle \\
= & \langle\big((\textbf{fn}\ \mathrm{y{:}int} \Rightarrow 7+\mathrm{y})\big)\ 5\ , s\rangle \\
\longrightarrow & \langle 7+5\ , s\rangle \\
\longrightarrow & \langle 12\ , s\rangle
\end{array}$$

$$(\textbf{fn}\ \mathrm{f{:}int} \rightarrow \mathrm{int} \Rightarrow \mathrm{f}\ 3)\ (\textbf{fn}\ \mathrm{x{:}int} \Rightarrow (1+2)+\mathrm{x})$$

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Function Behaviour. Choice 2: Call-by-name

Informally: reduce left-hand-side of application to a **fn**-term; then replace all occurrences of the formal parameter in the **fn**-term by the argument.

$e = (\textbf{fn}\ \mathrm{x}{:}\mathsf{unit} \Rightarrow (l := 1); \mathrm{x})\ (l := 2)$

$$\begin{array}{rlll}
\langle e, \{l \mapsto 0\}\rangle & \longrightarrow & \langle (l := 1); l := 2, & \{l \mapsto 0\}\rangle \\
 & \longrightarrow & \langle \textbf{skip} \quad ; l := 2, & \{l \mapsto 1\}\rangle \\
 & \longrightarrow & \langle l := 2 & , \{l \mapsto 1\}\rangle \\
 & \longrightarrow & \langle \textbf{skip} & , \{l \mapsto 2\}\rangle
\end{array}$$

This is the foundation of 'lazy' functional languages - eg Haskell

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L2 Call-by-name

(same typing rules as before)

(CBN-app) $$\frac{\langle e_1, s\rangle \longrightarrow \langle e_1', s'\rangle}{\langle e_1\, e_2, s\rangle \longrightarrow \langle e_1'\, e_2, s'\rangle}$$

(CBN-fn) $\langle (\textbf{fn}\ x{:}T \Rightarrow e) e_2, s\rangle \longrightarrow \langle \{e_2/x\}e, s\rangle$

Here, don't evaluate the argument at all if it isn't used

$$\begin{array}{rll}
 & \langle (\textbf{fn}\ \mathrm{x}{:}\mathsf{unit} \Rightarrow \textbf{skip})(l := 2), & \{l \mapsto 0\}\rangle \\
\longrightarrow & \langle \{l := 2/\mathrm{x}\}\textbf{skip} & , \{l \mapsto 0\}\rangle \\
= & \langle \textbf{skip} & , \{l \mapsto 0\}\rangle
\end{array}$$

but if it is, end up evaluating it repeatedly.

Haskell uses a refined variant – call-by-need – in which the first time the argument evaluated we 'overwrite' all other copies by that value.

That lets you do some very nice programming, e.g. with potentially-infinite datastructures.

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Call-By-Need Example (Haskell)

```
let notdivby x y = y `mod` x /= 0
    enumFrom n   = n : (enumFrom (n+1))
    sieve (x:xs) =
      x : sieve (filter (notdivby x) xs)
in
sieve (enumFrom 2)
==>
[2,3,5,7,11,13,17,19,23,29,31,37,41,43,47,53,
 59,61,67,71,73,79,83,89,97,101,103,107,109,
 113,127,131,137,139,149,151,157,163,167,173,
 179,181,191,193,197,199,211,223,227,229,233,
 ,,Interrupted!
```

But, it becomes rather hard to understand what order your code is going to be run in! Hence, non-strict languages typically don't allow unrestricted side effects (our combination of store and CBN is *pretty odd*). Instead, Haskell encourages *pure* programming, without effects (store operations, IO, etc.) except where really necessary. Where they *are* necessary, it uses a fancy type system to give you some control of evaluation order.

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Purity

Note that CBV and CBN are distinguishable even if there is no store – consider applying a function to a non-terminating argument, eg (**fn** x:unit ⇒ **skip**) (**while true do skip**).

Call-by-Name and Call-by-Need are not distinguishable except by performance properties – but those really matter.

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Function Behaviour. Choice 3: Full beta

Allow both left and right-hand sides of application to reduce. At any point where the left-hand-side has reduced to a **fn**-term, replace all occurrences of the formal parameter in the **fn**-term by the argument. Allow reduction inside lambdas.

$(\textbf{fn}\ \mathrm{x{:}int} \Rightarrow 2+2) \longrightarrow (\textbf{fn}\ \mathrm{x{:}int} \Rightarrow 4)$

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L2 Beta

(beta-app1) $$\frac{\langle e_1, s\rangle \longrightarrow \langle e_1', s'\rangle}{\langle e_1\, e_2, s\rangle \longrightarrow \langle e_1'\, e_2, s'\rangle}$$

(beta-app2) $$\frac{\langle e_2, s\rangle \longrightarrow \langle e_2', s'\rangle}{\langle e_1\, e_2, s\rangle \longrightarrow \langle e_1\, e_2', s'\rangle}$$

(beta-fn1) $$\langle (\textbf{fn}\ x{:}T \Rightarrow e)e_2, s\rangle \longrightarrow \langle \{e_2/x\}e, s\rangle$$

(beta-fn2) $$\frac{\langle e, s\rangle \longrightarrow \langle e', s'\rangle}{\langle \textbf{fn}\ x{:}T \Rightarrow e, s\rangle \longrightarrow \langle \textbf{fn}\ x{:}T \Rightarrow e', s'\rangle}$$

This reduction relation includes the CBV and CBN relations, and also reduction inside lambdas.

This ain't much good for a programming language... why? (if you've got any non-terminating computation Ω, then $(\lambda x.y)$ Ω might terminate or not, depending on the implementation) (in pure lambda you do have *confluence*, which saves you – at least mathematically)

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Function Behaviour. Choice 4: Normal-order reduction

Leftmost, outermost variant of full beta.

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Back to CBV (from now on).

Properties of Typing

As before, but only interested in executing *closed* programs.

Theorem 11 (Progress) *If* e *closed and* $\Gamma \vdash e{:}T$ *and* $\mathit{dom}(\Gamma) \subseteq \mathit{dom}(s)$ *then either* e *is a value or there exist* e', s' *such that* $\langle e, s\rangle \longrightarrow \langle e', s'\rangle$.

Note there are now more stuck configurations, e.g.$((3)\ (4))$

Theorem 12 (Type Preservation) *If* e *closed and* $\Gamma \vdash e{:}T$ *and* $\mathit{dom}(\Gamma) \subseteq \mathit{dom}(s)$ *and* $\langle e, s\rangle \longrightarrow \langle e', s'\rangle$ *then* $\Gamma \vdash e'{:}T$ *and* e' *closed and* $\mathit{dom}(\Gamma) \subseteq \mathit{dom}(s')$.

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Proving Type Preservation

Theorem 14 (Type Preservation) *If e closed and $\Gamma \vdash e{:}T$ and $dom(\Gamma) \subseteq dom(s)$ and $\langle e, s\rangle \longrightarrow \langle e', s'\rangle$ then $\Gamma \vdash e'{:}T$ and e' closed and $dom(\Gamma) \subseteq dom(s')$.*

Taking

$$\begin{array}{l}
\Phi(e, s, e', s') = \\
\quad \forall \Gamma, T. \\
\qquad \Gamma \vdash e{:}T \wedge \mathsf{closed}(e) \wedge \mathsf{dom}(\Gamma) \subseteq \mathsf{dom}(s) \\
\qquad\qquad \Rightarrow \\
\qquad \Gamma \vdash e'{:}T \wedge \mathsf{closed}(e') \wedge \mathsf{dom}(\Gamma) \subseteq \mathsf{dom}(s')
\end{array}$$

we show $\forall e, s, e', s'.\langle e, s\rangle \longrightarrow \langle e', s'\rangle \Rightarrow \Phi(e, s, e', s')$ by rule induction.

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To prove this one uses:

Lemma 7 (Substitution) *If $\Gamma \vdash e{:}T$ and $\Gamma, x{:}T \vdash e'{:}T'$ with $x \notin dom(\Gamma)$ then $\Gamma \vdash \{e/x\}e'{:}T'$.*

Determinacy and type inference properties also hold.

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Normalisation

Theorem 13 (Normalisation) *In the sublanguage without while loops or store operations, if* $\Gamma \vdash e{:}T$ *and* e *closed then there does not exist an infinite reduction sequence* $\langle e, \{\}\rangle \longrightarrow \langle e_1, \{\}\rangle \longrightarrow \langle e_2, \{\}\rangle \longrightarrow \ldots$

Proof ? can't do a simple induction, as reduction can make terms grow. See Pierce Ch.12 (the details are not in the scope of this course). □

4.4 Local Definitions and Recursive Functions

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Local definitions

For readability, want to be able to *name* definitions, and to *restrict* their scope, so add:

$$e \quad ::= \quad \ldots \mid \textbf{let val } x{:}T = e_1 \textbf{ in } e_2 \textbf{ end}$$

this x is a binder, binding any free occurrences of x in e_2.

Can regard just as *syntactic sugar*:

$$\textbf{let val } x{:}T = e_1 \textbf{ in } e_2 \textbf{ end} \quad \rightsquigarrow \quad (\textbf{fn } x{:}T \Rightarrow e_2)e_1$$

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Local definitions – derived typing and reduction rules (CBV)

$$\textbf{let val } x{:}T = e_1 \textbf{ in } e_2 \textbf{ end} \quad \rightsquigarrow \quad (\textbf{fn } x{:}T \Rightarrow e_2)\, e_1$$

(let) $$\frac{\Gamma \vdash e_1{:}T \qquad \Gamma, x{:}T \vdash e_2{:}T'}{\Gamma \vdash \textbf{let val } x{:}T = e_1 \textbf{ in } e_2 \textbf{ end}{:}T'}$$

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(let1)

$$\frac{\langle e_1, s\rangle \longrightarrow \langle e_1', s'\rangle}{\langle \textbf{let val } x{:}T = e_1 \textbf{ in } e_2 \textbf{ end}, s\rangle \longrightarrow \langle \textbf{let val } x{:}T = e_1' \textbf{ in } e_2 \textbf{ end}, s'\rangle}$$

(let2)

$$\langle \textbf{let val } x{:}T = v \textbf{ in } e_2 \textbf{ end}, s\rangle \longrightarrow \langle \{v/x\}e_2, s\rangle$$

Our alpha convention means this really is a local definition – there is no way to refer to the locally-defined variable outside the **let val** .

$$\mathrm{x} + \textbf{let val } \mathrm{x{:}int} = \mathrm{x} \textbf{ in } (\mathrm{x} + 2) \textbf{ end} \quad = \quad \mathrm{x} + \textbf{let val } \mathrm{y{:}int} = \mathrm{x} \textbf{ in } (\mathrm{y} + 2) \textbf{ end}$$

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Recursive definitions – first attempt

How about

$$\mathrm{x} = (\textbf{fn}\ \mathrm{y}{:}\mathsf{int} \Rightarrow \textbf{if}\ \mathrm{y} \geq 1\ \textbf{then}\ \mathrm{y} + (\mathrm{x}\ (\mathrm{y} + -1))\ \textbf{else}\ 0)$$

where we use x within the definition of x? Think about evaluating $\mathrm{x}\ 3$.

Could add something like this:

$$e \quad ::= \quad \ldots \mid \textbf{let val rec}\ x{:}T = e\ \textbf{in}\ e'\ \textbf{end}$$

(here the x binds in both e and e') then say

$$\begin{array}{l} \textbf{let val rec}\ \mathrm{x}{:}\mathsf{int} \rightarrow \mathsf{int} = \\ \quad (\textbf{fn}\ \mathrm{y}{:}\mathsf{int} \Rightarrow \textbf{if}\ \mathrm{y} \geq 1\ \textbf{then}\ \mathrm{y} + (\mathrm{x}(\mathrm{y} + -1))\ \textbf{else}\ 0) \\ \ \textbf{in}\ \mathrm{x}\ 3\ \textbf{end} \end{array}$$

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But...

What about

$\textbf{let val rec}\ \mathrm{x} = (\mathrm{x}, \mathrm{x})\ \textbf{in}\ \mathrm{x}\ \textbf{end}$?

Have some rather weird things, eg

$\textbf{let val rec}\ \mathrm{x}{:}\mathsf{int\ list} = 3 :: \mathrm{x}\ \textbf{in}\ \mathrm{x}\ \textbf{end}$

does that terminate? if so, is it equal to
$\textbf{let val rec}\ \mathrm{x}{:}\mathsf{int\ list} = 3 :: 3 :: \mathrm{x}\ \textbf{in}\ \mathrm{x}\ \textbf{end}$? does
$\textbf{let val rec}\ \mathrm{x}{:}\mathsf{int\ list} = 3 :: (\mathrm{x} + 1)\ \textbf{in}\ \mathrm{x}\ \textbf{end}$ terminate?

In a CBN language, it is reasonable to allow this kind of thing, as will only compute as much as needed. In a CBV language, would *usually* disallow, allowing recursive definitions only of functions...

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Recursive Functions

So, specialise the previous **let val rec** construct to

$$\begin{array}{rcll} T & = & T_1 \to T_2 & \text{recursion only at function types} \\ e & = & \textbf{fn}\ y{:}T_1 \Rightarrow e_1 & \text{and only of function values} \end{array}$$

$$e \quad ::= \quad \ldots \mid \textbf{let val rec}\ x{:}T_1 \to T_2 = (\textbf{fn}\ y{:}T_1 \Rightarrow e_1)\ \textbf{in}\ e_2\ \textbf{end}$$

(here the y binds in e_1; the x binds in $(\textbf{fn}\ y{:}T \Rightarrow e_1)$ and in e')

(let rec fn) $$\frac{\Gamma, x{:}T_1 \to T_2, y{:}T_1 \vdash e_1{:}T_2 \qquad \Gamma, x{:}T_1 \to T_2 \vdash e_2{:}T}{\Gamma \vdash \textbf{let val rec}\ x{:}T_1 \to T_2 = (\textbf{fn}\ y{:}T_1 \Rightarrow e_1)\ \textbf{in}\ e_2\ \textbf{end}{:}T}$$

Concrete syntax: In ML can write **let fun** $f(x{:}T_1){:}T_2 = e_1$ **in** e_2 **end**, or even **let fun** $f(x) = e_1$ **in** e_2 **end**, for **let val rec** $f{:}T_1 \to T_2 =$ **fn** $x{:}T_1 \Rightarrow e_1$ **in** e_2 **end**.

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Recursive Functions – Semantics

(letrecfn) $\quad \textbf{let val rec}\ x{:}T_1 \to T_2 = (\textbf{fn}\ y{:}T_1 \Rightarrow e_1)\ \textbf{in}\ e_2\ \textbf{end}$

$\longrightarrow$

$\{(\textbf{fn}\ y{:}T_1 \Rightarrow \textbf{let val rec}\ x{:}T_1 \to T_2 = (\textbf{fn}\ y{:}T_1 \Rightarrow e_1)\ \textbf{in}\ e_1\ \textbf{end})/x\}e_2$

(sometimes use $\textbf{fix}{:}((T_1 \to T_2) \to (T_1 \to T_2)) \to (T_1 \to T_2)$ – cf. Y)

For example:

```
let val rec x:int → int =
   (fn y:int ⇒ if y ≥ 1 then y + (x(y + −1)) else 0)
in
   x 3
end
⟶        (letrecfn)
(fn y:int ⇒
   let val rec x:int → int =
      (fn y:int ⇒ if y ≥ 1 then y + (x(y + −1)) else 0)
   in
      if y ≥ 1 then y + (x(y + −1)) else 0
   end) 3
⟶        (app)
let val rec x:int → int =
   (fn y:int ⇒ if y ≥ 1 then y + (x(y + −1)) else 0)
in
   if 3 ≥ 1 then 3 + (x(3 + −1)) else 0)
end
⟶        (letrecfn)
if 3 ≥ 1 then
   3 + ((fn y:int ⇒
      let val rec x:int → int =
         (fn y:int ⇒ if y ≥ 1 then y + (x(y + −1)) else 0)
      in
         if y ≥ 1 then y + (x(y + −1)) else 0
      end)(3 + −1))
 else
0
⟶ ...
```

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Recursive Functions – Minimisation Example

Below, in the context of the **let val rec** , x f n finds the smallest $n' \geq n$ for which f n' evaluates to some $m' \leq 0$.

```
let val rec x:(int → int) → int → int
   = fn f:int → int ⇒ fn z:int ⇒ if (f z) ≥ 1 then x f (z + 1) else z
in
   let val f:int → int
      = (fn z:int ⇒ if z ≥ 3 then (if 3 ≥ z then 0 else 1) else 1)
   in
      x f 0
   end
end
```

Slide 134

As a test case, we apply it to the function $(\textbf{fn}\ z{:}\mathrm{int} \Rightarrow \textbf{if}\ z \geq 3\ \textbf{then}\ (\textbf{if}\ 3 \geq z\ \textbf{then}\ 0\ \textbf{else}\ 1)\ \textbf{else}\ 1)$, which is 0 for argument 3 and 1 elsewhere.

Slide 135

More Syntactic Sugar

Do we need $e_1; e_2$?

No: Could encode by $e_1; e_2 \rightsquigarrow (\textbf{fn}\ y{:}\mathrm{unit} \Rightarrow e_2)\, e_1$

Do we need $\textbf{while}\ e_1\ \textbf{do}\ e_2$?

No: could encode by $\textbf{while}\ e_1\ \textbf{do}\ e_2 \rightsquigarrow$

$$
\begin{array}{l}
\textbf{let val rec}\ \mathrm{w}{:}\mathrm{unit} \rightarrow \mathrm{unit} = \\
\quad \textbf{fn}\ \mathrm{y}{:}\mathrm{unit} \Rightarrow \textbf{if}\ e_1\ \textbf{then}\ (e_2; (\mathrm{w}\ \textbf{skip}))\ \textbf{else skip} \\
\textbf{in} \\
\quad \mathrm{w}\ \textbf{skip} \\
\textbf{end}
\end{array}
$$

for fresh w and y not in $\mathrm{fv}(e_1) \cup \mathrm{fv}(e_2)$.

In each case typing is the same (more precisely?); reduction is ‘essentially’ the same. What does that mean? More later, on contextual equivalence.

Slide 136

OTOH, Could we encode recursion in the language without?

We know at least that you can't in the language without **while** or store, as had normalisation theorem there and can write

let val rec x:int $\longrightarrow$ int $=$ **fn** y:int $\Rightarrow \mathrm{x}(\mathrm{y}+1)$ **in** $\mathrm{x}\ 0$ **end**

here.

4.5 Implementation

Slide 137

Implementation

There is an implementation of L2 on the course web page.

See especially `Syntax.sml` and `Semantics.sml`. It uses a front end written with mosmllex and mosmlyac.

Also, as on Slide 54, L2 expressions can be executed directly in a mosml context.

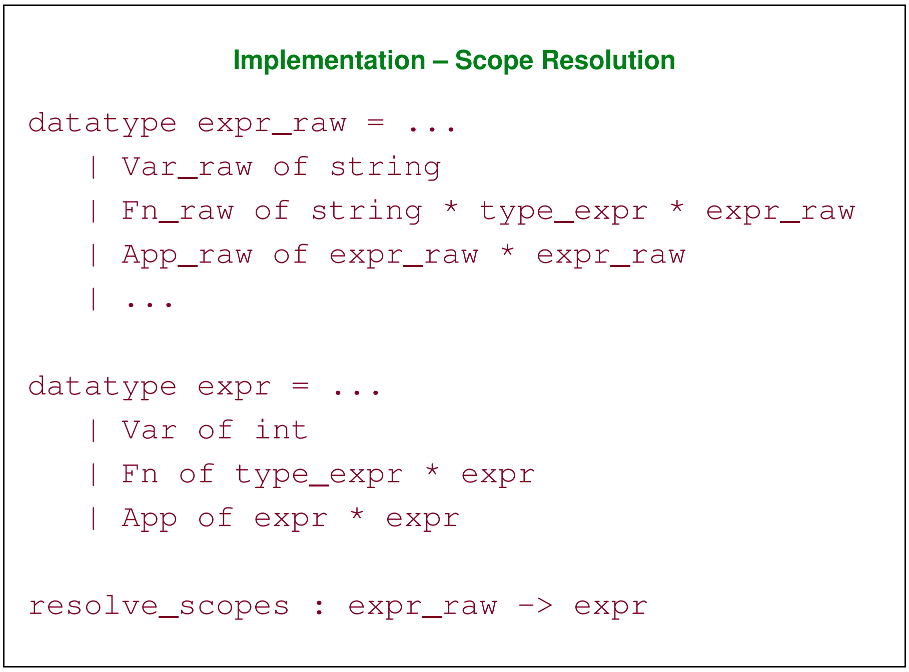
The README file says:

```
(* 2002-11-08 -- Time-stamp: <2003-04-25 17:28:25 pes20>    *)
(* Peter Sewell                                             *)

This directory contains an interpreter, pretty-printer and
type-checker for the language L2.

To make it go, copy it into a working directory, ensure Moscow ML
is available (including mosmllex and mosmlyac), and type
```

```
  make
  mosml
  load "Main";

It prompts you for an L2 expression (terminated by RETURN, no terminating
semicolons) and then for an initial store.  For the latter, if you
just press RETURN you get a default store in which all the locations
mentioned in your expression are mapped to 0.

Watch out for the parsing - it is not quite the same as (eg) mosml, so
you need to parenthesise more.

The source files are:

  Main.sml         the top-level loop
  Syntax.sml       datatypes for raw and de-bruijn expressions
  Lexer.lex        the lexer (input to mosmllex)
  Parser.grm       the grammar (input to mosmlyac)
  Semantics.sml    scope resolution, the interpreter, and the typechecker
  PrettyPrint.sml  pretty-printing code

  Examples.l2      some handy examples for cut-and-pasting into the
                     top-level loop

of these, you're most likely to want to look at, and change, Semantics.sml.
You should first also look at Syntax.sml.
```

Type in L2 expressions and initial stores and watch them resolve, type-check, and reduce.

Slide 138



Implementation – Scope Resolution

```
datatype expr_raw = ...
    | Var_raw of string
    | Fn_raw of string * type_expr * expr_raw
    | App_raw of expr_raw * expr_raw
    | ...

datatype expr = ...
    | Var of int
    | Fn of type_expr * expr
    | App of expr * expr

resolve_scopes : expr_raw -> expr
```

(it raises an exception if the expression has any free variables)

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Implementation – Substitution

`subst : expr -> int -> expr -> expr`

`subst e 0 e'` substitutes `e` for the outermost var in `e'`.

(the definition is only sensible if `e` is closed, but that's ok – we only evaluate whole programs. For a general definition, see [Pierce, Ch. 6])

```
fun subst e n (Var n1)        = if n=n1 then e else Var n1
  | subst e n (Fn(t,e1))      = Fn(t,subst e (n+1) e1)
  | subst e n (App(e1,e2))    = App(subst e n e1,subst e n e2)
  | subst e n (Let(t,e1,e2))
    = Let (t,subst e n e1,subst e (n+1) e2)

  | subst e n (Letrecfn (tx,ty,e1,e2))
    = Letrecfn (tx,ty,subst e (n+2) e1,subst e (n+1) e2)
  | ...
```

If `e'` represents a closed term **fn** $x{:}T \Rightarrow e'_1$ then `e'` = `Fn(t,e1')` for `t` and `e1'` representing T and e'_1. If also `e` represents a closed term e then `subst e 0 e1'` represents $\{e/x\}e'_1$.

Slide 140

Implementation – CBV reduction

```
reduce (App (e1,e2),s) = (case e1 of
    Fn (t,e) =>
    (if (is_value e2) then
       SOME (subst e2 0 e,s)
     else
       (case reduce (e2,s) of
          SOME(e2',s') => SOME(App (e1,e2'),s')
        | NONE => NONE))
  | _ => (case reduce (e1,s) of
            SOME (e1',s')=>SOME(App(e1',e2),s')
          | NONE => NONE ))
```

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Implementation – Type Inference

```
type typeEnv
       = (loc*type_loc) list * type_expr list

inftype gamma (Var n) = nth (#2 gamma) n
inftype gamma (Fn (t,e))
= (case inftype (#1 gamma, t::(#2 gamma)) e of
     SOME t’ => SOME (func(t,t’) )
   | NONE => NONE )
inftype gamma (App (e1,e2))
= (case (inftype gamma e1, inftype gamma e2) of
     (SOME (func(t1,t1’)), SOME t2) =>
        if t1=t2 then SOME t1’ else NONE
   | _ => NONE )
```

Slide 142

Implementation – Closures

Naively implementing substitution is expensive. An efficient implementation would use *closures* instead – cf. Compiler Construction.

We could give a more concrete semantics, closer to implementation, in terms of closures, and then prove it corresponds to the original semantics...

(if you get that wrong, you end up with dynamic scoping, as in original LISP)

Slide 143

Aside: Small-step vs Big-step Semantics

Throughout this course we use *small-step* semantics, $\langle e, s\rangle \longrightarrow \langle e', s'\rangle$.

There is an alternative style, of *big-step* semantics $\langle e, s\rangle \Downarrow \langle v, s'\rangle$, for example

$$\frac{}{\langle n, s\rangle \Downarrow \langle n, s\rangle} \qquad \frac{\langle e_1, s\rangle \Downarrow \langle n_1, s'\rangle \quad \langle e_2, s'\rangle \Downarrow \langle n_2, s''\rangle}{\langle e_1 + e_2, s\rangle \Downarrow \langle n, s''\rangle \quad n = n_1 + n_2}$$

(see the notes from earlier courses by Andy Pitts).

For sequential languages, it doesn't make a major difference. When we come to add concurrency, small-step is more convenient.

4.6 L2: Collected Definition

Syntax

Booleans $b \in \mathbb{B} = \{\textbf{true}, \textbf{false}\}$
Integers $n \in \mathbb{Z} = \{..., -1, 0, 1, ...\}$
Locations $\ell \in \mathbb{L} = \{l, l_0, l_1, l_2, ...\}$
Variables $x \in \mathbb{X}$ for a set $\mathbb{X} = \{\mathrm{x}, \mathrm{y}, \mathrm{z}, ...\}$

Operations $op ::= + \mid \geq$

Types

$$\begin{array}{lcl} T & ::= & \mathsf{int} \mid \mathsf{bool} \mid \mathsf{unit} \mid T_1 \to T_2 \\ T_{loc} & ::= & \mathsf{intref} \end{array}$$

Expressions

$$\begin{array}{lcl} e & ::= & n \mid b \mid e_1\ op\ e_2 \mid \textbf{if}\ e_1\ \textbf{then}\ e_2\ \textbf{else}\ e_3 \mid \\ & & \ell := e \mid !\ell \mid \\ & & \textbf{skip} \mid e_1; e_2 \mid \\ & & \textbf{while}\ e_1\ \textbf{do}\ e_2 \mid \\ & & \textbf{fn}\ x{:}T \Rightarrow e \mid e_1\ e_2 \mid x \mid \\ & & \textbf{let val}\ x{:}T = e_1\ \textbf{in}\ e_2\ \textbf{end} \mid \\ & & \textbf{let val rec}\ x{:}T_1 \to T_2 = (\textbf{fn}\ y{:}T_1 \Rightarrow e_1)\ \textbf{in}\ e_2\ \textbf{end} \end{array}$$

In expressions $\textbf{fn}\ x{:}T \Rightarrow e$ the x is a *binder*. In expressions $\textbf{let val}\ x{:}T = e_1\ \textbf{in}\ e_2\ \textbf{end}$ the x is a binder. In expressions $\textbf{let val rec}\ x{:}T_1 \to T_2 = (\textbf{fn}\ y{:}T_1 \Rightarrow e_1)\ \textbf{in}\ e_2\ \textbf{end}$ the y binds in e_1; the x binds in $(\textbf{fn}\ y{:}T \Rightarrow e_1)$ and in e_2.

Operational Semantics

Say *stores* s are finite partial functions from $\mathbb{L}$ to $\mathbb{Z}$. Values $v ::= b \mid n \mid \textbf{skip} \mid \textbf{fn}\ x{:}T \Rightarrow e$

$$\text{(op }+\text{)}\quad \langle n_1 + n_2, s\rangle \longrightarrow \langle n, s\rangle \quad \text{if } n = n_1 + n_2$$

$$\text{(op }\geq\text{)}\quad \langle n_1 \geq n_2, s\rangle \longrightarrow \langle b, s\rangle \quad \text{if } b = (n_1 \geq n_2)$$

$$\text{(op1)}\quad \frac{\langle e_1, s\rangle \longrightarrow \langle e_1', s'\rangle}{\langle e_1 \; op \; e_2, s\rangle \longrightarrow \langle e_1' \; op \; e_2, s'\rangle}$$

$$\text{(op2)}\quad \frac{\langle e_2, s\rangle \longrightarrow \langle e_2', s'\rangle}{\langle v \; op \; e_2, s\rangle \longrightarrow \langle v \; op \; e_2', s'\rangle}$$

$$\text{(deref)}\quad \langle !\ell, s\rangle \longrightarrow \langle n, s\rangle \quad \text{if } \ell \in \mathrm{dom}(s) \text{ and } s(\ell) = n$$

$$\text{(assign1)}\quad \langle \ell := n, s\rangle \longrightarrow \langle \mathbf{skip}, s + \{\ell \mapsto n\}\rangle \quad \text{if } \ell \in \mathrm{dom}(s)$$

$$\text{(assign2)}\quad \frac{\langle e, s\rangle \longrightarrow \langle e', s'\rangle}{\langle \ell := e, s\rangle \longrightarrow \langle \ell := e', s'\rangle}$$

$$\text{(seq1)}\quad \langle \mathbf{skip}; e_2, s\rangle \longrightarrow \langle e_2, s\rangle$$

$$\text{(seq2)}\quad \frac{\langle e_1, s\rangle \longrightarrow \langle e_1', s'\rangle}{\langle e_1; e_2, s\rangle \longrightarrow \langle e_1'; e_2, s'\rangle}$$

$$\text{(if1)}\quad \langle \mathbf{if\ true\ then}\ e_2\ \mathbf{else}\ e_3, s\rangle \longrightarrow \langle e_2, s\rangle$$

$$\text{(if2)}\quad \langle \mathbf{if\ false\ then}\ e_2\ \mathbf{else}\ e_3, s\rangle \longrightarrow \langle e_3, s\rangle$$

$$\text{(if3)}\quad \frac{\langle e_1, s\rangle \longrightarrow \langle e_1', s'\rangle}{\langle \mathbf{if}\ e_1\ \mathbf{then}\ e_2\ \mathbf{else}\ e_3, s\rangle \longrightarrow \langle \mathbf{if}\ e_1'\ \mathbf{then}\ e_2\ \mathbf{else}\ e_3, s'\rangle}$$

(while)

$$\langle \mathbf{while}\ e_1\ \mathbf{do}\ e_2, s\rangle \longrightarrow \langle \mathbf{if}\ e_1\ \mathbf{then}\ (e_2; \mathbf{while}\ e_1\ \mathbf{do}\ e_2)\ \mathbf{else\ skip}, s\rangle$$

$$\text{(app1)}\quad \frac{\langle e_1, s\rangle \longrightarrow \langle e_1', s'\rangle}{\langle e_1\ e_2, s\rangle \longrightarrow \langle e_1'\ e_2, s'\rangle}$$

$$\text{(app2)}\quad \frac{\langle e_2, s\rangle \longrightarrow \langle e_2', s'\rangle}{\langle v\ e_2, s\rangle \longrightarrow \langle v\ e_2', s'\rangle}$$

$$\text{(fn)}\quad \langle (\mathbf{fn}\ x{:}T \Rightarrow e)\ v, s\rangle \longrightarrow \langle \{v/x\}e, s\rangle$$

(let1)

$$\frac{\langle e_1, s\rangle \longrightarrow \langle e_1', s'\rangle}{\langle \mathbf{let\ val}\ x{:}T = e_1\ \mathbf{in}\ e_2\ \mathbf{end}, s\rangle \longrightarrow \langle \mathbf{let\ val}\ x{:}T = e_1'\ \mathbf{in}\ e_2\ \mathbf{end}, s'\rangle}$$

(let2)

$$\langle \mathbf{let\ val}\ x{:}T = v\ \mathbf{in}\ e_2\ \mathbf{end}, s\rangle \longrightarrow \langle \{v/x\}e_2, s\rangle$$

$$\text{(letrecfn)}\quad \mathbf{let\ val\ rec}\ x{:}T_1 \to T_2 = (\mathbf{fn}\ y{:}T_1 \Rightarrow e_1)\ \mathbf{in}\ e_2\ \mathbf{end}$$

$$\longrightarrow$$

$$\{(\mathbf{fn}\ y{:}T_1 \Rightarrow \mathbf{let\ val\ rec}\ x{:}T_1 \to T_2 = (\mathbf{fn}\ y{:}T_1 \Rightarrow e_1)\ \mathbf{in}\ e_1\ \mathbf{end})/x\}e_2$$

Typing

Take $\Gamma \in \mathrm{TypeEnv2}$, the finite partial functions from $\mathbb{L} \cup \mathbb{X}$ to $\mathrm{T}_{\mathrm{loc}} \cup \mathrm{T}$ such that

$\forall \ell \in \mathrm{dom}(\Gamma).\Gamma(\ell) \in \mathrm{T}_{\mathrm{loc}}$

$\forall x \in \mathrm{dom}(\Gamma).\Gamma(x) \in \mathrm{T}$

$$(\mathrm{int}) \quad \Gamma \vdash n{:}\mathsf{int} \quad \text{for } n \in \mathbb{Z}$$

$$(\mathrm{bool}) \quad \Gamma \vdash b{:}\mathsf{bool} \quad \text{for } b \in \{\mathbf{\mathsf{true}}, \mathbf{\mathsf{false}}\}$$

$$(\mathrm{op}\ +) \quad \frac{\begin{array}{c}\Gamma \vdash e_1{:}\mathsf{int}\\ \Gamma \vdash e_2{:}\mathsf{int}\end{array}}{\Gamma \vdash e_1 + e_2{:}\mathsf{int}} \qquad (\mathrm{op}\ \geq) \quad \frac{\begin{array}{c}\Gamma \vdash e_1{:}\mathsf{int}\\ \Gamma \vdash e_2{:}\mathsf{int}\end{array}}{\Gamma \vdash e_1 \geq e_2{:}\mathsf{bool}}$$

$$(\mathrm{if}) \quad \frac{\begin{array}{c}\Gamma \vdash e_1{:}\mathsf{bool}\\ \Gamma \vdash e_2{:}T\\ \Gamma \vdash e_3{:}T\end{array}}{\Gamma \vdash \mathbf{\mathsf{if}}\ e_1\ \mathbf{\mathsf{then}}\ e_2\ \mathbf{\mathsf{else}}\ e_3{:}T}$$

$$(\mathrm{assign}) \quad \frac{\begin{array}{c}\Gamma(\ell) = \mathsf{intref}\\ \Gamma \vdash e{:}\mathsf{int}\end{array}}{\Gamma \vdash \ell := e{:}\mathsf{unit}}$$

$$(\mathrm{deref}) \quad \frac{\Gamma(\ell) = \mathsf{intref}}{\Gamma \vdash !\ell{:}\mathsf{int}}$$

$$(\mathrm{skip}) \quad \Gamma \vdash \mathbf{\mathsf{skip}}{:}\mathsf{unit}$$

$$(\mathrm{seq}) \quad \frac{\begin{array}{c}\Gamma \vdash e_1{:}\mathsf{unit}\\ \Gamma \vdash e_2{:}T\end{array}}{\Gamma \vdash e_1; e_2{:}T}$$

$$(\mathrm{while}) \quad \frac{\begin{array}{c}\Gamma \vdash e_1{:}\mathsf{bool}\\ \Gamma \vdash e_2{:}\mathsf{unit}\end{array}}{\Gamma \vdash \mathbf{\mathsf{while}}\ e_1\ \mathbf{\mathsf{do}}\ e_2{:}\mathsf{unit}}$$

$$(\mathrm{var}) \quad \Gamma \vdash x{:}T \quad \text{if } \Gamma(x) = T$$

$$(\mathrm{fn}) \quad \frac{\Gamma, x{:}T \vdash e{:}T'}{\Gamma \vdash \mathbf{\mathsf{fn}}\ x{:}T \Rightarrow e : T \to T'}$$

$$(\mathrm{app}) \quad \frac{\Gamma \vdash e_1{:}T \to T' \qquad \Gamma \vdash e_2{:}T}{\Gamma \vdash e_1\ e_2{:}T'}$$

$$(\mathrm{let}) \quad \frac{\Gamma \vdash e_1{:}T \qquad \Gamma, x{:}T \vdash e_2{:}T'}{\Gamma \vdash \mathbf{\mathsf{let\ val}}\ x{:}T = e_1\ \mathbf{\mathsf{in}}\ e_2\ \mathbf{\mathsf{end}}{:}T'}$$

$$(\mathrm{let\ rec\ fn}) \quad \frac{\Gamma, x{:}T_1 \to T_2, y{:}T_1 \vdash e_1{:}T_2 \qquad \Gamma, x{:}T_1 \to T_2 \vdash e_2{:}T}{\Gamma \vdash \mathbf{\mathsf{let\ val\ rec}}\ x{:}T_1 \to T_2 = (\mathbf{\mathsf{fn}}\ y{:}T_1 \Rightarrow e_1)\ \mathbf{\mathsf{in}}\ e_2\ \mathbf{\mathsf{end}}{:}T}$$

4.7 Exercises

Exercise 19 ★*What are the free variables of the following?*

1. $x + ((\textbf{fn}\ y{:}\mathsf{int} \Rightarrow z)\ 2)$
2. $x + (\textbf{fn}\ y{:}\mathsf{int} \Rightarrow z)$
3. $\textbf{fn}\ y{:}\mathsf{int} \Rightarrow \textbf{fn}\ y{:}\mathsf{int} \Rightarrow \textbf{fn}\ y{:}\mathsf{int} \Rightarrow y$
4. $!l_0$
5. $\textbf{while}\ !l_0 \geq y\ \textbf{do}\ l_0 := x$

Draw their abstract syntax trees.

Exercise 20 ★*What are the following?*

1. $\{\textbf{fn}\ x{:}\mathsf{int} \Rightarrow y/z\}\textbf{fn}\ y{:}\mathsf{int} \Rightarrow z\ y$
2. $\{\textbf{fn}\ x{:}\mathsf{int} \Rightarrow x/x\}\textbf{fn}\ y{:}\mathsf{int} \Rightarrow x\ y$
3. $\{\textbf{fn}\ x{:}\mathsf{int} \Rightarrow x/x\}\textbf{fn}\ x{:}\mathsf{int} \Rightarrow x\ x$

Exercise 21 ★*Give typing derivations, or show why no derivation exists, for:*

1. $\textbf{if}\ 6\ \textbf{then}\ 7\ \textbf{else}\ 8$
2. $\textbf{fn}\ x{:}\mathsf{int} \Rightarrow x + (\textbf{fn}\ x{:}\mathsf{bool} \Rightarrow \textbf{if}\ x\ \textbf{then}\ 3\ \textbf{else}\ 4)\textbf{true}$

Exercise 22 ★★*Write a function of type* $\mathsf{unit} \to \mathsf{bool}$ *that, when applied to* $()$, *returns* **true** *in the CBV semantics and* **false** *in the CBN semantics. Can you do it without using the store?*

Exercise 23 ★★*Prove Lemma 7 (Substitution) on Slide 122.*

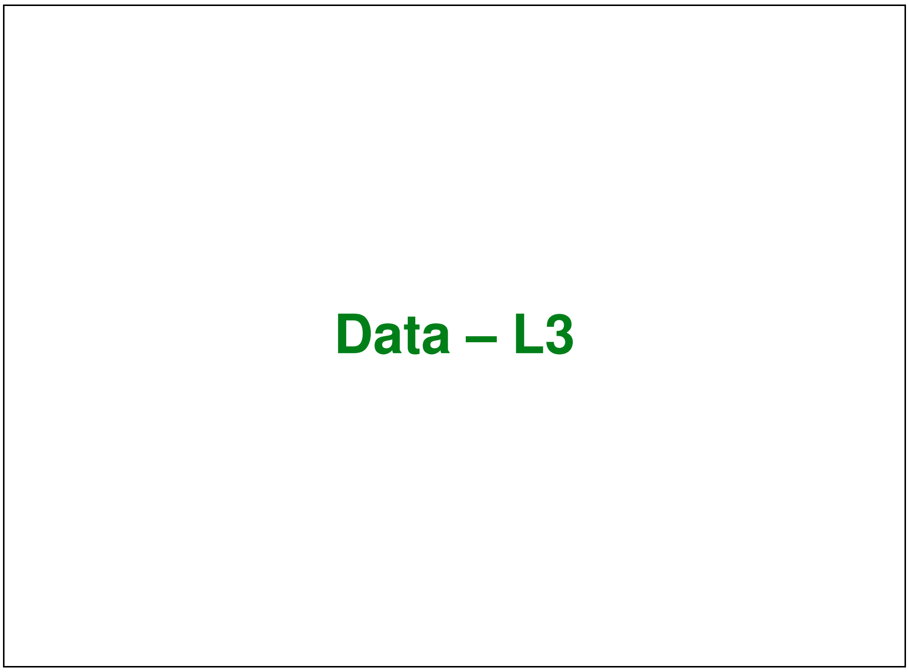
Exercise 24 ★★*Prove Theorem 14 (Type Preservation) on Slide 122.*

Exercise 25 ★★*Adapt the L2 implementation to CBN functions. Think of a few good test cases and check them in the new and old code.*

Exercise 26 ★★★*Re-implement the L2 interpreter to use closures instead of substitution.*

5 Data

Slide 144



Data – L3

5.1 Products, Sums, and Records

Slide 145

Products

$$T \quad ::= \quad ... \mid T_1 * T_2$$

$$e \quad ::= \quad ... \mid (e_1, e_2) \mid \#1\ e \mid \#2\ e$$

Design choices:

- pairs, not arbitray tuples – have $\mathsf{int} * (\mathsf{int} * \mathsf{int})$ and $(\mathsf{int} * \mathsf{int}) * \mathsf{int}$, but (a) they're different, and (b) we don't have $(\mathsf{int} * \mathsf{int} * \mathsf{int})$. In a full language you'd likely allow (b) (and still have it be a different type from the other two).
- have projections $\#1$ and $\#2$, not pattern matching **fn** $(x, y) \Rightarrow e$. A full language should allow the latter, as it often makes for much more elegant code.
- don't have $\#e\ e'$ (couldn't typecheck!).

Slide 146

Products - typing

(pair) $$\frac{\Gamma \vdash e_1{:}T_1 \qquad \Gamma \vdash e_2{:}T_2}{\Gamma \vdash (e_1, e_2){:}T_1 * T_2}$$

(proj1) $$\frac{\Gamma \vdash e{:}T_1 * T_2}{\Gamma \vdash \#1\ e{:}T_1}$$

(proj2) $$\frac{\Gamma \vdash e{:}T_1 * T_2}{\Gamma \vdash \#2\ e{:}T_2}$$

Slide 147

Products - reduction

$$v \quad ::= \quad ... \mid (v_1, v_2)$$

(pair1) $$\frac{\langle e_1, s\rangle \longrightarrow \langle e_1', s'\rangle}{\langle (e_1, e_2), s\rangle \longrightarrow \langle (e_1', e_2), s'\rangle}$$

(pair2) $$\frac{\langle e_2, s\rangle \longrightarrow \langle e_2', s'\rangle}{\langle (v_1, e_2), s\rangle \longrightarrow \langle (v_1, e_2'), s'\rangle}$$

(proj1) $\langle \#1 (v_1, v_2), s\rangle \longrightarrow \langle v_1, s\rangle$ (proj2) $\langle \#2 (v_1, v_2), s\rangle \longrightarrow \langle v_2, s\rangle$

(proj3) $$\frac{\langle e, s\rangle \longrightarrow \langle e', s'\rangle}{\langle \#1\ e, s\rangle \longrightarrow \langle \#1\ e', s'\rangle}$$ (proj4) $$\frac{\langle e, s\rangle \longrightarrow \langle e', s'\rangle}{\langle \#2\ e, s\rangle \longrightarrow \langle \#2\ e', s'\rangle}$$

Again, have to choose evaluation strategy (CBV) and evaluation order (left-to-right, for consistency).

Slide 148

Sums (or Variants, or Tagged Unions)

$$\begin{aligned} T \quad &::= \quad ... \mid T_1 + T_2 \\ e \quad &::= \quad ... \mid \textbf{inl}\ e{:}T \mid \textbf{inr}\ e{:}T \mid \\ &\qquad \textbf{case}\ e\ \textbf{of inl}\ (x_1{:}T_1) \Rightarrow e_1 \mid \textbf{inr}\ (x_2{:}T_2) \Rightarrow e_2 \end{aligned}$$

Those xs are binders.

Here we diverge slightly from mosml syntax - our $T_1 + T_2$ corresponds to the mosml `(T1,T2) Sum` in the context of the declaration

```
datatype ('a,'b) Sum = inl of 'a | inr of 'b;
```

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Sums - typing

(inl) $$\frac{\Gamma \vdash e{:}T_1}{\Gamma \vdash \textbf{inl}\ e{:}T_1 + T_2{:}T_1 + T_2}$$

(inr) $$\frac{\Gamma \vdash e{:}T_2}{\Gamma \vdash \textbf{inr}\ e{:}T_1 + T_2{:}T_1 + T_2}$$

(case) $$\frac{\begin{array}{l}\Gamma \vdash e{:}T_1 + T_2\\ \Gamma, x{:}T_1 \vdash e_1{:}T\\ \Gamma, y{:}T_2 \vdash e_2{:}T\end{array}}{\Gamma \vdash \textbf{case}\ e\ \textbf{of inl}\ (x{:}T_1) \Rightarrow e_1 \mid \textbf{inr}\ (y{:}T_2) \Rightarrow e_2{:}T}$$

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Why do we have these irritating type annotations? To maintain the unique typing property, as otherwise

$$\textbf{inl}\ 3{:}\mathsf{int} + \mathsf{int}$$

and

$$\textbf{inl}\ 3{:}\mathsf{int} + \mathsf{bool}$$

You might:

- have a compiler use a type inference algorithm that can infer them.
- require every sum type in a program to be declared, each with different names for the constructors **inl** , **inr** (cf OCaml).
- ...

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Sums - reduction

$$v \quad ::= \quad \ldots \mid \textbf{inl}\ v{:}T \mid \textbf{inr}\ v{:}T$$

(inl) $$\frac{\langle e, s\rangle \longrightarrow \langle e', s'\rangle}{\langle \textbf{inl}\ e{:}T, s\rangle \longrightarrow \langle \textbf{inl}\ e'{:}T, s'\rangle}$$

(case1) $$\frac{\langle e, s\rangle \longrightarrow \langle e', s'\rangle}{\begin{array}{l}\langle \textbf{case}\ e\ \textbf{of inl}\ (x{:}T_1) \Rightarrow e_1 \mid \textbf{inr}\ (y{:}T_2) \Rightarrow e_2, s\rangle \\ \longrightarrow \langle \textbf{case}\ e'\ \textbf{of inl}\ (x{:}T_1) \Rightarrow e_1 \mid \textbf{inr}\ (y{:}T_2) \Rightarrow e_2, s'\rangle\end{array}}$$

(case2) $\langle \textbf{case inl}\ v{:}T\ \textbf{of inl}\ (x{:}T_1) \Rightarrow e_1 \mid \textbf{inr}\ (y{:}T_2) \Rightarrow e_2, s\rangle$
$\longrightarrow \langle \{v/x\}e_1, s\rangle$

(inr) and (case3) like (inl) and (case2)

(inr) $$\frac{\langle e, s\rangle \longrightarrow \langle e', s'\rangle}{\langle \textbf{inr}\ e{:}T, s\rangle \longrightarrow \langle \textbf{inr}\ e'{:}T, s'\rangle}$$

(case3) $\langle \textbf{case inr}\ v{:}T\ \textbf{of inl}\ (x{:}T_1) \Rightarrow e_1 \mid \textbf{inr}\ (y{:}T_2) \Rightarrow e_2, s\rangle$
$\longrightarrow \langle \{v/y\}e_2, s\rangle$

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Constructors and Destructors

type	constructors	destructors
$T \rightarrow T$	**fn** $x{:}T \Rightarrow$ _	_ e
$T * T$	$(\_, \_)$	$\#1$ _ $\quad \#2$ _
$T + T$	**inl** (_) **inr** (_)	**case**
bool	**true** **false**	**if**

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The Curry-Howard Isomorphism

(var) $\Gamma, x{:}T \vdash x{:}T$

(fn) $$\frac{\Gamma, x{:}T \vdash e{:}T'}{\Gamma \vdash \textbf{fn}\ x{:}T \Rightarrow e : T \to T'}$$

(app) $$\frac{\Gamma \vdash e_1{:}T \to T' \qquad \Gamma \vdash e_2{:}T}{\Gamma \vdash e_1\ e_2{:}T'}$$

(pair) $$\frac{\Gamma \vdash e_1{:}T_1 \qquad \Gamma \vdash e_2{:}T_2}{\Gamma \vdash (e_1, e_2){:}T_1 * T_2}$$

(proj1) $$\frac{\Gamma \vdash e{:}T_1 * T_2}{\Gamma \vdash \#1\ e{:}T_1}$$ (proj2) $$\frac{\Gamma \vdash e{:}T_1 * T_2}{\Gamma \vdash \#2\ e{:}T_2}$$

(inl) $$\frac{\Gamma \vdash e{:}T_1}{\Gamma \vdash \textbf{inl}\ e{:}T_1 + T_2{:}T_1 + T_2}$$

(inr), (case), (unit), (zero), etc.. – but not (letrec)

$\Gamma, P \vdash P$

$$\frac{\Gamma, P \vdash P'}{\Gamma \vdash P \to P'}$$

$$\frac{\Gamma \vdash P \to P' \qquad \Gamma \vdash P}{\Gamma \vdash P'}$$

$$\frac{\Gamma \vdash P_1 \quad \Gamma \vdash P_2}{\Gamma \vdash P_1 \wedge P_2}$$

$$\frac{\Gamma \vdash P_1 \wedge P_2}{\Gamma \vdash P_1} \qquad \frac{\Gamma \vdash P_1 \wedge P_2}{\Gamma \vdash P_2}$$

$$\frac{\Gamma \vdash P_1}{\Gamma \vdash P_1 \vee P_2}$$

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ML Datatypes

Datatypes in ML generalise both sums and products, in a sense

```
datatype IntList = Null of unit
                 | Cons of Int * IntList
```

is (roughly!) like saying

```
IntList = unit + (Int * IntList)
```

Note (a) this involves recursion at the type level (e.g. types for binary trees), (b) it introduces constructors (`Null` and `Cons`) for each summand, and (c) it's *generative* - two different declarations of `IntList` will make different types. Making all that precise is beyond the scope of this course.

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Records

A mild generalisation of products that'll be handy later.

Take field labels

Labels $lab \in \mathbb{LAB}$ for a set $\mathbb{LAB} = \{\mathrm{p}, \mathrm{q}, ...\}$

$$\begin{array}{lcl} T & ::= & ... \mid \{lab_1{:}T_1, .., lab_k{:}T_k\} \\ e & ::= & ... \mid \{lab_1 = e_1, .., lab_k = e_k\} \mid \#lab\ e \end{array}$$

(where in each record (type or expression) no lab occurs more than once)

Note:

- The condition on record formation means that our syntax is no longer 'free'. Formally, we should have a well-formedness judgment on types.
- Labels are not the same syntactic class as variables, so $(\textbf{fn}\ \mathrm{x}{:}T \Rightarrow \{\mathrm{x} = 3\})$ is not an expression.
- Does the order of fields matter? Can you use reuse labels in different record types? The typing rules will fix an answer.
- In ML a pair $(\textbf{true}, \textbf{fn}\ \mathrm{x}{:}\mathsf{int} \Rightarrow \mathrm{x})$ is actually syntactic sugar for a record $\{1 = \textbf{true}, 2 = \textbf{fn}\ \mathrm{x}{:}\mathsf{int} \Rightarrow \mathrm{x}\}$.
- Note that $\#lab\ e$ is not an application, it just looks like one in the concrete syntax.
- Again we will choose a left-to-right evaluation order for consistency.

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Records - typing

(record) $$\frac{\Gamma \vdash e_1{:}T_1 \quad .. \quad \Gamma \vdash e_k{:}T_k}{\Gamma \vdash \{lab_1 = e_1, .., lab_k = e_k\}{:}\{lab_1{:}T_1, .., lab_k{:}T_k\}}$$

(recordproj) $$\frac{\Gamma \vdash e{:}\{lab_1{:}T_1, .., lab_k{:}T_k\}}{\Gamma \vdash \#lab_i\ e{:}T_i}$$

- Here the field order matters, so $(\textbf{fn}\ \mathrm{x}{:}\{\mathit{foo}{:}\mathsf{int}, \mathit{bar}{:}\mathsf{bool}\} \Rightarrow \mathrm{x})\{\mathit{bar} = \textbf{true}, \mathit{foo} = 17\}$ does not typecheck. In ML, though, the order doesn't matter – so mosml will accept strictly more programs in this syntax than this type system allows.
- Here and in mosml can reuse labels, so $\{\} \vdash (\{\mathit{foo} = 17\}, \{\mathit{foo} = \textbf{true}\}){:}\{\mathit{foo}{:}\mathsf{int}\} * \{\mathit{foo}{:}\mathsf{bool}\}$ is legal, but in some languages (e.g. OCaml) you can't.

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Records - reduction

$$v \quad ::= \quad ... \mid \{lab_1 = v_1, .., lab_k = v_k\}$$

(record1) $$\frac{\langle e_i, s\rangle \longrightarrow \langle e_i', s'\rangle}{\begin{array}{l}\langle \{lab_1 = v_1, .., lab_i = e_i, .., lab_k = e_k\}, s\rangle \\ \longrightarrow \langle \{lab_1 = v_1, .., lab_i = e_i', .., lab_k = e_k\}, s'\rangle\end{array}}$$

(record2) $$\langle \#lab_i\ \{lab_1 = v_1, .., lab_k = v_k\}, s\rangle \longrightarrow \langle v_i, s\rangle$$

(record3) $$\frac{\langle e, s\rangle \longrightarrow \langle e', s'\rangle}{\langle \#lab_i\ e, s\rangle \longrightarrow \langle \#lab_i\ e', s'\rangle}$$

5.2 Mutable Store

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Mutable Store

Most languages have some kind of mutable store. Two main choices:

1 What we've got in L1 and L2:

$$e \quad ::= \quad ... \mid \ell := e \mid !\ell \mid x$$

- locations store mutable values
- variables refer to a previously-calculated value, immutably
- explicit dereferencing and assignment operators for locations
  $\textbf{fn}\ \mathrm{x}{:}\mathsf{int} \Rightarrow l := (!l) + \mathrm{x}$

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2 The C-way (also Java etc).

- variables let you refer to a previously calculated value *and* let you overwrite that value with another.
- implicit dereferencing and assignment,

```
void foo(x:int) {
  l =  l + x
  ...}
```

- have some limited type machinery (const qualifiers) to limit mutability.

– pros and cons: ....

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References

Staying with 1 here. But, those L1/L2 references are very limited:

- can only store ints - for uniformity, would like to store any value
- cannot create new locations (all must exist at beginning)
- cannot write functions that abstract on locations **fn** l:intref $\Rightarrow !l$

So, generalise.

T ::= ... | T ref

T_{loc} ::= ~~intref~~ T ref

e ::= ... | ~~$\ell := e$~~ | ~~$!\ell$~~

| $e_1 := e_2$ | $!e$ | ref e | ℓ

Have locations in the expression syntax, but that is just so we can express the intermediate states of computations – whole programs now should have no locations in at the start, but can create them with ref. They can have variables of T ref type, e.g.**fn** x:int ref $\Rightarrow$!x.

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References - Typing

(ref) $$\frac{\Gamma \vdash e{:}T}{\Gamma \vdash \ \mathsf{ref}\, e : T \ \mathsf{ref}}$$

(assign) $$\frac{\begin{array}{c}\Gamma \vdash e_1{:}T \ \mathsf{ref} \\ \Gamma \vdash e_2{:}T\end{array}}{\Gamma \vdash e_1 := e_2{:}\mathsf{unit}}$$

(deref) $$\frac{\Gamma \vdash e{:}T \ \mathsf{ref}}{\Gamma \vdash !e{:}T}$$

(loc) $$\frac{\Gamma(\ell) = T \ \mathsf{ref}}{\Gamma \vdash \ell{:}T \ \mathsf{ref}}$$

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References – Reduction

A location is a value:

$$v \quad ::= \quad ... \mid \ell$$

Stores s were finite partial maps from $\mathbb{L}$ to $\mathbb{Z}$. From now on, take them to be finite partial maps from $\mathbb{L}$ to the set of all values.

(ref1) $$\langle \ \mathsf{ref}\, v, s\rangle \longrightarrow \langle \ell, s + \{\ell \mapsto v\}\rangle \quad \ell \notin \mathsf{dom}(s)$$

(ref2) $$\frac{\langle e, s\rangle \longrightarrow \langle e', s'\rangle}{\langle \ \mathsf{ref}\, e, s\rangle \longrightarrow \langle \ \mathsf{ref}\, e', s'\rangle}$$

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(deref1) $\langle !\ell, s\rangle \longrightarrow \langle v, s\rangle$ if $\ell \in \mathsf{dom}(s)$ and $s(\ell) = v$

(deref2) $$\frac{\langle e, s\rangle \longrightarrow \langle e', s'\rangle}{\langle !e, s\rangle \longrightarrow \langle !e', s'\rangle}$$

(assign1) $\langle \ell := v, s\rangle \longrightarrow \langle \mathbf{skip}, s + \{\ell \mapsto v\}\rangle$ if $\ell \in \mathsf{dom}(s)$

(assign2) $$\frac{\langle e, s\rangle \longrightarrow \langle e', s'\rangle}{\langle \ell := e, s\rangle \longrightarrow \langle \ell := e', s'\rangle}$$

(assign3) $$\frac{\langle e, s\rangle \longrightarrow \langle e', s'\rangle}{\langle e := e_2, s\rangle \longrightarrow \langle e' := e_2, s'\rangle}$$

- ref *has* to do something – (ref 0, ref 0) should return a pair of two new locations, each containing 0, not a pair of one location repeated.
- Note the typing and this dynamics permit locations to contain locations, e.g. ref(ref3)
- This semantics no longer has determinacy, for a technical reason – new locations are chosen arbitrarily. At the cost of some slight semantic complexity, we could regain determinacy by working 'up to alpha for locations'.
- What *is* the store:
  1. an array of bytes,
  2. an array of values, or
  3. a partial function from locations to values?

  We take the third, most abstract option. Within the language one cannot do arithmetic on locations (just as well!) (can in C, can't in Java) or test whether one is bigger than another (in presence of garbage collection, they may not stay that way). Might or might not even be able to test them for equality (can in ML, cannot in L3).
- This store just grows during computation – an implementation can garbage collect (in many fancy ways), but platonic memory is free.

  We *don't* have an explicit deallocation operation – if you do, you need a very baroque type system to prevent dangling pointers being dereferenced. We don't have uninitialised locations (cf. null pointers), so don't have to worry about dereferencing null.

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Type-checking the store

For L1, our type properties used $\mathsf{dom}(\Gamma) \subseteq \mathsf{dom}(s)$ to express the condition 'all locations mentioned in Γ exist in the store s'.

Now need more: for each $\ell \in \mathsf{dom}(s)$ need that $s(\ell)$ is typable.

Moreover, $s(\ell)$ might contain some other locations...

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Type-checking the store – Example

Consider

$$\begin{aligned} e \quad = \quad & \textbf{let val}\ \mathrm{x}{:}(\mathsf{int} \rightarrow \mathsf{int})\ \mathsf{ref} = \ \mathsf{ref}(\textbf{fn}\ \mathrm{z}{:}\mathsf{int} \Rightarrow \mathrm{z})\ \textbf{in} \\ & (\mathrm{x} := (\textbf{fn}\ \mathrm{z}{:}\mathsf{int} \Rightarrow \textbf{if}\ \mathrm{z} \geq 1\ \textbf{then}\ \mathrm{z} + ((!\mathrm{x})\ (\mathrm{z} + -1))\ \textbf{else}\ 0); \\ & (!\mathrm{x})\ 3)\ \textbf{end} \end{aligned}$$

which has reductions

$$\begin{aligned} & \langle e, \{\}\rangle \longrightarrow^* \\ & \langle e_1, \{l_1 \mapsto (\textbf{fn}\ \mathrm{z}{:}\mathsf{int} \Rightarrow \mathrm{z})\}\rangle \longrightarrow^* \\ & \langle e_2, \{l_1 \mapsto (\textbf{fn}\ \mathrm{z}{:}\mathsf{int} \Rightarrow \textbf{if}\ \mathrm{z} \geq 1\ \textbf{then}\ \mathrm{z} + ((!l_1)\ (\mathrm{z} + -1))\ \textbf{else}\ 0)\}\rangle \\ & \longrightarrow^* \langle 6, ...\rangle \end{aligned}$$

For reference, e_1 and e_2 are

$$\begin{aligned} e_1 \quad &= \quad l_1 := (\textbf{fn}\ \mathrm{z}{:}\mathsf{int} \Rightarrow \textbf{if}\ \mathrm{z} \geq 1\ \textbf{then}\ \mathrm{z} + ((!l_1)\ (\mathrm{z} + -1))\ \textbf{else}\ 0); \\ & \quad\ ((!l_1)\ 3) \\ e_2 \quad &= \quad \textbf{skip}; ((!l_1)\ 3) \end{aligned}$$

Have made a recursive function by 'tying the knot by hand', not using **let val rec** .

To do this we needed to store function values – couldn't do this in L2, so this doesn't contradict the normalisation theorem we had there.

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So, say $\Gamma \vdash s$ if $\forall \ell \in \mathrm{dom}(s).\exists T.\Gamma(\ell) = T\ \mathsf{ref} \wedge \Gamma \vdash s(\ell){:}T$.

The statement of type preservation will then be:

Theorem 14 (Type Preservation) *If e closed and $\Gamma \vdash e{:}T$ and $\Gamma \vdash s$ and $\langle e, s\rangle \longrightarrow \langle e', s'\rangle$ then for some Γ' with disjoint domain to Γ we have $\Gamma, \Gamma' \vdash e'{:}T$ and $\Gamma, \Gamma' \vdash s'$.*

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Implementation

The collected definition so far is in the notes, called L3.

It is again a mosml fragment (modulo the syntax for $T + T$ of Slide 148), so you can run programs. The mosml record typing is more liberal that that of L3, though.

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Evaluation Contexts

Define *evaluation contexts*

$$
\begin{aligned}
E \quad ::= \quad & \_\ op\ e \mid v\ op\ \_ \mid \textbf{if}\ \_\ \textbf{then}\ e\ \textbf{else}\ e \mid \\
& \_ ; e \mid \\
& \_\ e \mid v\ \_ \mid \\
& \textbf{let val}\ x{:}T = \_\ \textbf{in}\ e_2\ \textbf{end} \mid \\
& (\_, e) \mid (v, \_) \mid \#1\ \_ \mid \#2\ \_ \mid \\
& \textbf{inl}\ \_{:}T \mid \textbf{inr}\ \_{:}T \mid \\
& \textbf{case}\ \_\ \textbf{of inl}\ (x{:}T) \Rightarrow e \mid \textbf{inr}\ (x{:}T) \Rightarrow e \mid \\
& \{lab_1 = v_1, .., lab_i = \_, .., lab_k = e_k\} \mid \#lab\ \_ \mid \\
& \_ := e \mid v := \_ \mid !\_ \mid \ \textsf{ref}\_
\end{aligned}
$$

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and have the single *context* rule

$$
\text{(eval)} \quad \frac{\langle e, s\rangle \longrightarrow \langle e', s'\rangle}{\langle E[e], s\rangle \longrightarrow \langle E[e'], s'\rangle}
$$

replacing the rules (all those with ≥ 1 premise) (op1), (op2), (seq2), (if3), (app1), (app2), (let1), (pair1), (pair2), (proj3), (proj4), (inl), (inr), (case1), (record1), (record3), (ref2), (deref2), (assign2), (assign3).

To (eval) we add all the *computation* rules (all the rest) (op $+$), (op $\geq$), (seq1), (if1), (if2), (while), (fn), (let2), (letrecfn), (proj1), (proj2), (case2), (case3), (record2), (ref1), (deref1), (assign1).

Theorem 15 *The two definitions of $\longrightarrow$ define the same relation.*

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A Little (Oversimplified!) History

Formal logic	1880–
Untyped lambda-calculus	1930s
Simply-typed lambda calculus	1940s
Fortran	1950s
Curry-Howard, Algol 60, Algol 68, SECD machine (64)	1960s
Pascal, Polymorphism, ML, PLC	1970s
Structured Operational Semantics	1981–
Standard ML definition	1985
Haskell	1987
Subtyping	1980s
Module systems	1980–
Object calculus	1990–
Typed assembly and intermediate languages	1990–

And now? module systems, distribution, mobility, reasoning about objects, security, typed compilation,

approximate analyses,.......

5.3 L3: Collected Definition

L3 Syntax

Booleans $b \in \mathbb{B} = \{\mathbf{true}, \mathbf{false}\}$
Integers $n \in \mathbb{Z} = \{..., -1, 0, 1, ...\}$
Locations $\ell \in \mathbb{L} = \{l, l_0, l_1, l_2, ...\}$
Variables $x \in \mathbb{X}$ for a set $\mathbb{X} = \{\mathrm{x}, \mathrm{y}, \mathrm{z}, ...\}$
Labels $lab \in \mathbb{LAB}$ for a set $\mathbb{LAB} = \{\mathrm{p}, \mathrm{q}, ...\}$

Operations $op ::= + \mid \geq$

Types:

$$T \quad ::= \quad \mathsf{int} \mid \mathsf{bool} \mid \mathsf{unit} \mid T_1 \rightarrow T_2 | T_1 * T_2 | T_1 + T_2 | \{lab_1{:}T_1, .., lab_k{:}T_k\} | T\ \mathsf{ref}$$

Expressions

$$\begin{array}{lll}
e & ::= & n \mid b \mid e_1\ op\ e_2 \mid \mathbf{if}\ e_1\ \mathbf{then}\ e_2\ \mathbf{else}\ e_3 \mid \\
 & & e_1 := e_2 \mid !e \mid \mathsf{ref}\,e \mid \ell \mid \\
 & & \mathbf{skip} \mid e_1; e_2 \mid \\
 & & \mathbf{while}\ e_1\ \mathbf{do}\ e_2 \mid \\
 & & \mathbf{fn}\ x{:}T \Rightarrow e \mid e_1\ e_2 \mid x \mid \\
 & & \mathbf{let\ val}\ x{:}T = e_1\ \mathbf{in}\ e_2\ \mathbf{end} \mid \\
 & & \mathbf{let\ val\ rec}\ x{:}T_1 \rightarrow T_2 = (\mathbf{fn}\ y{:}T_1 \Rightarrow e_1)\ \mathbf{in}\ e_2\ \mathbf{end} \mid \\
 & & (e_1, e_2) \mid \#1\ e \mid \#2\ e \mid \\
 & & \mathbf{inl}\ e{:}T \mid \mathbf{inr}\ e{:}T \mid \\
 & & \mathbf{case}\ e\ \mathbf{of\ inl}\ (x_1{:}T_1) \Rightarrow e_1 \mid \mathbf{inr}\ (x_2{:}T_2) \Rightarrow e_2 \mid \\
 & & \{lab_1 = e_1, .., lab_k = e_k\} \mid \#lab\ e
\end{array}$$

(where in each record (type or expression) no lab occurs more than once)

In expressions $\mathbf{fn}\ x{:}T \Rightarrow e$ the x is a *binder*. In expressions $\mathbf{let\ val}\ x{:}T = e_1\ \mathbf{in}\ e_2\ \mathbf{end}$ the x is a binder. In expressions $\mathbf{let\ val\ rec}\ x{:}T_1 \rightarrow T_2 = (\mathbf{fn}\ y{:}T_1 \Rightarrow e_1)\ \mathbf{in}\ e_2\ \mathbf{end}$ the y binds in e_1; the x binds in $(\mathbf{fn}\ y{:}T \Rightarrow e_1)$ and in e_2. In $\mathbf{case}\ e\ \mathbf{of\ inl}\ (x_1{:}T_1) \Rightarrow e_1 \mid \mathbf{inr}\ (x_2{:}T_2) \Rightarrow e_2$ the x_1 binds in e_1 and the x_2 binds in e_2.

L3 Semantics

Stores s were finite partial maps from $\mathbb{L}$ to $\mathbb{Z}$. From now on, take them to be finite partial maps from $\mathbb{L}$ to the set of all values.

Values $v ::= b \mid n \mid \textbf{skip} \mid \textbf{fn}\ x{:}T \Rightarrow e | (v_1, v_2) | \textbf{inl}\ v{:}T \mid \textbf{inr}\ v{:}T | \{lab_1 = v_1, .., lab_k = v_k\} | \ell$

$$(\text{op } +) \quad \langle n_1 + n_2, s\rangle \longrightarrow \langle n, s\rangle \quad \text{if } n = n_1 + n_2$$

$$(\text{op } \geq) \quad \langle n_1 \geq n_2, s\rangle \longrightarrow \langle b, s\rangle \quad \text{if } b = (n_1 \geq n_2)$$

$$(\text{op1}) \quad \frac{\langle e_1, s\rangle \longrightarrow \langle e_1', s'\rangle}{\langle e_1\ op\ e_2, s\rangle \longrightarrow \langle e_1'\ op\ e_2, s'\rangle}$$

$$(\text{op2}) \quad \frac{\langle e_2, s\rangle \longrightarrow \langle e_2', s'\rangle}{\langle v\ op\ e_2, s\rangle \longrightarrow \langle v\ op\ e_2', s'\rangle}$$

$$(\text{seq1}) \quad \langle \textbf{skip}; e_2, s\rangle \longrightarrow \langle e_2, s\rangle$$

$$(\text{seq2}) \quad \frac{\langle e_1, s\rangle \longrightarrow \langle e_1', s'\rangle}{\langle e_1; e_2, s\rangle \longrightarrow \langle e_1'; e_2, s'\rangle}$$

$$(\text{if1}) \quad \langle \textbf{if true then}\ e_2\ \textbf{else}\ e_3, s\rangle \longrightarrow \langle e_2, s\rangle$$

$$(\text{if2}) \quad \langle \textbf{if false then}\ e_2\ \textbf{else}\ e_3, s\rangle \longrightarrow \langle e_3, s\rangle$$

$$(\text{if3}) \quad \frac{\langle e_1, s\rangle \longrightarrow \langle e_1', s'\rangle}{\langle \textbf{if}\ e_1\ \textbf{then}\ e_2\ \textbf{else}\ e_3, s\rangle \longrightarrow \langle \textbf{if}\ e_1'\ \textbf{then}\ e_2\ \textbf{else}\ e_3, s'\rangle}$$

(while)
$$\langle \textbf{while}\ e_1\ \textbf{do}\ e_2, s\rangle \longrightarrow \langle \textbf{if}\ e_1\ \textbf{then}\ (e_2; \textbf{while}\ e_1\ \textbf{do}\ e_2)\ \textbf{else skip}, s\rangle$$

$$(\text{app1}) \quad \frac{\langle e_1, s\rangle \longrightarrow \langle e_1', s'\rangle}{\langle e_1\ e_2, s\rangle \longrightarrow \langle e_1'\ e_2, s'\rangle}$$

$$(\text{app2}) \quad \frac{\langle e_2, s\rangle \longrightarrow \langle e_2', s'\rangle}{\langle v\ e_2, s\rangle \longrightarrow \langle v\ e_2', s'\rangle}$$

$$(\text{fn}) \quad \langle (\textbf{fn}\ x{:}T \Rightarrow e)\ v, s\rangle \longrightarrow \langle \{v/x\}e, s\rangle$$

(let1)
$$\frac{\langle e_1, s\rangle \longrightarrow \langle e_1', s'\rangle}{\langle \textbf{let val}\ x{:}T = e_1\ \textbf{in}\ e_2\ \textbf{end}, s\rangle \longrightarrow \langle \textbf{let val}\ x{:}T = e_1'\ \textbf{in}\ e_2\ \textbf{end}, s'\rangle}$$

(let2)
$$\langle \textbf{let val}\ x{:}T = v\ \textbf{in}\ e_2\ \textbf{end}, s\rangle \longrightarrow \langle \{v/x\}e_2, s\rangle$$

(letrecfn) $\textbf{let val rec}\ x{:}T_1 \to T_2 = (\textbf{fn}\ y{:}T_1 \Rightarrow e_1)\ \textbf{in}\ e_2\ \textbf{end}$
$\longrightarrow$
$\{(\textbf{fn}\ y{:}T_1 \Rightarrow \textbf{let val rec}\ x{:}T_1 \to T_2 = (\textbf{fn}\ y{:}T_1 \Rightarrow e_1)\ \textbf{in}\ e_1\ \textbf{end})/x\}e_2$

(pair1) $$\frac{\langle e_1, s\rangle \longrightarrow \langle e_1', s'\rangle}{\langle (e_1, e_2), s\rangle \longrightarrow \langle (e_1', e_2), s'\rangle}$$

(pair2) $$\frac{\langle e_2, s\rangle \longrightarrow \langle e_2', s'\rangle}{\langle (v_1, e_2), s\rangle \longrightarrow \langle (v_1, e_2'), s'\rangle}$$

(proj1) $\langle \#1 (v_1, v_2), s\rangle \longrightarrow \langle v_1, s\rangle$ (proj2) $\langle \#2 (v_1, v_2), s\rangle \longrightarrow \langle v_2, s\rangle$

(proj3) $\dfrac{\langle e, s\rangle \longrightarrow \langle e', s'\rangle}{\langle \#1\ e, s\rangle \longrightarrow \langle \#1\ e', s'\rangle}$ (proj4) $\dfrac{\langle e, s\rangle \longrightarrow \langle e', s'\rangle}{\langle \#2\ e, s\rangle \longrightarrow \langle \#2\ e', s'\rangle}$

(inl) $$\frac{\langle e, s\rangle \longrightarrow \langle e', s'\rangle}{\langle \textsf{\textbf{inl}}\ e{:}T, s\rangle \longrightarrow \langle \textsf{\textbf{inl}}\ e'{:}T, s'\rangle}$$

(case1) $$\frac{\langle e, s\rangle \longrightarrow \langle e', s'\rangle}{\begin{array}{l}\langle \textsf{\textbf{case}}\ e\ \textsf{\textbf{of}}\ \textsf{\textbf{inl}}\ (x{:}T_1) \Rightarrow e_1 \mid \textsf{\textbf{inr}}\ (y{:}T_2) \Rightarrow e_2, s\rangle\\ \longrightarrow \langle \textsf{\textbf{case}}\ e'\ \textsf{\textbf{of}}\ \textsf{\textbf{inl}}\ (x{:}T_1) \Rightarrow e_1 \mid \textsf{\textbf{inr}}\ (y{:}T_2) \Rightarrow e_2, s'\rangle\end{array}}$$

(case2) $\langle \textsf{\textbf{case}}\ \textsf{\textbf{inl}}\ v{:}T\ \textsf{\textbf{of}}\ \textsf{\textbf{inl}}\ (x{:}T_1) \Rightarrow e_1 \mid \textsf{\textbf{inr}}\ (y{:}T_2) \Rightarrow e_2, s\rangle$
$\longrightarrow \langle \{v/x\}e_1, s\rangle$

(inr) and (case3) like (inl) and (case2)

(inr) $$\frac{\langle e, s\rangle \longrightarrow \langle e', s'\rangle}{\langle \textsf{\textbf{inr}}\ e{:}T, s\rangle \longrightarrow \langle \textsf{\textbf{inr}}\ e'{:}T, s'\rangle}$$

(case3) $\langle \textsf{\textbf{case}}\ \textsf{\textbf{inr}}\ v{:}T\ \textsf{\textbf{of}}\ \textsf{\textbf{inl}}\ (x{:}T_1) \Rightarrow e_1 \mid \textsf{\textbf{inr}}\ (y{:}T_2) \Rightarrow e_2, s\rangle$
$\longrightarrow \langle \{v/y\}e_2, s\rangle$

(record1) $$\frac{\langle e_i, s\rangle \longrightarrow \langle e_i', s'\rangle}{\begin{array}{l}\langle \{lab_1 = v_1, .., lab_i = e_i, .., lab_k = e_k\}, s\rangle\\ \longrightarrow \langle \{lab_1 = v_1, .., lab_i = e_i', .., lab_k = e_k\}, s'\rangle\end{array}}$$

(record2) $\langle \#lab_i\ \{lab_1 = v_1, .., lab_k = v_k\}, s\rangle \longrightarrow \langle v_i, s\rangle$

(record3) $$\frac{\langle e, s\rangle \longrightarrow \langle e', s'\rangle}{\langle \#lab_i\ e, s\rangle \longrightarrow \langle \#lab_i\ e', s'\rangle}$$

(ref1) $\langle\ \textsf{ref}\, v, s\rangle \longrightarrow \langle \ell, s + \{\ell \mapsto v\}\rangle \quad \ell \notin \mathrm{dom}(s)$

(ref2) $$\frac{\langle e, s\rangle \longrightarrow \langle e', s'\rangle}{\langle\ \textsf{ref}\, e, s\rangle \longrightarrow \langle\ \textsf{ref}\, e', s'\rangle}$$

(deref1) $\langle !\ell, s\rangle \longrightarrow \langle v, s\rangle$ if $\ell \in \mathrm{dom}(s)$ and $s(\ell) = v$

(deref2) $$\frac{\langle e, s\rangle \longrightarrow \langle e', s'\rangle}{\langle !e, s\rangle \longrightarrow \langle !e', s'\rangle}$$

(assign1) $\langle \ell := v, s\rangle \longrightarrow \langle \mathbf{skip}, s + \{\ell \mapsto v\}\rangle$ if $\ell \in \mathrm{dom}(s)$

(assign2) $$\frac{\langle e, s\rangle \longrightarrow \langle e', s'\rangle}{\langle \ell := e, s\rangle \longrightarrow \langle \ell := e', s'\rangle}$$

(assign3) $$\frac{\langle e, s\rangle \longrightarrow \langle e', s'\rangle}{\langle e := e_2, s\rangle \longrightarrow \langle e' := e_2, s'\rangle}$$

L3 Typing

Take $\Gamma \in \mathrm{TypeEnv2}$, the finite partial functions from $\mathbb{L} \cup \mathbb{X}$ to $\mathrm{T}_{\mathrm{loc}} \cup \mathrm{T}$ such that

$\forall \ell \in \mathrm{dom}(\Gamma).\Gamma(\ell) \in \mathrm{T}_{\mathrm{loc}}$

$\forall x \in \mathrm{dom}(\Gamma).\Gamma(x) \in \mathrm{T}$

(int) $\Gamma \vdash n{:}\mathsf{int}$ for $n \in \mathbb{Z}$

(bool) $\Gamma \vdash b{:}\mathsf{bool}$ for $b \in \{\mathbf{true}, \mathbf{false}\}$

(op $+$) $$\frac{\begin{array}{c}\Gamma \vdash e_1{:}\mathsf{int}\\ \Gamma \vdash e_2{:}\mathsf{int}\end{array}}{\Gamma \vdash e_1 + e_2{:}\mathsf{int}}$$

(op $\geq$) $$\frac{\begin{array}{c}\Gamma \vdash e_1{:}\mathsf{int}\\ \Gamma \vdash e_2{:}\mathsf{int}\end{array}}{\Gamma \vdash e_1 \geq e_2{:}\mathsf{bool}}$$

(if) $$\frac{\begin{array}{c}\Gamma \vdash e_1{:}\mathsf{bool}\\ \Gamma \vdash e_2{:}T\\ \Gamma \vdash e_3{:}T\end{array}}{\Gamma \vdash \mathbf{if}\ e_1\ \mathbf{then}\ e_2\ \mathbf{else}\ e_3{:}T}$$

(assign) $$\frac{\begin{array}{c}\Gamma(\ell) = \mathsf{intref}\\ \Gamma \vdash e{:}\mathsf{int}\end{array}}{\Gamma \vdash \ell := e{:}\mathsf{unit}}$$

(deref) $$\frac{\Gamma(\ell) = \mathsf{intref}}{\Gamma \vdash !\ell{:}\mathsf{int}}$$

(skip) $\Gamma \vdash \mathbf{skip}{:}\mathsf{unit}$

(seq) $$\frac{\begin{array}{c}\Gamma \vdash e_1{:}\mathsf{unit}\\ \Gamma \vdash e_2{:}T\end{array}}{\Gamma \vdash e_1; e_2{:}T}$$

(while) $$\frac{\begin{array}{c}\Gamma \vdash e_1{:}\mathsf{bool}\\ \Gamma \vdash e_2{:}\mathsf{unit}\end{array}}{\Gamma \vdash \mathbf{while}\ e_1\ \mathbf{do}\ e_2{:}\mathsf{unit}}$$

$$\text{(var)} \quad \Gamma \vdash x{:}T \quad \text{ if } \Gamma(x) = T$$

$$\text{(fn)} \quad \frac{\Gamma, x{:}T \vdash e{:}T'}{\Gamma \vdash \mathbf{fn}\ x{:}T \Rightarrow e : T \to T'}$$

$$\text{(app)} \quad \frac{\Gamma \vdash e_1{:}T \to T' \qquad \Gamma \vdash e_2{:}T}{\Gamma \vdash e_1\ e_2{:}T'}$$

$$\text{(let)} \quad \frac{\Gamma \vdash e_1{:}T \qquad \Gamma, x{:}T \vdash e_2{:}T'}{\Gamma \vdash \mathbf{let\ val}\ x{:}T = e_1\ \mathbf{in}\ e_2\ \mathbf{end}{:}T'}$$

$$\text{(let rec fn)} \quad \frac{\Gamma, x{:}T_1 \to T_2, y{:}T_1 \vdash e_1{:}T_2 \qquad \Gamma, x{:}T_1 \to T_2 \vdash e_2{:}T}{\Gamma \vdash \mathbf{let\ val\ rec}\ x{:}T_1 \to T_2 = (\mathbf{fn}\ y{:}T_1 \Rightarrow e_1)\ \mathbf{in}\ e_2\ \mathbf{end}{:}T}$$

$$\text{(pair)} \quad \frac{\Gamma \vdash e_1{:}T_1 \qquad \Gamma \vdash e_2{:}T_2}{\Gamma \vdash (e_1, e_2){:}T_1 * T_2}$$

$$\text{(proj1)} \quad \frac{\Gamma \vdash e{:}T_1 * T_2}{\Gamma \vdash \#1\ e{:}T_1}$$

$$\text{(proj2)} \quad \frac{\Gamma \vdash e{:}T_1 * T_2}{\Gamma \vdash \#2\ e{:}T_2}$$

$$\text{(inl)} \quad \frac{\Gamma \vdash e{:}T_1}{\Gamma \vdash \mathbf{inl}\ e{:}T_1 + T_2{:}T_1 + T_2}$$

$$\text{(inr)} \quad \frac{\Gamma \vdash e{:}T_2}{\Gamma \vdash \mathbf{inr}\ e{:}T_1 + T_2{:}T_1 + T_2}$$

$$\text{(case)} \quad \frac{\begin{array}{l}\Gamma \vdash e{:}T_1 + T_2\\ \Gamma, x{:}T_1 \vdash e_1{:}T\\ \Gamma, y{:}T_2 \vdash e_2{:}T\end{array}}{\Gamma \vdash \mathbf{case}\ e\ \mathbf{of\ inl}\ (x{:}T_1) \Rightarrow e_1 \mid \mathbf{inr}\ (y{:}T_2) \Rightarrow e_2{:}T}$$

$$\text{(record)} \quad \frac{\Gamma \vdash e_1{:}T_1 \quad .. \quad \Gamma \vdash e_k{:}T_k}{\Gamma \vdash \{lab_1 = e_1, .., lab_k = e_k\}{:}\{lab_1{:}T_1, .., lab_k{:}T_k\}}$$

$$\text{(recordproj)} \quad \frac{\Gamma \vdash e{:}\{lab_1{:}T_1, .., lab_k{:}T_k\}}{\Gamma \vdash \#lab_i\ e{:}T_i}$$

$$\text{(ref)} \quad \frac{\Gamma \vdash e{:}T}{\Gamma \vdash\ \mathsf{ref}\, e : T\ \mathsf{ref}}$$

$$\text{(assign)} \quad \frac{\begin{array}{l}\Gamma \vdash e_1{:}T\ \mathsf{ref}\\ \Gamma \vdash e_2{:}T\end{array}}{\Gamma \vdash e_1 := e_2{:}\mathsf{unit}}$$

$$\text{(deref)} \quad \frac{\Gamma \vdash e{:}T\ \mathsf{ref}}{\Gamma \vdash !e{:}T}$$

$$\text{(loc)} \quad \frac{\Gamma(\ell) = T\ \mathsf{ref}}{\Gamma \vdash \ell{:}T\ \mathsf{ref}}$$

5.4 Exercises

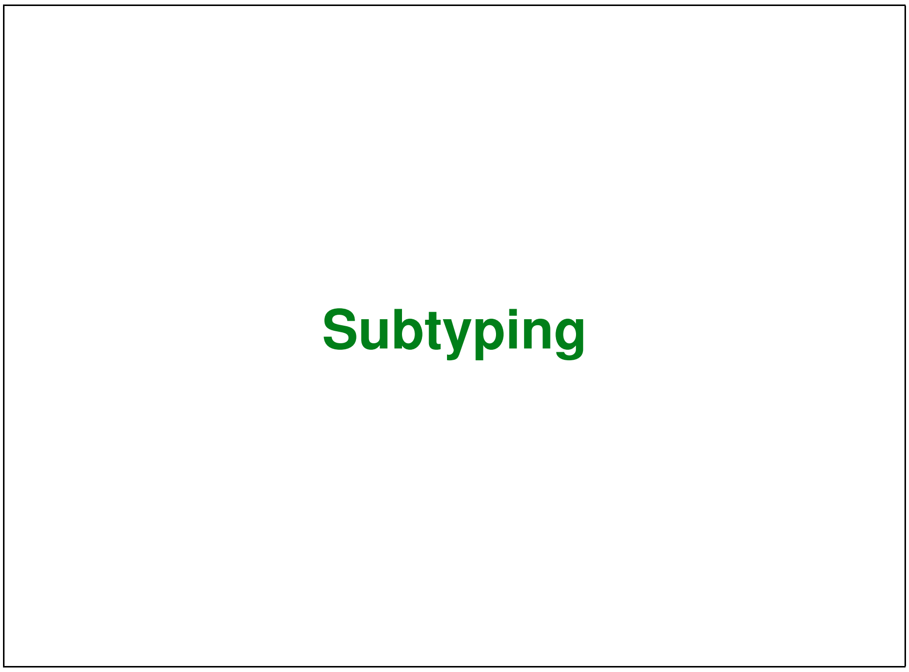
Exercise 27 ★★*Design abstract syntax, type rules and evaluation rules for labelled variants, analogously to the way in which records generalise products.*

Exercise 28 ★★*Design type rules and evaluation rules for ML-style exceptions. Start with exceptions that do not carry any values. Hint 1: take care with nested handlers within recursive functions. Hint 2: you might want to express your semantics using evaluation contexts.*

Exercise 29 ★★★*Extend the L2 implementation to cover all of L3.*

6 Subtyping

Slide 171



Subtyping

Our type systems so far would all be annoying to use, as they're quite rigid (Pascal-like). There is no support for code reuse, e.g. you would have to have different sorting code for int lists and int ∗ int lists.

Slide 172

Polymorphism

Ability to use expressions at many different types.

- Ad-hoc polymorphism (overloading).

  e.g. in mosml the built-in $+$ can be used to add two integers or to add two reals. (see Haskell *type classes*)

- Parametric Polymorphism – as in ML. See the Part II Types course.

  can write a function that for any type α takes an argument of type α list and computes its length (parametric - uniform in whatever α is)

- Subtype polymorphism – as in various OO languages. See here.

  Dating back to the 1960s (Simula etc); formalised in 1980,1984,...

Slide 173

Subtyping – Motivation

Recall

$$\text{(app)}\ \frac{\Gamma \vdash e_1{:}T \to T' \qquad \Gamma \vdash e_2{:}T}{\Gamma \vdash e_1\, e_2{:}T'}$$

so can't type

$$\nvdash (\textbf{fn}\ \mathrm{x}{:}\{\mathrm{p}{:}\mathsf{int}\} \Rightarrow \#\mathrm{p}\ \mathrm{x})\ \{\mathrm{p} = 3, \mathrm{q} = 4\} : \mathsf{int}$$

even though we're giving the function a *better* argument, with more structure, than it needs.

Slide 174

Subsumption

'Better'? Any value of type $\{\mathrm{p}{:}\mathsf{int}, \mathrm{q}{:}\mathsf{int}\}$ can be used wherever a value of type $\{\mathrm{p}{:}\mathsf{int}\}$ is expected. (*)

Introduce a *subtyping relation* between types, written $T <: T'$, read as T is a subtype of T' (a T is useful in more contexts than a T').

Will define it on the next slides, but it will include
$\{\mathrm{p}{:}\mathsf{int}, \mathrm{q}{:}\mathsf{int}\} <: \{\mathrm{p}{:}\mathsf{int}\} <: \{\}$

Introduce a *subsumption rule*

$$\text{(sub)} \quad \frac{\Gamma \vdash e{:}T \qquad T <: T'}{\Gamma \vdash e{:}T'}$$

allowing subtyping to be used, capturing (*).

Can then deduce $\{\mathrm{p} = 3, \mathrm{q} = 4\}{:}\{\mathrm{p}{:}\mathsf{int}\}$, hence can type the example.

Example:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathrm{x}{:}\{\mathrm{p}{:}\mathsf{int}\} \vdash \mathrm{x}{:}\{\mathrm{p}{:}\mathsf{int}\}}\,(\mathrm{var})}{\mathrm{x}{:}\{\mathrm{p}{:}\mathsf{int}\} \vdash \#\mathrm{p}\ \mathrm{x}{:}\mathsf{int}}\,(\mathrm{record\text{-}proj})}{\{\} \vdash (\mathbf{fn}\ \mathrm{x}{:}\{\mathrm{p}{:}\mathsf{int}\} \Rightarrow \#\mathrm{p}\ \mathrm{x}){:}\{\mathrm{p}{:}\mathsf{int}\} \to \mathsf{int}}\,(\mathrm{fn}) \qquad \dfrac{\dfrac{\dfrac{}{\{\} \vdash 3{:}\mathsf{int}}\,(\mathrm{var}) \quad \dfrac{}{\{\} \vdash 4{:}\mathsf{int}}\,(\mathrm{var})}{\{\} \vdash \{\mathrm{p} = 3, \mathrm{q} = 4\}{:}\{\mathrm{p}{:}\mathsf{int}, \mathrm{q}{:}\mathsf{int}\}}\,(\mathrm{record}) \qquad (\bigstar)}{\{\} \vdash \{\mathrm{p} = 3, \mathrm{q} = 4\}{:}\{\mathrm{p}{:}\mathsf{int}\}}\,(\mathrm{sub})}{\{\} \vdash (\mathbf{fn}\ \mathrm{x}{:}\{\mathrm{p}{:}\mathsf{int}\} \Rightarrow \#\mathrm{p}\ \mathrm{x})\{\mathrm{p} = 3, \mathrm{q} = 4\}{:}\mathsf{int}}\,(\mathrm{app})$$

where $(\bigstar)$ is $\{\mathrm{p}{:}\mathsf{int}, \mathrm{q}{:}\mathsf{int}\} <: \{\mathrm{p}{:}\mathsf{int}\}$

Now, how do we define the subtype relation? First:

Slide 175

The Subtype Relation $\boxed{T <: T'}$

$$\text{(s-refl)} \quad \frac{}{T <: T}$$

$$\text{(s-trans)} \quad \frac{T <: T' \qquad T' <: T''}{T <: T''}$$

Now have to look at each type

Slide 176

Subtyping – Records

Forgetting fields on the right:

$$\{lab_1{:}T_1, .., lab_k{:}T_k, lab_{k+1}{:}T_{k+1}, .., lab_{k+k'}{:}T_{k+k'}\}$$

$<:$ (s-record-width)

$$\{lab_1{:}T_1, .., lab_k{:}T_k\}$$

Allowing subtyping within fields:

$$\text{(s-record-depth)}\quad \dfrac{T_1 <: T'_1 \quad .. \quad T_k <: T'_k}{\{lab_1{:}T_1, .., lab_k{:}T_k\} <: \{lab_1{:}T'_1, .., lab_k{:}T'_k\}}$$

Combining these:

$$\dfrac{\dfrac{}{\{\mathrm{p}{:}\mathsf{int}, \mathrm{q}{:}\mathsf{int}\} <: \{\mathrm{p}{:}\mathsf{int}\}}\ \text{(s-record-width)} \qquad \dfrac{}{\{\mathrm{r}{:}\mathsf{int}\} <: \{\}}\ \text{(s-record-width)}}{\{\mathrm{x}{:}\{\mathrm{p}{:}\mathsf{int}, \mathrm{q}{:}\mathsf{int}\}, \mathrm{y}{:}\{\mathrm{r}{:}\mathsf{int}\}\} <: \{\mathrm{x}{:}\{\mathrm{p}{:}\mathsf{int}\}, \mathrm{y}{:}\{\}\}}\ \text{(s-record-depth)}$$

Another example:

$$\dfrac{\dfrac{}{\{\mathrm{x}{:}\{\mathrm{p}{:}\mathsf{int}, \mathrm{q}{:}\mathsf{int}\}, \mathrm{y}{:}\{\mathrm{r}{:}\mathsf{int}\}\} <: \{\mathrm{x}{:}\{\mathrm{p}{:}\mathsf{int}, \mathrm{q}{:}\mathsf{int}\}\}}\ \text{(s-record-width)} \qquad \dfrac{\dfrac{}{\{\mathrm{p}{:}\mathsf{int}, \mathrm{q}{:}\mathsf{int}\} <: \{\mathrm{p}{:}\mathsf{int}\}}\ \text{(s-record-width)}}{\{\mathrm{x}{:}\{\mathrm{p}{:}\mathsf{int}, \mathrm{q}{:}\mathsf{int}\}\} <: \{\mathrm{x}{:}\{\mathrm{p}{:}\mathsf{int}\}\}}\ \text{(s-record-depth)}}{\{\mathrm{x}{:}\{\mathrm{p}{:}\mathsf{int}, \mathrm{q}{:}\mathsf{int}\}, \mathrm{y}{:}\{\mathrm{r}{:}\mathsf{int}\}\} <: \{\mathrm{x}{:}\{\mathrm{p}{:}\mathsf{int}\}\}}\ \text{(s-trans)}$$

Slide 177

Allowing reordering of fields:

$$\text{(s-record-order)}\quad \dfrac{\pi \text{ a permutation of } 1, .., k}{\{lab_1{:}T_1, .., lab_k{:}T_k\} <: \{lab_{\pi(1)}{:}T_{\pi(1)}, .., lab_{\pi(k)}{:}T_{\pi(k)}\}}$$

(the subtype order is *not* anti-symmetric – it is a preorder, not a partial order)

Slide 178

Subtyping - Functions

$$\text{(s-fn)}\quad \frac{T_1' <: T_1 \qquad T_2 <: T_2'}{T_1 \to T_2 <: T_1' \to T_2'}$$

contravariant on the left of $\to$

covariant on the right of $\to$ (like (s-record-depth))

Slide 179

If $f{:}T_1 \to T_2$ then we can give f any argument which is a subtype of T_1; we can regard the result of f as any supertype of T_2. e.g., for

$$f = \textbf{fn}\ \mathrm{x}{:}\{\mathrm{p}{:}\mathsf{int}\} \Rightarrow \{\mathrm{p} = \#\mathrm{p}\ \mathrm{x}, \mathrm{q} = 28\}$$

we have

$$\{\} \vdash f{:}\{\mathrm{p}{:}\mathsf{int}\} \to \{\mathrm{p}{:}\mathsf{int}, \mathrm{q}{:}\mathsf{int}\}$$
$$\{\} \vdash f{:}\{\mathrm{p}{:}\mathsf{int}\} \to \{\mathrm{p}{:}\mathsf{int}\}$$
$$\{\} \vdash f{:}\{\mathrm{p}{:}\mathsf{int}, \mathrm{q}{:}\mathsf{int}\} \to \{\mathrm{p}{:}\mathsf{int}, \mathrm{q}{:}\mathsf{int}\}$$
$$\{\} \vdash f{:}\{\mathrm{p}{:}\mathsf{int}, \mathrm{q}{:}\mathsf{int}\} \to \{\mathrm{p}{:}\mathsf{int}\}$$

as

$$\{\mathrm{p}{:}\mathsf{int}, \mathrm{q}{:}\mathsf{int}\} <: \{\mathrm{p}{:}\mathsf{int}\}$$

Slide 180

On the other hand, for

$$\textbf{fn}\ \mathrm{x}{:}\{\mathrm{p}{:}\mathsf{int}, \mathrm{q}{:}\mathsf{int}\} \Rightarrow \{\mathrm{p} = (\#\mathrm{p}\ \mathrm{x}) + (\#\mathrm{q}\ \mathrm{x})\}$$

we have

$$\{\} \vdash f{:}\{\mathrm{p}{:}\mathsf{int}, \mathrm{q}{:}\mathsf{int}\} \rightarrow \{\mathrm{p}{:}\mathsf{int}\}$$

$$\{\} \nvdash f{:}\{\mathrm{p}{:}\mathsf{int}\} \rightarrow T \quad \text{for any } T$$

$$\{\} \nvdash f{:}T \rightarrow \{\mathrm{p}{:}\mathsf{int}, \mathrm{q}{:}\mathsf{int}\} \quad \text{for any } T$$

Slide 181

Subtyping – Products

Just like (s-record-depth)

$$\text{(s-pair)} \quad \frac{T_1 <: T_1' \quad T_2 <: T_2'}{T_1 * T_2 <: T_1' * T_2'}$$

Subtyping – Sums

Exercise.

Slide 182

Subtyping – References

Are either of these any good?

$$\frac{T <: T'}{T \text{ ref} <: T' \text{ ref}} \qquad \frac{T' <: T}{T \text{ ref} <: T' \text{ ref}}$$

No...

Slide 183

Subtyping – Down-casts

The subsumption rule (sub) permits up-casting at any point. How about down-casting? We could add

$$e \quad ::= \quad \ldots \mid (T)e$$

with typing rule

$$\frac{\Gamma \vdash e{:}T'}{\Gamma \vdash (T)e{:}T}$$

then you need a dynamic type-check...

This gives flexibility, but at the cost of many potential run-time errors. Many uses might be better handled by Parametric Polymorphism, aka Generics. (cf. work by Martin Odersky, to be in Java 1.5)

Slide 184

Semantics

No change (note that we've not changed the expression grammar).

Properties

Have Type Preservation and Progress.

Implementation

Type inference is more subtle, as the rules are no longer syntax-directed.

Getting a good runtime implementation is also tricky, especially with field re-ordering.

The following development is taken from [Pierce, Chapter 18], where you can find more details (including a treatment of self and a direct semantics for a 'featherweight' fragment of Java).

Slide 185

(Very Simple) Objects

```
let val c:{get:unit → int, inc:unit → unit} =
  let val x:int ref = ref 0 in
    {get = fn y:unit ⇒!x,
     inc = fn y:unit ⇒ x := 1+!x}
  end
in
  (#inc c)(); (#get c)()
end
```

$Counter = \{\mathrm{get}{:}\mathsf{unit} \rightarrow \mathsf{int},\ \mathrm{inc}{:}\mathsf{unit} \rightarrow \mathsf{unit}\}$.

Slide 186

Using Subtyping

```
let val c:{get:unit → int, inc:unit → unit, reset:unit → unit} =
  let val x:int ref = ref 0 in
    {get = fn y:unit ⇒!x,
     inc = fn y:unit ⇒ x := 1+!x,
     reset = fn y:unit ⇒ x := 0}
  end
in
  (#inc c)(); (#get c)()
end
```

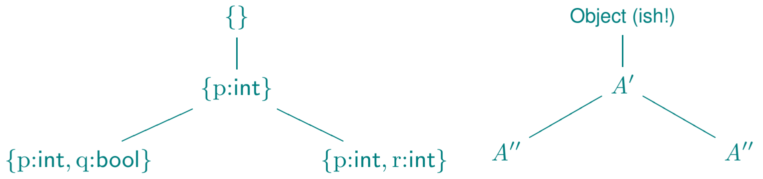
$\mathit{ResetCounter} = \{\text{get:unit} \rightarrow \text{int}, \text{inc:unit} \rightarrow \text{unit}, \text{reset:unit} \rightarrow \text{unit}\}$

$<: \ \mathit{Counter} = \{\text{get:unit} \rightarrow \text{int}, \ \text{inc:unit} \rightarrow \text{unit}\}$.

Slide 187

Subtyping – Structural vs Named

$$\begin{aligned} A' &= \{\} \text{ with } \{\text{p:int}\} \\ A'' &= A' \text{ with } \{\text{q:bool}\} \\ A''' &= A' \text{ with } \{\text{r:int}\} \end{aligned}$$



{}

{p:int}

{p:int, q:bool}

{p:int, r:int}

Object (ish!)

A'

A''

A''

Slide 188

Object Generators

```
let val newCounter:unit → {get:unit → int, inc:unit → unit} =
  fn y:unit ⇒
    let val x:int ref = ref 0 in
      {get = fn y:unit ⇒!x,
       inc = fn y:unit ⇒ x := 1+!x}
    end
in
  (#inc (newCounter ())) ()
end
```

and onwards to simple classes...

Slide 189

Reusing Method Code (Simple Classes)

Recall *Counter* = {get:unit → int, inc:unit → unit}.

First, make the internal state into a record. *CounterRep* = {p:int ref}.

```
let val counterClass:CounterRep → Counter =
  fn x:CounterRep ⇒
    {get = fn y:unit ⇒!(#p x),
     inc = fn y:unit ⇒ (#p x) := 1+!(#p x)}


let val newCounter:unit → Counter =
  fn y:unit ⇒
    let val x:CounterRep = {p = ref 0} in
      counterClass x
```

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Slide 191

Reusing Method Code (Simple Classes)

```
let val resetCounterClass: CounterRep → ResetCounter =
  fn x: CounterRep ⇒
    let val super = counterClass x in
      {get = #get super,
       inc = #inc super,
       reset = fn y:unit ⇒ (#p x) := 0}
```

CounterRep = {p:int ref}.

Counter = {get:unit → int, inc:unit → unit}.

ResetCounter = {get:unit → int, inc:unit → unit, reset:unit → unit}.

Slide 192

Reusing Method Code (Simple Classes)

```
class Counter
  {  protected int p;
     Counter() { this.p=0; }
     int get () { return this.p; }
     void inc () { this.p++ ; }
     };

class ResetCounter
  extends Counter
  {  void reset () {this.p=0;}
     };
```

6.1 Exercises

Exercise 30 ★*For each of the following, either give a type derivation or explain why it is untypable.*

1. $\{\} \vdash \{\mathrm{p}{:}\{\mathrm{p}{:}\{\mathrm{p}{:}\{\mathrm{p}{:}\mathsf{int}\}\}\}\}{:}\{\mathrm{p}{:}\{\}\}$
2. $\{\} \vdash \mathbf{fn}\ \mathrm{x}{:}\{\mathrm{p}{:}\mathsf{bool}, \mathrm{q}{:}\{\mathrm{p}{:}\mathsf{int}, \mathrm{q}{:}\mathsf{bool}\}\} \Rightarrow \#\mathrm{q}\ \#\mathrm{p}\ \mathrm{x} : ?$
3. $\{\} \vdash \mathbf{fn}\ \mathrm{f}{:}\{\mathrm{p}{:}\mathsf{int}\} \rightarrow \mathsf{int} \Rightarrow (\mathrm{f}\ \{\mathrm{q} = 3\}) + (\mathrm{f}\ \{\mathrm{p} = 4\}) : ?$
4. $\{\} \vdash \mathbf{fn}\ \mathrm{f}{:}\{\mathrm{p}{:}\mathsf{int}\} \rightarrow \mathsf{int} \Rightarrow (\mathrm{f}\ \{\mathrm{q} = 3, \mathrm{p} = 2\}) + (\mathrm{f}\ \{\mathrm{p} = 4\}) : ?$

Exercise 31 ★*For each of the two bogus T ref subtype rules on Slide 187, give an example program that is typable with that rule but gets stuck at runtime.*

Exercise 32 ★★*What should the subtype rules for sums $T + T'$ be?*

Exercise 33 ★★*...and for* **let** *and* **let rec** *?*

7 Low-level semantics

Slide 193

Low-level semantics

Slide 194

Can usefully apply semantics not just to high-level languages but to

- Intermediate Languages (e.g. Java Bytecode, MS IL, C−−)
- Assembly languages (esp. for use as a compilation target)
- C-like languages (cf. Cyclone)

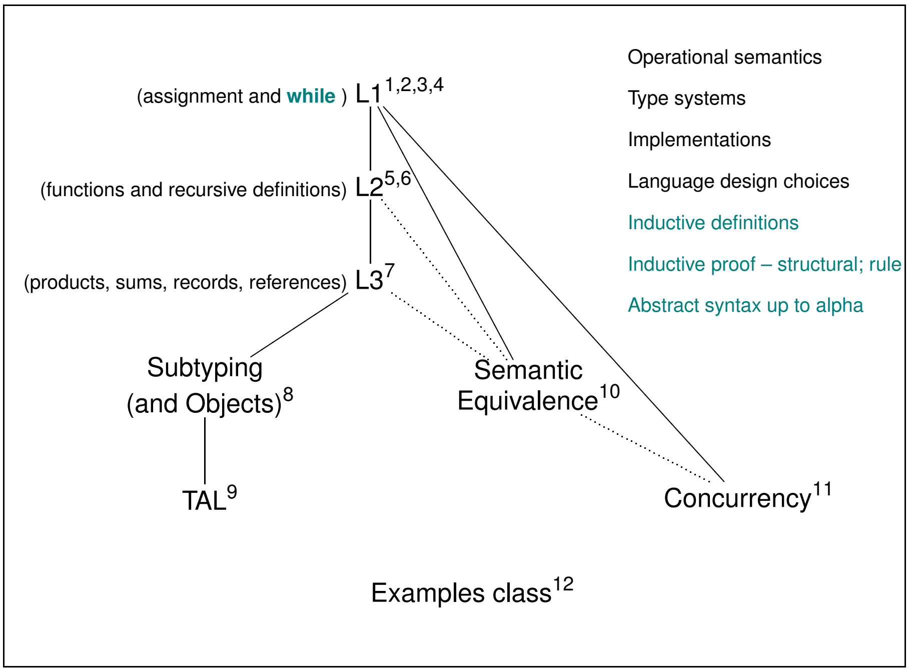
By making these type-safe we can make more robust systems.

Next: a guest lecture by Greg Morrisett from Cornell University.

`http://www.cs.cornell.edu/home/jgm/`

(see separate handout)

Slide 195



(assignment and **while**) L1[1,2,3,4]

(functions and recursive definitions) L2[5,6]

(products, sums, records, references) L3[7]

Subtyping (and Objects)[8]

TAL[9]

Semantic Equivalence[10]

Concurrency[11]

Examples class[12]

Operational semantics

Type systems

Implementations

Language design choices

Inductive definitions

Inductive proof – structural; rule

Abstract syntax up to alpha

8 Semantic Equivalence

Slide 196

Semantic Equivalence

Slide 197

$$2+2 \overset{?}{\simeq} 4$$

In what sense are these two expressions the same?

They have different abstract syntax trees.

They have different reduction sequences.

But, you'd hope that in any program you could replace one by the other without affecting the result....

$$\int_0^{2+2} e^{\sin(x)} dx = \int_0^{4} e^{\sin(x)} dx$$

Slide 198

How about $(l := 0; 4) \overset{?}{\simeq} (l := 1; 3+!l)$

They will produce the same result (in any store), but you *cannot* replace one by the other in an arbitrary program context. For example:

$C = \_+!l$

$$\begin{array}{rl} C[l := 0; 4] = & (l := 0; 4)+!l \\ & \not\simeq \\ C[l := 1; 3+!l] = & (l := 1; 3+!l)+!l \end{array}$$

On the other hand, consider

$(l :=!l + 1); (l :=!l - 1) \overset{?}{\simeq} (l :=!l)$

Slide 199

Those were all particular expressions – may want to know that some *general laws* are valid for all $e_1, e_2, \ldots$. How about these:

$$e_1; (e_2; e_3) \overset{?}{\simeq} (e_1; e_2); e_3$$

$$(\textbf{if } e_1 \textbf{ then } e_2 \textbf{ else } e_3); e \overset{?}{\simeq} \textbf{if } e_1 \textbf{ then } e_2; e \textbf{ else } e_3; e$$

$$e; (\textbf{if } e_1 \textbf{ then } e_2 \textbf{ else } e_3) \overset{?}{\simeq} \textbf{if } e_1 \textbf{ then } e; e_2 \textbf{ else } e; e_3$$

$$e; (\textbf{if } e_1 \textbf{ then } e_2 \textbf{ else } e_3) \overset{?}{\simeq} \textbf{if } e; e_1 \textbf{ then } e_2 \textbf{ else } e_3$$

Slide 200

$$\textbf{let val } \mathrm{x} = \ \mathsf{ref}\ 0 \textbf{ in fn } \mathrm{y}{:}\mathsf{int} \Rightarrow (\mathrm{x} := !\mathrm{x} + \mathrm{y}); !\mathrm{x}$$

$$\overset{?}{\simeq}$$

$$\textbf{let val } \mathrm{x} = \ \mathsf{ref}\ 0 \textbf{ in fn } \mathrm{y}{:}\mathsf{int} \Rightarrow (\mathrm{x} := !\mathrm{x} - \mathrm{y}); (0 - !\mathrm{x})$$

Slide 201

Temporarily extend L3 with pointer equality

$$op \; ::= \ldots \mid =$$

$$(\text{op} =) \quad \frac{\Gamma \vdash e_1{:}T\ \mathsf{ref} \qquad \Gamma \vdash e_2{:}T\ \mathsf{ref}}{\Gamma \vdash e_1 = e_2{:}\mathsf{bool}}$$

$$(\text{op} =) \quad \langle \ell = \ell', s\rangle \longrightarrow \langle b, s\rangle \quad \text{if } b = (\ell = \ell')$$

Slide 202

$$\begin{aligned} f = \;& \textbf{let val } \mathrm{x} = \ \mathsf{ref}\ 0 \textbf{ in} \\ & \textbf{let val } \mathrm{y} = \ \mathsf{ref}\ 0 \textbf{ in} \\ & \textbf{fn } \mathrm{z}{:}\mathsf{int\ ref} \Rightarrow \textbf{if } \mathrm{z} = \mathrm{x} \textbf{ then } \mathrm{y} \textbf{ else } \mathrm{x} \end{aligned}$$

$$\begin{aligned} g = \;& \textbf{let val } \mathrm{x} = \ \mathsf{ref}\ 0 \textbf{ in} \\ & \textbf{let val } \mathrm{y} = \ \mathsf{ref}\ 0 \textbf{ in} \\ & \textbf{fn } \mathrm{z}{:}\mathsf{int\ ref} \Rightarrow \textbf{if } \mathrm{z} = \mathrm{y} \textbf{ then } \mathrm{y} \textbf{ else } \mathrm{x} \end{aligned}$$

$$f \overset{?}{\simeq} g$$

The last two examples are taken from A.M. Pitts, Operational Semantics and Program Equivalence. In: G. Barthe, P. Dybjer and J. Saraiva (Eds), Applied Semantics. Lecture Notes in Computer Science, Tutorial, Volume 2395 (Springer-Verlag, 2002), pages 378-412. (Revised version of lectures at the International Summer School On Applied Semantics, APPSEM 2000, Caminha, Minho, Portugal, 9-15 September 2000.) `ftp://ftp.cl.cam.ac.uk/papers/amp12/opespe-lncs.pdf`

Slide 203

With a ‘good’ notion of semantic equivalence, we might:

1. prove that some particular expression (say an efficient algorithm) is equivalent to another (say a clear specification)
2. prove the soundness of general laws for equational reasoning about programs
3. prove some compiler optimisations are sound (source/IL/TAL)
4. understand the differences between languages

Slide 204

What does it mean for $\simeq$ to be ‘good’?

1. programs that result in observably-different values (in some initial store) must not be equivalent

   $(\exists s, s_1, s_2, v_1, v_2. \langle e_1, s\rangle \longrightarrow \langle v_1, s_1\rangle \wedge \langle e_2, s\rangle \longrightarrow \langle v_2, s_2\rangle$
   $\wedge\ v_1 \neq v_2) \Rightarrow e_1 \not\simeq e_2$
2. programs that terminate must not be equivalent to programs that don’t
3. $\simeq$ must be an equivalence relation

   $e \simeq e, \qquad e_1 \simeq e_2 \Rightarrow e_2 \simeq e_1, \qquad e_1 \simeq e_2 \simeq e_3 \implies e_1 \simeq e_3$
4. $\simeq$ must be a congruence

   if $e_1 \simeq e_2$ then for any context C we must have $C[e_1] \simeq C[e_2]$
5. $\simeq$ should relate as many programs as possible subject to the above.

Slide 205

Semantic Equivalence for L1

Consider Typed L1 again.

Define $e_1 \simeq^T_\Gamma e_2$ to hold iff forall s such that $\mathsf{dom}(\Gamma) \subseteq \mathsf{dom}(s)$, we have $\Gamma \vdash e_1{:}T$, $\Gamma \vdash e_2{:}T$, and either

1. $\langle e_1, s\rangle \longrightarrow^\omega$ and $\langle e_2, s\rangle \longrightarrow^\omega$, or
2. for some v, s' we have $\langle e_1, s\rangle \longrightarrow^* \langle v, s'\rangle$ and $\langle e_2, s\rangle \longrightarrow^* \langle v, s'\rangle$.

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If $T = \mathsf{unit}$ then $C = \_; !l$.

If $T = \mathsf{bool}$ then $C = \textbf{if}\ \_\ \textbf{then}\ !l\ \textbf{else}\ !l$.

If $T = \mathsf{int}$ then $C = l_1 := \_; !l$.

Slide 207

Congruence for Typed L1

The L1 contexts are:

$$
\begin{aligned}
C \quad ::= \quad & \_\ op\ e_2 \mid e_1\ op\ \_ \mid \\
& \textbf{if}\ \_\ \textbf{then}\ e_2\ \textbf{else}\ e_3 \mid \textbf{if}\ e_1\ \textbf{then}\ \_\ \textbf{else}\ e_3 \mid \textbf{if}\ e_1\ \textbf{then}\ e_2\ \textbf{else}\ \_ \mid \\
& \ell := \_ \mid \\
& \_; e_2 \mid e_1; \_ \mid \\
& \textbf{while}\ \_\ \textbf{do}\ e_2 \mid \textbf{while}\ e_1\ \textbf{do}\ \_
\end{aligned}
$$

Say $\simeq^T_\Gamma$ has the *congruence property* if whenever $e_1 \simeq^T_\Gamma e_2$ we have, for all C and T', if $\Gamma \vdash C[e_1]{:}T'$ and $\Gamma \vdash C[e_2]{:}T'$ then $C[e_1] \simeq^{T'}_\Gamma C[e_2]$.

Slide 208

Theorem 16 (Congruence for L1) *$\simeq^T_\Gamma$ has the congruence property.*

Proof Outline By case analysis, looking at each L1 context C in turn.

For each C (and for arbitrary e and s), consider the possible reduction sequences

$\langle C[e], s\rangle \longrightarrow \langle e_1, s_1\rangle \longrightarrow \langle e_2, s_2\rangle \longrightarrow \ldots$

For each such reduction sequence, deduce what behaviour of e was involved

$\langle e, s\rangle \longrightarrow \langle \hat{e}_1, \hat{s}_1\rangle \longrightarrow \ldots$

Using $e \simeq^T_\Gamma e'$ find a similar reduction sequence of e'.

Using the reduction rules construct a sequence of $C[e']$.

Slide 209

Theorem 17 (Congruence for L1) $\simeq^T_\Gamma$ *has the congruence property.*

By case analysis, looking at each L1 context in turn.

Case $C = (\ell := \_)$. Suppose $e \simeq^T_\Gamma e'$, $\Gamma \vdash \ell := e{:}T'$ and $\Gamma \vdash \ell := e'{:}T'$. By examining the typing rules $T = \mathsf{int}$ and $T' = \mathsf{unit}$.

To show $(\ell := e) \simeq^{T'}_\Gamma (\ell := e')$ we have to show for all s such that $\mathsf{dom}(\Gamma) \subseteq \mathsf{dom}(s)$, then $\Gamma \vdash \ell := e{:}T'$ ($\surd$), $\Gamma \vdash \ell := e'{:}T'$ ($\surd$), and either

1. $\langle \ell := e, s\rangle \longrightarrow^\omega$ and $\langle \ell := e', s\rangle \longrightarrow^\omega$, or
2. for some v, s' we have $\langle \ell := e, s\rangle \longrightarrow^* \langle v, s'\rangle$ and $\langle \ell := e', s\rangle \longrightarrow^* \langle v, s'\rangle$.

Slide 210

Consider the possible reduction sequences of a state $\langle \ell := e, s\rangle$. Either:

Case: $\langle \ell := e, s\rangle \longrightarrow^\omega$, i.e.

$\langle \ell := e, s\rangle \longrightarrow \langle e_1, s_1\rangle \longrightarrow \langle e_2, s_2\rangle \longrightarrow \ldots$

hence all these must be instances of (assign2), with

$\langle e, s\rangle \longrightarrow \langle \hat{e}_1, s_1\rangle \longrightarrow \langle \hat{e}_2, s_2\rangle \longrightarrow \ldots$

and $e_1 = (\ell := \hat{e}_1)$, $e_2 = (\ell := \hat{e}_2)$,...

Case: $\neg(\langle \ell := e, s\rangle \longrightarrow^\omega)$, i.e.

$\langle \ell := e, s\rangle \longrightarrow \langle e_1, s_1\rangle \longrightarrow \langle e_2, s_2\rangle \ldots \longrightarrow \langle e_k, s_k\rangle \not\longrightarrow$

hence all these must be instances of (assign2) except the last, which must be an instance of (assign1), with

$\langle e, s\rangle \longrightarrow \langle \hat{e}_1, s_1\rangle \longrightarrow \langle \hat{e}_2, s_2\rangle \longrightarrow \ldots \longrightarrow \langle \hat{e}_{k-1}, s_{k-1}\rangle$

and $e_1 = (\ell := \hat{e}_1)$, $e_2 = (\ell := \hat{e}_2)$,..., $e_{k-1} = (\ell := \hat{e}_{k-1})$ and for some n we have $\hat{e}_{k-1} = n$, $e_k = \mathbf{skip}$, and $s_k = s_{k-1} + \{\ell \mapsto n\}$.

(the other possibility, of zero or more (assign1) reductions ending in a stuck state, is excluded by Theorems 2 and 3 (type preservation and progress))

Slide 211

Now, if $\langle \ell := e, s \rangle \longrightarrow^{\omega}$ we have $\langle e, s \rangle \longrightarrow^{\omega}$, so by $e \simeq^{T}_{\Gamma} e'$ we have $\langle e', s \rangle \longrightarrow^{\omega}$, so (using (assign2)) we have $\langle \ell := e', s \rangle \longrightarrow^{\omega}$.

On the other hand, if $\neg(\langle \ell := e, s \rangle \longrightarrow^{\omega})$ then by the above there is some n and s_{k-1} such that $\langle e, s \rangle \longrightarrow^{*} \langle n, s_{k-1} \rangle$ and $\langle \ell := e, s \rangle \longrightarrow \langle \textbf{skip}, s_{k-1} + \{\ell \mapsto n\} \rangle$.

By $e \simeq^{T}_{\Gamma} e'$ we have $\langle \ell := e', s \rangle \longrightarrow^{*} \langle n, s_{k-1} \rangle$.

Then using (assign1)

$\langle \ell := e', s \rangle \longrightarrow^{*} \langle \ell := n, s_{k-1} \rangle \longrightarrow \langle \textbf{skip}, s_{k-1} + \{\ell \mapsto n\} \rangle =$ $\langle e_k, s_k \rangle$ as required.

Theorem 17 (Congruence for L1) $\simeq^{T}_{\Gamma}$ *has the congruence property.*

Proof By case analysis, looking at each L1 context in turn. We give only one case here, leaving the others for the reader.

Case $C = (\ell := \_)$. Suppose $e \simeq^{T}_{\Gamma} e'$, $\Gamma \vdash \ell := e{:}T'$ and $\Gamma \vdash \ell := e'{:}T'$. By examining the typing rules we have $T = \mathsf{int}$ and $T' = \mathsf{unit}$.

To show $\ell := e \simeq^{T'}_{\Gamma} \ell := e'$ we have to show for all s such that $\mathrm{dom}(\Gamma) \subseteq \mathrm{dom}(s)$, then $\Gamma \vdash \ell := e{:}T'$ ($\surd$), $\Gamma \vdash \ell := e'{:}T'$ ($\surd$), and either

1. $\langle \ell := e, s \rangle \longrightarrow^{\omega}$ and $\langle \ell := e', s \rangle \longrightarrow^{\omega}$, or
2. for some v, s' we have $\langle \ell := e, s \rangle \longrightarrow^{*} \langle v, s' \rangle$ and $\langle \ell := e', s \rangle \longrightarrow^{*} \langle v, s' \rangle$.

Consider the possible reduction sequences of a state $\langle \ell := e, s \rangle$. Recall that (by examining the reduction rules), if $\langle \ell := e, s \rangle \longrightarrow \langle e_1, s_1 \rangle$ then either that is an instance of (assign1), with $\exists n. e = n \wedge \ell \in \mathrm{dom}(s) \wedge e_1 = \textbf{skip} \wedge s' = s + \{\ell \mapsto n\}$, or it is an instance of (assign2), with $\exists \hat{e}_1. \langle e, s \rangle \longrightarrow \langle \hat{e}_1, s_1 \rangle \wedge e_1 = (\ell := \hat{e}_1)$. We know also that $\langle \textbf{skip}, s \rangle$ does not reduce.

Now (using Determinacy), for any e and s we have either

Case: $\langle \ell := e, s \rangle \longrightarrow^{\omega}$, i.e.

$\langle \ell := e, s \rangle \longrightarrow \langle e_1, s_1 \rangle \longrightarrow \langle e_2, s_2 \rangle \longrightarrow \ldots$

hence all these must be instances of (assign2), with

$\langle e, s \rangle \longrightarrow \langle \hat{e}_1, s_1 \rangle \longrightarrow \langle \hat{e}_2, s_2 \rangle \longrightarrow \ldots$

and $e_1 = (\ell := \hat{e}_1)$, $e_2 = (\ell := \hat{e}_2)$,...

Case: $\neg(\langle \ell := e, s \rangle \longrightarrow^{\omega})$, i.e.

$\langle \ell := e, s \rangle \longrightarrow \langle e_1, s_1 \rangle \longrightarrow \langle e_2, s_2 \rangle \ldots \longrightarrow \langle e_k, s_k \rangle \not\longrightarrow$

hence all these must be instances of (assign2) except the last, which must be an instance of (assign1), with

$\langle e, s \rangle \longrightarrow \langle \hat{e}_1, s_1 \rangle \longrightarrow \langle \hat{e}_2, s_2 \rangle \longrightarrow \ldots \longrightarrow \langle \hat{e}_{k-1}, s_{k-1} \rangle$

and $e_1 = (\ell := \hat{e}_1)$, $e_2 = (\ell := \hat{e}_2)$,..., $e_{k-1} = (\ell := \hat{e}_{k-1})$ and for some n we have $\hat{e}_{k-1} = n$, $e_k = \textbf{skip}$, and $s_k = s_{k-1} + \{\ell \mapsto n\}$.

(the other possibility, of zero or more (assign1) reductions ending in a stuck state, is excluded by Theorems 2 and 3 (type preservation and progress))

Now, if $\langle \ell := e, s\rangle \longrightarrow^{\omega}$, by the above there is an infinite reduction sequence for $\langle e, s\rangle$, so by $e \simeq^{T}_{\Gamma} e'$ there is an infinite reduction sequence of $\langle e', s\rangle$, so (using (assign2)) there is an infinite reduction sequence of $\langle \ell := e', s\rangle$.

On the other hand, if $\neg(\langle \ell := e, s\rangle \longrightarrow^{\omega})$ then by the above there is some n and s_{k-1} such that $\langle e, s\rangle \longrightarrow^{*} \langle n, s_{k-1}\rangle$ and $\langle \ell := e, s\rangle \longrightarrow \langle \textbf{skip}, s_{k-1} + \{\ell \mapsto n\}\rangle$. By $e \simeq^{T}_{\Gamma} e'$ we have $\langle \ell := e', s\rangle \longrightarrow^{*} \langle n, s_{k-1}\rangle$. Then using (assign1) $\langle \ell := e', s\rangle \longrightarrow^{*} \langle \ell := n, s_{k-1}\rangle \longrightarrow \langle \textbf{skip}, s_{k-1} + \{\ell \mapsto n\} = \langle e_k, s_k\rangle$ as required.

□

Slide 212

Back to the Examples

We defined $e_1 \simeq^{T}_{\Gamma} e_2$ iff forall s such that $\mathsf{dom}(\Gamma) \subseteq \mathsf{dom}(s)$, we have $\Gamma \vdash e_1{:}T$, $\Gamma \vdash e_2{:}T$, and either

1. $\langle e_1, s\rangle \longrightarrow^{\omega}$ and $\langle e_2, s\rangle \longrightarrow^{\omega}$, or
2. for some v, s' we have $\langle e_1, s\rangle \longrightarrow^{*} \langle v, s'\rangle$ and $\langle e_2, s\rangle \longrightarrow^{*} \langle v, s'\rangle$.

So:

$2 + 2 \simeq^{\mathsf{int}}_{\Gamma} 4$ for any Γ

$(l := 0; 4) \not\simeq^{\mathsf{int}}_{\Gamma} (l := 1; 3{+}!l)$ for any Γ

$(l :=!l + 1); (l :=!l - 1) \simeq^{\mathsf{unit}}_{\Gamma} (l :=!l)$ for any Γ including $l{:}\mathsf{intref}$

Slide 213

And the general laws?

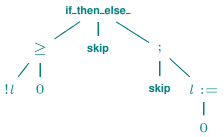
Conjecture 1 $e_1; (e_2; e_3) \simeq^{T}_{\Gamma} (e_1; e_2); e_3$ *for any* Γ, T, e_1, e_2 *and* e_3 *such that* $\Gamma \vdash e_1{:}\mathsf{unit}$, $\Gamma \vdash e_2{:}\mathsf{unit}$, *and* $\Gamma \vdash e_3{:}T$

Conjecture 2 $((\textbf{\textit{if}}\ e_1\ \textbf{\textit{then}}\ e_2\ \textbf{\textit{else}}\ e_3); e) \simeq^{T}_{\Gamma} (\textbf{\textit{if}}\ e_1\ \textbf{\textit{then}}\ e_2; e\ \textbf{\textit{else}}\ e_3; e)$ *for any* Γ, T, e, e_1, e_2 *and* e_3 *such that* $\Gamma \vdash e_1{:}\mathsf{bool}$, $\Gamma \vdash e_2{:}\mathsf{unit}$, $\Gamma \vdash e_3{:}\mathsf{unit}$, *and* $\Gamma \vdash e{:}T$

Conjecture 3 $(e; (\textbf{\textit{if}}\ e_1\ \textbf{\textit{then}}\ e_2\ \textbf{\textit{else}}\ e_3)) \simeq^{T}_{\Gamma} (\textbf{\textit{if}}\ e_1\ \textbf{\textit{then}}\ e; e_2\ \textbf{\textit{else}}\ e; e_3)$ *for any* Γ, T, e, e_1, e_2 *and* e_3 *such that* $\Gamma \vdash e{:}\mathsf{unit}$, $\Gamma \vdash e_1{:}\mathsf{bool}$, $\Gamma \vdash e_2{:}T$, *and* $\Gamma \vdash e_3{:}T$

Slide 214

Q: Is a typed expression $\Gamma \vdash e{:}T$, e.g.
$l{:}\mathsf{intref} \vdash \textbf{if } !l \geq 0 \textbf{ then skip else } (\textbf{skip}; l := 0){:}\mathsf{unit}$:

1. a list of characters `[‘i’, ‘f’, ‘_’, ‘!’, ‘l’, ..];`
2. a list of tokens `[ IF, DEREF, LOC "l", GTEQ, ..];`
3. an abstract syntax tree 

; or
4. - the equivalence class $\{e' \mid e \simeq^T_\Gamma e'\}$
   - the partial function $[\![e]\!]_\Gamma$ that takes any store s with $\mathsf{dom}(s) = \mathsf{dom}(\Gamma)$ and either is undefined, if $\langle e, s\rangle \longrightarrow^\omega$, or is $\langle v, s'\rangle$, if $\langle e, s\rangle \longrightarrow^* \langle v, s'\rangle$

(the Determinacy theorem tells us that this is a definition of a function).

Slide 215

Suppose $\Gamma \vdash e_1{:}\mathsf{unit}$ and $\Gamma \vdash e_2{:}\mathsf{unit}$.

When *is* $e_1; e_2 \simeq^{\mathsf{unit}}_\Gamma e_2; e_1$?

A sufficient condition: they don't mention any locations (but not necessary... e.g. if e_1 does but e_2 doesn't)

A weaker sufficient condition: they don't mention any of the same locations. (but not necessary... e.g. if they both just read l)

An even weaker sufficient condition: we can regard each expression as a partial function over stores with domain $\mathrm{dom}(\Gamma)$. Say $[\![e_i]\!]_\Gamma$ is the function that takes a store s with $\mathrm{dom}(s) = \mathrm{dom}(\Gamma)$ and either is undefined, if $\langle e_i, s\rangle \longrightarrow^\omega$, or is s', if $\langle e_i, s\rangle \longrightarrow^* \langle (), s'\rangle$ (the Determinacy theorem tells us that this is a definition of a function).

For each location ℓ in $\mathrm{dom}(\Gamma)$, say e_i *semantically depends on* ℓ if there exists s, n such that $[\![e_i]\!]_\Gamma(s) \neq [\![e_i]\!]_\Gamma(s + \{\ell \mapsto n\})$. (note this is much weaker than "e_i contains an dereference of ℓ")

Say e_i *semantically affects* ℓ if there exists s such that $[\![e_i]\!]_\Gamma(s)(\ell) \neq [\![e_i]\!]_\Gamma(s)(\ell)$. (note this is much weaker than "e_i contains an assignment to ℓ")

Now $e_1; e_2 \simeq^{\mathsf{unit}}_\Gamma e_2; e_1$ if there is no ℓ that is depended on by one e_i and affected by the other.

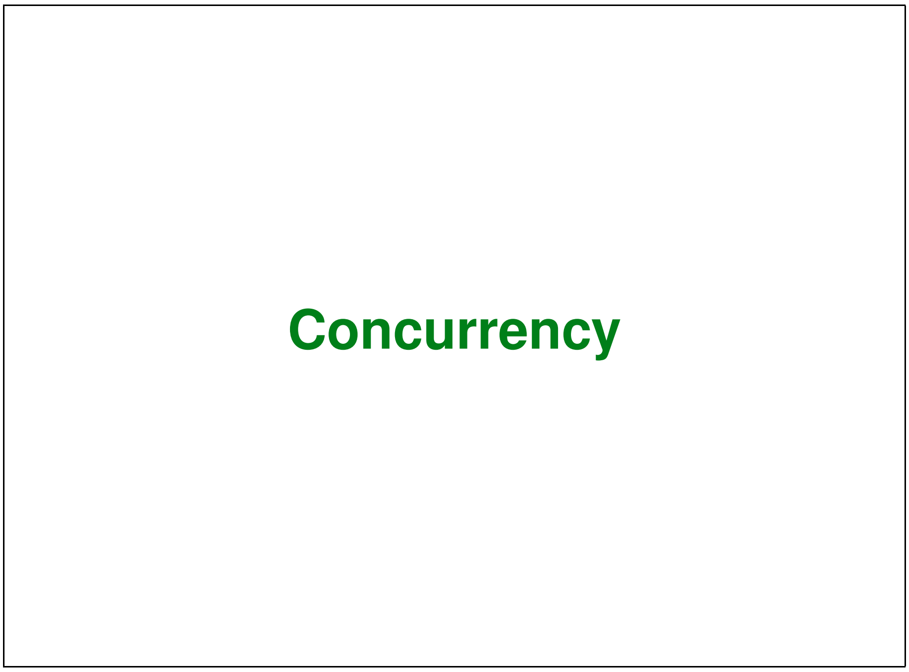
(sill not necessary...?)

8.1 Exercises

Exercise 34 ★★*Prove some of the other cases of the Congruence theorem.*

9 Concurrency

Slide 216



Concurrency

Slide 217

Our focus so far has been on semantics for *sequential* computation. But the world is not sequential...

- hardware is intrinsically parallel (fine-grain, across words, to coarse-grain, e.g. multiple execution units)
- multi-processor machines
- multi-threading (perhaps on a single processor)
- networked machines

Slide 218

Problems

- the state-spaces of our systems become *large*, with the *combinatorial explosion* – with n threads, each of which can be in 2 states, the system has 2^n states.
- the state-spaces become *complex*
- computation becomes *nondeterministic* (unless synchrony is imposed), as different threads/machines/... operate at different speeds.
- parallel components competing for access to resources may *deadlock* or suffer *starvation*. Need *mutual exclusion* between components accessing a resource.

Slide 219

More Problems!

- *partial failure* (of some processes, of some machines in a network, of some persistent storage devices). Need *transactional mechanisms*.
- *communication betweeen different environments* (with different local resources (e.g. different local stores, or libraries, or...)
- *partial version change*
- communication between administrative regions with *partial trust* (or, indeed, *no trust*); protection against mailicious attack.
- dealing with contingent complexity (embedded historical accidents; upwards-compatible deltas)

Slide 220

Theme: as for sequential languages, but much more so, it's a complicated world.

Aim of this lecture: just to give you a taste of how a little semantics can be used to express some of the fine distinctions. Primarily (1) to boost your intuition for informal reasoning, but also (2) this can support rigorous proof about really hairy crypto protocols, cache-coherency protocols, comms, database transactions,....

Going to define the simplest possible (well, almost) concurrent language, call it $\mathsf{L1}_{|}$, and explore a few issues. You've seen most of them informally in CSAA.

Slide 221

Booleans $b \in \mathbb{B} = \{\textbf{true}, \textbf{false}\}$

Integers $n \in \mathbb{Z} = \{..., -1, 0, 1, ...\}$

Locations $\ell \in \mathbb{L} = \{l, l_0, l_1, l_2, ...\}$

Operations $op ::= + \mid \geq$

Expressions

$$\begin{array}{rcl} e & ::= & n \mid b \mid e_1 \; op \; e_2 \mid \textbf{if } e_1 \textbf{ then } e_2 \textbf{ else } e_3 \mid \\ & & \ell := e \mid !\ell \mid \\ & & \textbf{skip} \mid e_1; e_2 \mid \\ & & \textbf{while } e_1 \textbf{ do } e_2 \mid \\ & & e_1 \boldsymbol{|} e_2 \end{array}$$

Slide 222

$$\begin{array}{rcl} T & ::= & \mathsf{int} \mid \mathsf{bool} \mid \mathsf{unit} \mid \mathsf{proc} \\ T_{loc} & ::= & \mathsf{intref} \end{array}$$

Slide 223

Parallel Composition: Typing and Reduction

(thread) $\dfrac{\Gamma \vdash e{:}\mathsf{unit}}{\Gamma \vdash e{:}\mathsf{proc}}$

(parallel) $\dfrac{\Gamma \vdash e_1{:}\mathsf{proc} \qquad \Gamma \vdash e_2{:}\mathsf{proc}}{\Gamma \vdash e_1 \,|\, e_2{:}\mathsf{proc}}$

(parallel1) $\dfrac{\langle e_1, s\rangle \longrightarrow \langle e_1', s'\rangle}{\langle e_1 \,|\, e_2, s\rangle \longrightarrow \langle e_1' \,|\, e_2, s'\rangle}$

(parallel2) $\dfrac{\langle e_2, s\rangle \longrightarrow \langle e_2', s'\rangle}{\langle e_1 \,|\, e_2, s\rangle \longrightarrow \langle e_1 \,|\, e_2', s'\rangle}$

Slide 224

Parallel Composition: Design Choices

- threads don't return a value
- threads don't have an identity
- termination of a thread cannot be observed within the language
- threads aren't partitioned into 'processes' or machines
- threads can't be killed externally

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Threads execute asynchronously – the semantics allows any interleaving of the reductions of the threads.

All threads can read and write the shared memory.

$$\begin{tikzcd}
 & \langle () \mid l := 2, \{l \mapsto 1\}\rangle \arrow[r] & \langle () \mid (), \{l \mapsto 2\}\rangle \\
\langle l := 1 \mid l := 2, \{l \mapsto 0\}\rangle \arrow[ur] \arrow[dr] & & \\
 & \langle l := 1 \mid (), \{l \mapsto 2\}\rangle \arrow[r] & \langle () \mid (), \{l \mapsto 1\}\rangle
\end{tikzcd}$$

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But, assignments and dereferencing are *atomic*. For example,

$\langle l := 3498734590879238429384 \mid l := 7, \{l \mapsto 0\}\rangle$

will reduce to a state with l either 3498734590879238429384 or 7, not something with the first word of one and the second word of the other. Implement?

But but, in $(l := e) \mid e'$, the steps of evaluating e and e' can be interleaved.

Think of $(l := 1 + !l) \mid (l := 7 + !l)$ – there are races....

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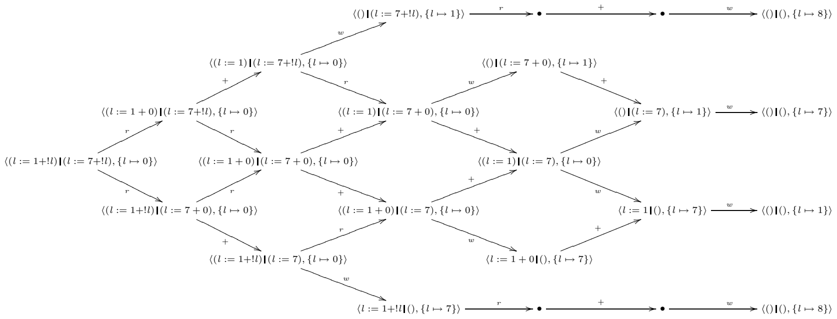
The behaviour of $(l := 1+!l) \mathbin{|} (l := 7+!l)$ for the initial store $\{l \mapsto 0\}$:


$\langle () \mathbin{|} (l := 7+!l), \{l \mapsto 1\}\rangle$
r
$+$
w
$\langle () \mathbin{|} (), \{l \mapsto 8\}\rangle$
w
$\langle (l := 1) \mathbin{|} (l := 7+!l), \{l \mapsto 0\}\rangle$
$\langle () \mathbin{|} (l := 7+0), \{l \mapsto 1\}\rangle$
$+$
r
w
$+$
$\langle (l := 1+0) \mathbin{|} (l := 7+!l), \{l \mapsto 0\}\rangle$
$\langle (l := 1) \mathbin{|} (l := 7+0), \{l \mapsto 0\}\rangle$
$\langle () \mathbin{|} (l := 7), \{l \mapsto 1\}\rangle$
w
$\langle () \mathbin{|} (), \{l \mapsto 7\}\rangle$
r
r
$+$
$+$
w
$\langle (l := 1+!l) \mathbin{|} (l := 7+!l), \{l \mapsto 0\}\rangle$
$\langle (l := 1+0) \mathbin{|} (l := 7+0), \{l \mapsto 0\}\rangle$
$\langle (l := 1) \mathbin{|} (l := 7), \{l \mapsto 0\}\rangle$
r
r
$+$
$+$
w
$\langle (l := 1+!l) \mathbin{|} (l := 7+0), \{l \mapsto 0\}\rangle$
$\langle (l := 1+0) \mathbin{|} (l := 7), \{l \mapsto 0\}\rangle$
$\langle l := 1 \mathbin{|} (), \{l \mapsto 7\}\rangle$
w
$\langle () \mathbin{|} (), \{l \mapsto 1\}\rangle$
$+$
r
w
$+$
$\langle (l := 1+!l) \mathbin{|} (l := 7), \{l \mapsto 0\}\rangle$
$\langle l := 1+0 \mathbin{|} (), \{l \mapsto 7\}\rangle$
w
$\langle l := 1+!l \mathbin{|} (), \{l \mapsto 7\}\rangle$
r
$+$
w
$\langle () \mathbin{|} (), \{l \mapsto 8\}\rangle$

Note that the labels $+$, w and r in this picture are just informal hints as to how those transitions were derived – they are not actually part of the reduction relation.

Some of the nondeterministic choices "don't matter", as you can get back to the same state. Others do...

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Morals

- There really is a combinatorial explosion – and you don't want to be standing next to it...
- Drawing state-space diagrams only works for really tiny examples – we need better techniques for analysis.
- Almost certainly you (as the programmer) didn't want all those 3 outcomes to be possible – need better idioms or constructs for programming.

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So, how do we get anything coherent done?

Need some way(s) to synchronise between threads, so can enforce *mutual exclusion* for shared data.

cf. Lamport's "Bakery" algorithm from Concurrent Systems and Applications. Can you code that in L1$_{l}$? If not, what's the smallest extension required?

Usually, though, you can depend on built-in support from the scheduler, e.g. for *mutexes* and *condition variables* (or, at a lower level, `tas` or `cas`).

See this – in the library – for a good discussion of mutexes and condition variables. A. Birrell, J. Guttag, J. Horning, and R. Levin. *Thread synchronization: a Formal Specification.* In G. Nelson, editor, System Programming with Modula-3, chapter 5, pages 119-129. Prentice-Hall, 1991.

See N. Lynch. *Distributed Algorithms* for other mutual exclusion algorithms (and much else besides).

Consider simple mutexes, with commands to lock an unlocked mutex and to unlock a locked mutex (and do nothing for an unlock of an unlocked mutex).

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Adding Primitive Mutexes

Mutex names $m \in \mathbb{M} = \{\mathrm{m}, \mathrm{m}_1, ...\}$

Configurations $\langle e, s, M \rangle$ where $M{:}\mathbb{M} \rightarrow \mathbb{B}$ is the mutex state

Expressions $e ::= ... \mid \textbf{lock}\ m \mid \textbf{unlock}\ m$

(lock) $\dfrac{}{\Gamma \vdash \textbf{lock}\ m{:}\mathsf{unit}}$ (unlock) $\dfrac{}{\Gamma \vdash \textbf{unlock}\ m{:}\mathsf{unit}}$

(lock) $\langle \textbf{lock}\ m, s, M \rangle \longrightarrow \langle (), s, M + \{m \mapsto \textbf{true}\} \rangle$ if $\neg M(m)$

(unlock) $\langle \textbf{unlock}\ m, s, M \rangle \longrightarrow \langle (), s, M + \{m \mapsto \textbf{false}\} \rangle$

Note that (lock) *atomically* (a) checks the mutex is currently false, (b) changes its state, and (c) lets the thread proceed.

Also, there is no record of which thread is holding a locked mutex.

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Need to adapt all the other semantic rules to carry the mutex state M around. For example, replace

$$\text{(op2)}\quad \frac{\langle e_2, s\rangle \longrightarrow \langle e_2', s'\rangle}{\langle v\ op\ e_2, s\rangle \longrightarrow \langle v\ op\ e_2', s'\rangle}$$

by

$$\text{(op2)}\quad \frac{\langle e_2, s, M\rangle \longrightarrow \langle e_2', s', M'\rangle}{\langle v\ op\ e_2, s, M\rangle \longrightarrow \langle v\ op\ e_2', s', M'\rangle}$$

(note, the M and s must behave the same wrt evaluation order).

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Using a Mutex

Consider

$e = (\textbf{lock}\ \mathrm{m}; l := 1{+}!l; \textbf{unlock}\ \mathrm{m}) \,|\, (\textbf{lock}\ \mathrm{m}; l := 7{+}!l; \textbf{unlock}\ \mathrm{m})$

The behaviour of $\langle e, s, M\rangle$, with the initial store $s = \{l \mapsto 0\}$ and initial mutex state $M_0 = \lambda m \in \mathbb{M}.\textbf{false}$, is:

$$\begin{tikzcd}
 & \langle (l := 1{+}!l; \textbf{unlock}\ \mathrm{m}) \,|\, (\textbf{lock}\ \mathrm{m}; l := 7{+}!l; \textbf{unlock}\ \mathrm{m}), s, M'\rangle \arrow[dr, "12"] & \\
\langle e, s, M_0\rangle \arrow[ur, "\textbf{lock}\ \mathrm{m}"] \arrow[dr, "\textbf{lock}\ \mathrm{m}"'] & & \langle () \,|\, (), \{l \mapsto 8\}, M\rangle \\
 & \langle (\textbf{lock}\ \mathrm{m}; l := 1{+}!l; \textbf{unlock}\ \mathrm{m}) \,|\, (l := 7{+}!l; \textbf{unlock}\ \mathrm{m}), s, M'\rangle \arrow[ur, "12"'] &
\end{tikzcd}$$

(where $M' = M_0 + \{\mathrm{m} \mapsto \textbf{true}\}$)

In all the intervening states (until the first **unlock**) the second **lock** can't proceed.

Look back to behaviour of the program without mutexes. We've essentially cut down to the top and bottom paths (and also added some extra reductions for **lock** , **unlock** , and ;).

In this example, $l := 1{+}!l$ and $l := 7{+}!l$ *commute*, so we end up in the same final state whichever got the lock first. In general, that won't be the case.

On the downside, we've also lost any performance benefits of concurrency (for this program that's fine, but in general there's some other computation that wouldn't conflict and so could be done in parallel).

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Using Several Mutexes

lock m can block (that's the point). Hence, you can *deadlock*.

$$\begin{array}{rcl} e = & & (\textbf{lock}\ \mathrm{m}_1; \textbf{lock}\ \mathrm{m}_2; l_1 :=!l_2; \textbf{unlock}\ \mathrm{m}_1; \textbf{unlock}\ \mathrm{m}_2) \\ & \mid & (\textbf{lock}\ \mathrm{m}_2; \textbf{lock}\ \mathrm{m}_1; l_2 :=!l_1; \textbf{unlock}\ \mathrm{m}_1; \textbf{unlock}\ \mathrm{m}_2) \end{array}$$

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Locking Disciplines

So, suppose we have several programs $e_1, ..., e_k$, all well-typed with $\Gamma \vdash e_i$:unit, that we want to execute concurrently without 'interference' (whatever that is). Think of them as transaction bodies.

There are many possible locking disciplines. We'll focus on one, to see how it – and the properties it guarantees – can be made precise and proved.

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An Ordered 2PL Discipline, Informally

Fix an association between locations and mutexes. For simplicity, make it 1:1 – associate l with m, l_1 with m_1, etc.

Fix a lock acquisition order. For simplicity, make it $\mathrm{m}, \mathrm{m}_0, \mathrm{m}_1, \mathrm{m}_2, \ldots$.

Require that each e_i

- acquires the lock m_j for each location l_j it uses, before it uses it
- acquires and releases each lock in a properly-bracketed way
- does not acquire any lock after it's released any lock (two-phase)
- acquires locks in increasing order

Then, informally, $(e_1 \mid \ldots \mid e_k)$ should (a) never deadlock, and (b) be *serialisable* – any execution of it should be 'equivalent' to an execution of $e_{\pi(1)}; \ldots; e_{\pi(k)}$ for some permuation π.

These are *semantic properties* again. In general, it won't be computable whether they hold. For simple e_i, though, it's often obvious. Further, one can construct syntactic disciplines that are checkable and are sufficient to guarantee these.

See *Transactional Information Systems*, Gerhard Weikum and Gottfried Vossen, for much more detail on locking disciplines etc. (albeit not from a programming-language semantics perspective).

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Problem: Need a Thread-Local Semantics

Our existing semantics defines the behaviour only of global configurations $\langle e, s, M\rangle$. To state properties of subexpressions, e.g.

- e_i acquires the lock m_j for each location l_j it uses, before it uses it

which really means

- in any execution of $\langle (e_1 \mid \ldots \mid e_i \mid \ldots \mid e_k), s, M\rangle$, e_i acquires the lock m_j for each location l_j it uses, before it uses it

we need some notion of the behaviour of the thread e_i on its own

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Solution: Write One Down

Instead of only defining the global $\langle e, s, M\rangle \longrightarrow \langle e', s', M'\rangle$, with rules

$$\text{(assign1)}\quad \langle \ell := n, s, M\rangle \longrightarrow \langle \textbf{skip}, s + \{\ell \mapsto n\}, M\rangle \quad \text{if } \ell \in \mathrm{dom}(s)$$

$$\text{(parallel1)}\quad \frac{\langle e_1, s, M\rangle \longrightarrow \langle e_1', s', M'\rangle}{\langle e_1 \mathbin{|} e_2, s, M\rangle \longrightarrow \langle e_1' \mathbin{|} e_2, s', M'\rangle}$$

define a per-thread $e \xrightarrow{a} e'$ and use that to define $\langle e, s, M\rangle \longrightarrow \langle e', s', M'\rangle$, with rules like

$$\text{(t-assign1)}\quad \ell := n \xrightarrow{\ell:=n} \textbf{skip}$$

$$\text{(t-parallel1)}\quad \frac{e_1 \xrightarrow{a} e_1'}{e_1 \mathbin{|} e_2 \xrightarrow{a} e_1' \mathbin{|} e_2}$$

$$\text{(c-assign)}\quad \frac{e \xrightarrow{\ell:=n} e' \quad \ell \in \mathrm{dom}(s)}{\langle e, s, M\rangle \longrightarrow \langle e', s + \{\ell \mapsto n\}, M\rangle}$$

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Note the per-thread rules don't mention s or M. Instead, we record in the label a what interactions with the store or mutexes it has.

$$a \quad ::= \quad \tau \mid \ell := n \mid !\ell = n \mid \textbf{lock } m \mid \textbf{unlock } m$$

Conventionally, τ (tau), stands for "no interactions", so $e \xrightarrow{\tau} e'$ if e does an internal step, not involving the store or mutexes.

Theorem 18 (Coincidence of global and thread-local semantics) *The two definitions of $\longrightarrow$ agree exactly.*

Proof strategy: a couple of rule inductions.

Write $\mathbb{A}$ for the set of all actions (explicitly, $\mathbb{A} = \{\tau\} \cup \{\ell := n \mid \ell \in \mathbb{L} \wedge n \in \mathbb{N}\} \cup \{!\ell = n \mid \ell \in \mathbb{L} \wedge n \in \mathbb{N}\} \cup \{\textbf{lock } m \mid m \in \mathbb{M}\} \cup \{\textbf{unlock } m \mid m \in M\}$).

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Global Semantics

(op $+$) $\langle n_1 + n_2, s, M\rangle \longrightarrow \langle n, s, M\rangle$ if $n = n_1 + n_2$

(op $\geq$) $\langle n_1 \geq n_2, s, M\rangle \longrightarrow \langle b, s, M\rangle$ if $b = (n_1 \geq n_2)$

(op1) $$\frac{\langle e_1, s, M\rangle \longrightarrow \langle e_1', s', M'\rangle}{\langle e_1\ op\ e_2, s, M\rangle \longrightarrow \langle e_1'\ op\ e_2, s', M'\rangle}$$

(op2) $$\frac{\langle e_2, s, M\rangle \longrightarrow \langle e_2', s', M'\rangle}{\langle v\ op\ e_2, s, M\rangle \longrightarrow \langle v\ op\ e_2', s', M'\rangle}$$

(deref) $\langle !\ell, s, M\rangle \longrightarrow \langle n, s, M\rangle$ if $\ell \in \mathsf{dom}(s)$ and $s(\ell) = n$

(assign1) $\langle \ell := n, s, M\rangle \longrightarrow \langle \textbf{skip}, s + \{\ell \mapsto n\}, M\rangle$ if $\ell \in \mathsf{dom}(s)$

(assign2) $$\frac{\langle e, s, M\rangle \longrightarrow \langle e', s', M'\rangle}{\langle \ell := e, s, M\rangle \longrightarrow \langle \ell := e', s', M'\rangle}$$

(seq1) $\langle \textbf{skip}; e_2, s, M\rangle \longrightarrow \langle e_2, s, M\rangle$

(seq2) $$\frac{\langle e_1, s, M\rangle \longrightarrow \langle e_1', s', M'\rangle}{\langle e_1; e_2, s, M\rangle \longrightarrow \langle e_1'; e_2, s', M'\rangle}$$

(if1) $\langle \textbf{if true then } e_2 \textbf{ else } e_3, s, M\rangle \longrightarrow \langle e_2, s, M\rangle$

(if2) $\langle \textbf{if false then } e_2 \textbf{ else } e_3, s, M\rangle \longrightarrow \langle e_3, s, M\rangle$

(if3) $$\frac{\langle e_1, s, M\rangle \longrightarrow \langle e_1', s', M'\rangle}{\langle \textbf{if } e_1 \textbf{ then } e_2 \textbf{ else } e_3, s, M\rangle \longrightarrow \langle \textbf{if } e_1' \textbf{ then } e_2 \textbf{ else } e_3, s', M'\rangle}$$

(while)
$\langle \textbf{while } e_1 \textbf{ do } e_2, s, M\rangle \longrightarrow \langle \textbf{if } e_1 \textbf{ then } (e_2; \textbf{while } e_1 \textbf{ do } e_2) \textbf{ else skip}, s, M\rangle$

(parallel1) $$\frac{\langle e_1, s, M\rangle \longrightarrow \langle e_1', s', M'\rangle}{\langle e_1 \,|\, e_2, s, M\rangle \longrightarrow \langle e_1' \,|\, e_2, s', M'\rangle}$$

(parallel2) $$\frac{\langle e_2, s, M\rangle \longrightarrow \langle e_2', s', M'\rangle}{\langle e_1 \,|\, e_2, s, M\rangle \longrightarrow \langle e_1 \,|\, e_2', s', M'\rangle}$$

(lock) $\langle \textbf{lock } m, s, M\rangle \longrightarrow \langle (), s, M + \{m \mapsto \textbf{true}\}\rangle$ if $\neg M(m)$

(unlock) $\langle \textbf{unlock } m, s, M\rangle \longrightarrow \langle (), s, M + \{m \mapsto \textbf{false}\}\rangle$

Thread-Local Semantics

(t-op $+$) $n_1 + n_2 \xrightarrow{\tau} n$ if $n = n_1 + n_2$

(t-op $\geq$) $n_1 \geq n_2 \xrightarrow{\tau} b$ if $b = (n_1 \geq n_2)$

(t-op1) $$\frac{e_1 \xrightarrow{a} e_1'}{e_1\ op\ e_2 \xrightarrow{a} e_1'\ op\ e_2}$$

(t-op2) $$\frac{e_2 \xrightarrow{a} e_2'}{v\ op\ e_2 \xrightarrow{a} v\ op\ e_2'}$$

(t-deref) $!\ell \xrightarrow{!\ell=n} n$

(t-assign1) $\ell := n \xrightarrow{\ell:=n} \textbf{skip}$

(t-assign2) $$\frac{e \xrightarrow{a} e'}{\ell := e \xrightarrow{a} \ell := e'}$$

(t-seq1) $\textbf{skip}; e_2 \xrightarrow{\tau} e_2$

(t-seq2) $$\frac{e_1 \xrightarrow{a} e_1'}{e_1; e_2 \xrightarrow{a} e_1'; e_2}$$

(t-if1) $\textbf{if true then } e_2 \textbf{ else } e_3 \xrightarrow{\tau} e_2$

(t-if2) $\textbf{if false then } e_2 \textbf{ else } e_3 \xrightarrow{\tau} e_3$

(t-if3) $$\frac{e_1 \xrightarrow{a} e_1'}{\textbf{if } e_1 \textbf{ then } e_2 \textbf{ else } e_3 \xrightarrow{a} \textbf{if } e_1' \textbf{ then } e_2 \textbf{ else } e_3}$$

(t-while)
$\textbf{while } e_1 \textbf{ do } e_2 \xrightarrow{\tau} \textbf{if } e_1 \textbf{ then } (e_2; \textbf{while } e_1 \textbf{ do } e_2) \textbf{ else skip}$

(t-parallel1) $$\frac{e_1 \xrightarrow{a} e_1'}{e_1 \,|\, e_2 \xrightarrow{a} e_1' \,|\, e_2}$$

(t-parallel2) $$\frac{e_2 \xrightarrow{a} e_2'}{e_1 \,|\, e_2 \xrightarrow{a} e_1 \,|\, e_2'}$$

(t-lock) $\textbf{lock } m \xrightarrow{\textbf{lock } m} ()$

(t-unlock) $\textbf{unlock } m \xrightarrow{\textbf{unlock } m} ()$

(c-tau) $$\frac{e \xrightarrow{\tau} e'}{\langle e, s, M\rangle \longrightarrow \langle e', s, M\rangle}$$

(c-assign) $$\frac{e \xrightarrow{\ell:=n} e' \quad \ell \in \mathsf{dom}(s)}{\langle e, s, M\rangle \longrightarrow \langle e', s + \{\ell \mapsto n\}, M\rangle}$$

(c-lock) $$\frac{e \xrightarrow{\textbf{lock } m} e' \quad \neg M(m)}{\langle e, s, M\rangle \longrightarrow \langle e', s, M + \{m \mapsto \textbf{true}\}\rangle}$$

(c-deref) $$\frac{e \xrightarrow{!\ell=n} e' \quad \ell \in \mathsf{dom}(s) \wedge s(\ell) = n}{\langle e, s, M\rangle \longrightarrow \langle e', s, M\rangle}$$

(c-unlock) $$\frac{e \xrightarrow{\textbf{unlock } m} e'}{\langle e, s, M\rangle \longrightarrow \langle e', s, M + \{m \mapsto \textbf{false}\}\rangle}$$

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Example of Thread-local transitions

For $e = (\textbf{lock}\ \mathrm{m}; (l := 1+!l; \textbf{unlock}\ \mathrm{m}))$ we have

$$
\begin{array}{lll}
e & \xrightarrow{\textbf{lock}\ \mathrm{m}} & \textbf{skip}; (l := 1+!l; \textbf{unlock}\ \mathrm{m}) \\
 & \xrightarrow{\tau} & (l := 1+!l; \textbf{unlock}\ \mathrm{m}) \\
 & \xrightarrow{!l=n} & (l := 1+n; \textbf{unlock}\ \mathrm{m}) \qquad \text{for any } n \in \mathbb{Z} \\
 & \xrightarrow{\tau} & (l := n'; \textbf{unlock}\ \mathrm{m}) \qquad \text{for } n' = 1+n \\
 & \xrightarrow{l:=n'} & \textbf{skip}; \textbf{unlock}\ \mathrm{m} \\
 & \xrightarrow{\tau} & \textbf{unlock}\ \mathrm{m} \\
 & \xrightarrow{\textbf{unlock}\ \mathrm{m}} & \textbf{skip}
\end{array}
$$

Hence, using (t-parallel) and the (c-*) rules, for $s' = s + \{l \mapsto 1 + s(l)\}$,
$\langle e \,|\, e', s, M_0\rangle \longrightarrow\longrightarrow\longrightarrow\longrightarrow\longrightarrow\longrightarrow\longrightarrow \langle \textbf{skip} \,|\, e', s', M_0\rangle$

(need $l \in \mathrm{dom}(s)$ also)

One often uses similar labelled transitions in defining *communication* between threads (or machines), and also in working with observational equivalences for concurrent languages (cf. *bisimulation*).

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Now can make the Ordered 2PL Discipline precise

Say e obeys the discipline if for any (finite or infinite)
$e \xrightarrow{a_1} e_1 \xrightarrow{a_2} e_2 \xrightarrow{a_3} \ldots$

- if a_i is $(l_j := n)$ or $(!l_j = n)$ then for some $k < i$ we have $a_k = \textbf{lock}\ m_j$ without an intervening **unlock** m_j.
- for each j, the subsequence of $a_1, a_2, \ldots$ with labels **lock** m_j and **unlock** m_j is a prefix of $((\textbf{lock}\ \mathrm{m}_j)(\textbf{unlock}\ \mathrm{m}_j))^*$. Moreover, if $\neg(e_k \xrightarrow{a}\ )$ then the subsequence does not end in a **lock** m_j.
- if $a_i = \textbf{lock}\ \mathrm{m}_j$ and $a_{i'} = \textbf{unlock}\ \mathrm{m}_{j'}$ then $i < i'$
- if $a_i = \textbf{lock}\ \mathrm{m}_j$ and $a_{i'} = \textbf{lock}\ \mathrm{m}_{j'}$ and $i < i'$ then $j < j'$

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... and make the guaranteed properties precise

Say $e_1, \ldots, e_k$ are *serialisable* if for any initial store s, if
$\langle (e_1 | \ldots | e_k), s, M_0\rangle \longrightarrow^* \langle e, s', M'\rangle \not\longrightarrow$ then for some permutation π
we have $\langle e_{\pi(1)}; \ldots; e_{\pi(k)}, s, M_0\rangle \longrightarrow^* \langle e', s', M'\rangle$.

Say they are *deadlock-free* if for any initial store s, if
$\langle (e_1 | \ldots | e_k), s, M_0\rangle \longrightarrow^* \langle e, s', M\rangle \not\longrightarrow$ then not $e \xrightarrow{\textbf{lock}\ m} e'$,
i.e. e does not contain any blocked **lock** m subexpressions.

(Warning: there are many subtle variations of these properties!)

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The Theorem

Conjecture 4 *If each e_i obeys the discipline, then $e_1, \ldots e_k$ are serialisable and deadlock-free.*

(may be false!)

Proof strategy: Consider a (derivation of a) computation

$\langle (e_1 | \ldots | e_k), s, M_0 \rangle \longrightarrow \langle \hat{e}_1, s_1, M_1 \rangle \longrightarrow \langle \hat{e}_2, s_2, M_2 \rangle \longrightarrow \ldots$

We know each $\hat{e}_i$ is a corresponding parallel composition. Look at the points at which each e_i acquires its final lock. That defines a serialisation order. In between times, consider commutativity of actions of the different e_i – the premises guarantee that many actions are semantically independent, and so can be permuted.

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We've not discussed *fairness* – the semantics allows any interleaving between parallel components, not only fair ones.

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Language Properties

(Obviously!) don't have Determinacy.

Still have Type Preservation.

Have Progress, but it has to be modified – a well-typed expression of type proc will reduce to some parallel composition of unit values.

Typing and type inference is scarcely changed.

(*very* fancy type systems can be used to enforce locking disciplines)

9.1 Exercises

Exercise 35 ★★*Are the mutexes specified here similar to those described in CSAA?*

Exercise 36 ★★*Can you show all the conditions for O2PL are necessary, by giving for each an example that satisfies all the others and either is not serialisable or deadlocks?*

Exercise 37 ★★★★*Prove the Conjecture about it.*

Exercise 38 ★★★*Write a semantics for an extension of L1 with threads that are more like Unix threads (e.g.* with *thread ids, fork, etc..). Include some of the various ways Unix threads can exchange information.*

10 Epilogue

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Epilogue

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Pre-hoc/Post-hoc Semantics

Pre-hoc: in the design stage. Most useful.

Post-hoc: retroactive semantics. Sometimes necessary.

Underlying both, you need mathematical structure which needs to be understood in its own terms.

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What can *you* use semantics for?

1. to understand a particular language - what you can depend on as a programmer; what you must provide as a compiler writer
2. as a tool for language design:
   (a) for expressing design choices, understanding language features and how they interact.
   (b) for proving properties of a language, eg type safety, decidability of type inference.
3. as a foundation for proving properties of particular programs

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The End

Global Semantics

(op +) $\langle n_1 + n_2, s, M\rangle \longrightarrow \langle n, s, M\rangle$ if $n = n_1 + n_2$

(op $\geq$) $\langle n_1 \geq n_2, s, M\rangle \longrightarrow \langle b, s, M\rangle$ if $b = (n_1 \geq n_2)$

(op1) $\dfrac{\langle e_1, s, M\rangle \longrightarrow \langle e_1', s', M'\rangle}{\langle e_1\ op\ e_2, s, M\rangle \longrightarrow \langle e_1'\ op\ e_2, s', M'\rangle}$

(op2) $\dfrac{\langle e_2, s, M\rangle \longrightarrow \langle e_2', s', M'\rangle}{\langle v\ op\ e_2, s, M\rangle \longrightarrow \langle v\ op\ e_2', s', M'\rangle}$

(deref) $\langle !\ell, s, M\rangle \longrightarrow \langle n, s, M\rangle$ if $\ell \in \mathrm{dom}(s)$ and $s(\ell) = n$

(assign1) $\langle \ell := n, s, M\rangle \longrightarrow \langle \textsf{\textbf{skip}}, s + \{\ell \mapsto n\}, M\rangle$ if $\ell \in \mathrm{dom}(s)$

(assign2) $\dfrac{\langle e, s, M\rangle \longrightarrow \langle e', s', M'\rangle}{\langle \ell := e, s, M\rangle \longrightarrow \langle \ell := e', s', M'\rangle}$

(seq1) $\langle \textsf{\textbf{skip}}; e_2, s, M\rangle \longrightarrow \langle e_2, s, M\rangle$

(seq2) $\dfrac{\langle e_1, s, M\rangle \longrightarrow \langle e_1', s', M'\rangle}{\langle e_1; e_2, s, M\rangle \longrightarrow \langle e_1'; e_2, s', M'\rangle}$

(if1) $\langle \textsf{\textbf{if true then}}\ e_2\ \textsf{\textbf{else}}\ e_3, s, M\rangle \longrightarrow \langle e_2, s, M\rangle$

(if2) $\langle \textsf{\textbf{if false then}}\ e_2\ \textsf{\textbf{else}}\ e_3, s, M\rangle \longrightarrow \langle e_3, s, M\rangle$

(if3) $\dfrac{\langle e_1, s, M\rangle \longrightarrow \langle e_1', s', M'\rangle}{\langle \textsf{\textbf{if}}\ e_1\ \textsf{\textbf{then}}\ e_2\ \textsf{\textbf{else}}\ e_3, s, M\rangle \longrightarrow \langle \textsf{\textbf{if}}\ e_1'\ \textsf{\textbf{then}}\ e_2\ \textsf{\textbf{else}}\ e_3, s', M'\rangle}$

(while)
$\langle \textsf{\textbf{while}}\ e_1\ \textsf{\textbf{do}}\ e_2, s, M\rangle \longrightarrow \langle \textsf{\textbf{if}}\ e_1\ \textsf{\textbf{then}}\ (e_2; \textsf{\textbf{while}}\ e_1\ \textsf{\textbf{do}}\ e_2)\ \textsf{\textbf{else skip}}, s, M\rangle$

(parallel1) $\dfrac{\langle e_1, s, M\rangle \longrightarrow \langle e_1', s', M'\rangle}{\langle e_1 \mid e_2, s, M\rangle \longrightarrow \langle e_1' \mid e_2, s', M'\rangle}$

(parallel2) $\dfrac{\langle e_2, s, M\rangle \longrightarrow \langle e_2', s', M'\rangle}{\langle e_1 \mid e_2, s, M\rangle \longrightarrow \langle e_1 \mid e_2', s', M'\rangle}$

(lock) $\langle \textsf{\textbf{lock}}\ m, s, M\rangle \longrightarrow \langle (), s, M + \{m \mapsto \textsf{\textbf{true}}\}\rangle$ if $\neg M(m)$

(unlock) $\langle \textsf{\textbf{unlock}}\ m, s, M\rangle \longrightarrow \langle (), s, M + \{m \mapsto \textsf{\textbf{false}}\}\rangle$

Thread-Local Semantics

(t-op +) $n_1 + n_2 \xrightarrow{\tau} n$ if $n = n_1 + n_2$

(t-op $\geq$) $n_1 \geq n_2 \xrightarrow{\tau} b$ if $b = (n_1 \geq n_2)$

(t-op1) $\dfrac{e_1 \xrightarrow{a} e_1'}{e_1\ op\ e_2 \xrightarrow{a} e_1'\ op\ e_2}$

(t-op2) $\dfrac{e_2 \xrightarrow{a} e_2'}{v\ op\ e_2 \xrightarrow{a} v\ op\ e_2'}$

(t-deref) $!\ell \xrightarrow{!\ell = n} n$

(t-assign1) $\ell := n \xrightarrow{\ell := n} \textsf{\textbf{skip}}$

(t-assign2) $\dfrac{e \xrightarrow{a} e'}{\ell := e \xrightarrow{a} \ell := e'}$

(t-seq1) $\textsf{\textbf{skip}}; e_2 \xrightarrow{\tau} e_2$

(t-seq2) $\dfrac{e_1 \xrightarrow{a} e_1'}{e_1; e_2 \xrightarrow{a} e_1'; e_2}$

(t-if1) $\textsf{\textbf{if true then}}\ e_2\ \textsf{\textbf{else}}\ e_3 \xrightarrow{\tau} e_2$

(t-if2) $\textsf{\textbf{if false then}}\ e_2\ \textsf{\textbf{else}}\ e_3 \xrightarrow{\tau} e_3$

(t-if3) $\dfrac{e_1 \xrightarrow{a} e_1'}{\textsf{\textbf{if}}\ e_1\ \textsf{\textbf{then}}\ e_2\ \textsf{\textbf{else}}\ e_3 \xrightarrow{a} \textsf{\textbf{if}}\ e_1'\ \textsf{\textbf{then}}\ e_2\ \textsf{\textbf{else}}\ e_3}$

(t-while)
$\textsf{\textbf{while}}\ e_1\ \textsf{\textbf{do}}\ e_2 \xrightarrow{\tau} \textsf{\textbf{if}}\ e_1\ \textsf{\textbf{then}}\ (e_2; \textsf{\textbf{while}}\ e_1\ \textsf{\textbf{do}}\ e_2)\ \textsf{\textbf{else skip}}$

(t-parallel1) $\dfrac{e_1 \xrightarrow{a} e_1'}{e_1 \mid e_2 \xrightarrow{a} e_1' \mid e_2}$

(t-parallel2) $\dfrac{e_2 \xrightarrow{a} e_2'}{e_1 \mid e_2 \xrightarrow{a} e_1 \mid e_2'}$

(t-lock) $\textsf{\textbf{lock}}\ m \xrightarrow{\textsf{\textbf{lock}}\ m} ()$

(t-unlock) $\textsf{\textbf{unlock}}\ m \xrightarrow{\textsf{\textbf{unlock}}\ m} ()$

(c-tau) $\dfrac{e \xrightarrow{\tau} e'}{\langle e, s, M\rangle \longrightarrow \langle e', s, M\rangle}$

(c-assign) $\dfrac{e \xrightarrow{\ell := n} e' \quad \ell \in \mathrm{dom}(s)}{\langle e, s, M\rangle \longrightarrow \langle e', s + \{\ell \mapsto n\}, M\rangle}$

(c-lock) $\dfrac{e \xrightarrow{\textsf{\textbf{lock}}\ m} e' \quad \neg M(m)}{\langle e, s, M\rangle \longrightarrow \langle e', s, M + \{m \mapsto \textsf{\textbf{true}}\}\rangle}$

(c-deref) $\dfrac{e \xrightarrow{!\ell = n} e' \quad \ell \in \mathrm{dom}(s) \wedge s(\ell) = n}{\langle e, s, M\rangle \longrightarrow \langle e', s, M\rangle}$

(c-unlock) $\dfrac{e \xrightarrow{\textsf{\textbf{unlock}}\ m} e'}{\langle e, s, M\rangle \longrightarrow \langle e', s, M + \{m \mapsto \textsf{\textbf{false}}\}\rangle}$

The behaviour of $(l := 1+!l) | (l := 7+!l)$ for the initial store $\{l \mapsto 0\}$:

$$\begin{tikzcd}
 & & & \langle () | (l := 7+!l), \{l \mapsto 1\}\rangle \arrow[r, "r"] & \bullet \arrow[r, "+"] & \bullet \arrow[r, "w"] & \langle () | (), \{l \mapsto 8\}\rangle \\
 & & \langle (l := 1) | (l := 7+!l), \{l \mapsto 0\}\rangle \arrow[ur, "w"] \arrow[dr, "r"] & & \langle () | (l := 7 + 0), \{l \mapsto 1\}\rangle \arrow[dr, "+"] & & \\
 & \langle (l := 1 + 0) | (l := 7+!l), \{l \mapsto 0\}\rangle \arrow[ur, "+"] \arrow[dr, "r"] & & \langle (l := 1) | (l := 7 + 0), \{l \mapsto 0\}\rangle \arrow[ur, "w"] \arrow[dr, "+"] & & \langle () | (l := 7), \{l \mapsto 1\}\rangle \arrow[r, "w"] & \langle () | (), \{l \mapsto 7\}\rangle \\
\langle (l := 1+!l) | (l := 7+!l), \{l \mapsto 0\}\rangle \arrow[ur, "r"] \arrow[dr, "r"] & & \langle (l := 1 + 0) | (l := 7 + 0), \{l \mapsto 0\}\rangle \arrow[ur, "+"] \arrow[dr, "+"] & & \langle (l := 1) | (l := 7), \{l \mapsto 0\}\rangle \arrow[ur, "w"] \arrow[dr, "w"] & & \\
 & \langle (l := 1+!l) | (l := 7 + 0), \{l \mapsto 0\}\rangle \arrow[ur, "r"] \arrow[dr, "+"] & & \langle (l := 1 + 0) | (l := 7), \{l \mapsto 0\}\rangle \arrow[ur, "+"] \arrow[dr, "w"] & & \langle l := 1 | (), \{l \mapsto 7\}\rangle \arrow[r, "w"] & \langle () | (), \{l \mapsto 1\}\rangle \\
 & & \langle (l := 1+!l) | (l := 7), \{l \mapsto 0\}\rangle \arrow[ur, "r"] \arrow[dr, "w"] & & \langle l := 1 + 0 | (), \{l \mapsto 7\}\rangle \arrow[ur, "+"] & & \\
 & & & \langle l := 1+!l | (), \{l \mapsto 7\}\rangle \arrow[r, "r"] & \bullet \arrow[r, "+"] & \bullet \arrow[r, "w"] & \langle () | (), \{l \mapsto 8\}\rangle
\end{tikzcd}$$