

COMPUTER SCIENCE TRIPOS, Part II (General)  
DIPLOMA IN COMPUTER SCIENCE

Mathematics for Computation Theory, Part A

(KM 2002)

**Examples.** (see also exercises in the notes)

- 1 Prove that  $[ \{ (a \rightarrow b) \vee (a \rightarrow d) \} \rightarrow (b \vee d) ] = a \vee b \vee d$ , where the symbol ' $\rightarrow$ ' stands for "implies", so that  $(a \rightarrow b) \equiv (\neg a \vee b)$ .
- 2 Consider the argument: "If Anna can cancan or Kant can't cant, then Greville will cavil vilely. If Greville will cavil vilely, Will will want. But Will *won't* want. Therefore, Kant *can* cant". By rewriting the statement in terms of four Boolean variables, show it is tautologous and hence a valid argument.
- 3 If  $|A| = k$ ,  $|B| = m$  and  $|C| = n$ , how many ternary relations may be defined on  $(A \times B \times C)$ ?
- 4 How many subsets of  $[1, n]$  are there of *even* size?
- 5 Let  $R$  and  $S$  be relations from  $A$  to  $B$ . Show that if  $R \subseteq S$  then  $R^{-1} \subseteq S^{-1}$ . Establish the identities  $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$ ,  $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$ .
- 6 Show that the smallest equivalence relation containing the two equivalence relations  $R$  and  $S$  is  $(R \cup S)^+$ .
- 7 Prove that  $\mathcal{P}(\{0,1,2\})$  and  $(\{0,1\}^3)$  are isomorphic posets, that is there is a bijection  $f$  between them for which both  $f$  and  $f^{-1}$  are monotonic.
- 8 Prove that  $(\mathbb{N} \times \mathbb{N})$  with the lexicographical order is well-ordered.
- 9 The Ackermann function  $a(m, n): (\mathbb{N} \times \mathbb{N}) \rightarrow \mathbb{N}$  is defined by
$$a(0, n) = n+1, \text{ for all } n \geq 0;$$
$$a(m+1, 0) = a(m, 1), \text{ for all } m \geq 0;$$
$$a(m+1, n+1) = a(m, a(m+1, n)), \text{ for all } m \geq 0, \text{ and } n \geq 0.$$
Prove that Ackermann's function is a total function on  $(\mathbb{N} \times \mathbb{N})$ . Show that  $a(1, n) = n+2$ ,  $a(2, n) = 2(n+3) - 3$ ,  $a(3, n) = 2^{n+3} - 3$ . What is  $a(4, n)$ ? What happens when you run a Java program that calculates  $a(m, n)$ ? What happens if you change  $n+1$  to  $n$  in the first line? What happens if you change  $a(m, 1)$  to  $a(m, 0)$  in the second line?
- 10 A *triomino* is an L-shaped pattern made from three square tiles. A  $(2^k \times 2^k)$  chessboard, whose squares are the same size as the tiles, has one of its squares painted purple. Show that the chessboard can be covered with triominoes so that only the purple square is exposed.
- 11 Identify and prove the uniqueness of the function  $f: \mathbb{N} \rightarrow \mathbb{N}$  on the natural numbers (including 0, of course) that has the following properties:
  - a)  $\forall n \geq 0, f(n+1) > f(n)$ ;
  - b)  $\forall n \geq 0, f\{f(n)\} = 3n$ .