Computer Systems Modelling

Notation summary sheet

The numbers in the second column refer to relevant pages in the slides

Queuing disciplines and Kendall notation

FCFS	(22)	First-come first-served queueing discipline
LCFS	(22)	Last-come first-served queueing discipline
SIRO	(22)	Service in random order queueing discipline
$M/M/1$	(108)	Exponentially distributed inter-arrival and service-time distributions, one server, no queue limit, infinite population
$M/M/m$	(116)	As $M/M/1$, but with m servers rather than a single server
$M/M/1/K$	(117)	As $M/M/1$, but with a queue limit of K customers
$M/M/1//N$	(118)	As $M/M/1$, but with a limited population of N customers (and hence implicitly a maximum queue length of N)
$M/M/m/m$	(119)	An 'm server loss system' in which there are m servers and no queueing capacity
$M/E_2/1$	(124)	As $M/M/1$, but with a two-stage Erlangian service-time distribution

Probability

$\mathbb{P}(x)$	(13)	The probability that an event x will occur
$F_X(x)$	(14)	Cumulative density function for the random variable X
$f_X(x)$	(14)	Probability density function for the random variable X
$\mathbb{E}(X)$	(15)	The expected value of the random variable X, sometimes called the *first moment* of the distribution, the *mean*, the *average*
$\mathbb{E}(X^n)$	(15)	The n^{th} moment of X
$\overline{(X-\overline{X})^n}$	(15)	The n^{th} central moment of X
σ_X^2	(16)	The *variance* of X ($\sigma_X^2 = \overline{X^2} - (\overline{X})^2$)
$\sigma_X \triangleq \sqrt{\sigma_X^2}$	(16)	The *standard deviation* of X
$\frac{\sigma_X}{\overline{X}}$	(16)	The *coefficient of variation* of X
z_α	(55)	The largest value z such that $\mathbb{P}(Z > z) = \alpha$ where Z is a unit normal random variable. Used when deriving confidence bounds.

Operational analysis

Subscript i refers to a per-device quantity, for example X is the system throughput and X_i is the throughput at device i.

Subscript b, and in particular V_b and S_b, refer to a *bottleneck* device.

T	(61)	Time we observe the system
A	(61)	Number of arrivals observed
C	(61)	Number of departures observed
W	(61)	Job-time product: the sum of the durations of all customers
λ	(62)	Arrival rate
X	(62)	Throughput – departures per unit time
N	(62)	Mean number of customers in the system
M	(69)	Number of customers in the system, when this remains fixed
R	(62)	Mean residence time in the system
Z	(67)	Average 'think time' in an interactive system – the time between completing one interaction and beginning the next

Additional per-device variables:

U_i	(63)	Utilisation – proportion of time a resource is busy
S_i	(63)	Service requirement – the time the resource spends on each completion
B_i	(61)	The time for which the resource is busy
V_i	(71)	The number of completions at the resource for every completion from the system
D_i	(72)	The service demand a customer makes on a resource

Queueing theory

λ		Arrivals rate into the system
μ		Service rate of a server
T	(11)	Mean time in the system
W	(12)	Mean waiting time in the system
N	(12)	Mean number in the system, sometime split into N_q (mean number in the queue) and N_s (mean number being served)
p_k	(106)	is the probability of being in state k in a queuing system – in particular p_0 is the probability of being in state 0 which conventionally represents an idle system
$Q = (q_{ij})$	(130)	Routing matrix in an queueing network. The q_ij are the *routing probabilities*

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Random Variables

Family		Range	Density	Mean	Variance	Generating function	Notes
Poisson	$\mathcal{P}(\lambda)$	$0,1,\ldots$	$\mathbb{P}(X=r)=\dfrac{\lambda^r e^{-\lambda}}{r!}$	λ	λ	$\mathbb{E}(s^X)=e^{\lambda(s-1)}$	$\sum\mathcal{P}(\lambda_i)\sim\mathcal{P}(\sum\lambda_i)$
Geometric	$\mathrm{Geom}(p)$	$1,2,\ldots$	$\mathbb{P}(X=r)=p(1-p)^{r-1}$	$\dfrac{1}{p}$	$\dfrac{1-p}{p^2}$	$\mathbb{E}(s^X)=\dfrac{ps}{1-s(1-p)}$	
Binomial	$\mathrm{Bin}(n,p)$	$0,1,\ldots n$	$\mathbb{P}(X=r)=\dbinom{n}{r}p^r(1-p)^{n-r}$	np	$np(1-p)$	$\mathbb{E}(s^X)=\big(ps+(1-p)\big)^n$	$\sum^n \mathrm{Bin}(1,p)\sim\mathrm{Bin}(n,p)$
Uniform	$\mathrm{U}[1,n]$	$1,2,\ldots n$	$\mathbb{P}(X=r)=\dfrac{1}{n}$	$\dfrac{n+1}{2}$	$\dfrac{n^2-1}{12}$	$\mathbb{E}(s^X)=\dfrac{s(1-s^n)}{n(1-s)}$	
Uniform	$\mathrm{U}[a,b]$	$[a,b]$	$f(x)=\dfrac{1}{b-a}$	$\dfrac{a+b}{2}$	$\dfrac{(b-a)^2}{12}$	$\mathbb{E}(e^{tX})=\dfrac{e^{bt}-e^{at}}{(b-a)t}$	
Exponential	$\mathrm{Exp}(\lambda)$	$\mathbb{R}^+$	$f(x)=\lambda e^{-\lambda x}$	$\dfrac{1}{\lambda}$	$\dfrac{1}{\lambda^2}$	$\mathbb{E}(e^{tX})=\dfrac{\lambda}{\lambda-t}$	$\sum^n \mathrm{Exp}(\lambda)\sim\Gamma(n,\lambda)$
Gamma	$\Gamma(\gamma,\lambda)$	$\mathbb{R}^+$	$f(x)=\dfrac{\lambda^\gamma x^{\gamma-1}e^{-\lambda x}}{\Gamma(\gamma)}$	$\dfrac{\gamma}{\lambda}$	$\dfrac{\gamma}{\lambda^2}$	$\mathbb{E}(e^{tX})=(1-t/\lambda)^{-\gamma}$	$\Gamma(n,\lambda)\le t\longleftrightarrow\mathcal{P}(\lambda t)\ge n$
Normal	$\mathrm{N}(\mu,\sigma^2)$	$\mathbb{R}$	$f(x)=\dfrac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	μ	σ^2	$\mathbb{E}(e^{tX})=\exp(t\mu+\frac{1}{2}t^2\sigma^2)$	$\sum\mathrm{N}(\mu_i,\sigma_i^2)\sim\mathrm{N}(\sum\mu_i,\sum\sigma_i^2)$
MVN	$\mathrm{N}(\mu,\Sigma)$	$\mathbb{R}^n$	$f(x)=\dfrac{\exp\left[-\frac{1}{2}(x-\mu)^\mathsf{T}\Sigma^{-1}(x-\mu)\right]}{\sqrt{(2\pi)^n\det\Sigma}}$	μ	Σ	$\mathbb{E}(e^{t^\mathsf{T}X})=\exp(t^\mathsf{T}\mu+\frac{1}{2}t^\mathsf{T}\Sigma t)$	$a+B\,\mathrm{N}(\mu,\Sigma)\sim\mathrm{N}(a+B\mu,B\Sigma B^\mathsf{T})$
Chi-squared	χ^2_n	$\mathbb{R}^+$	$f(x)=\dfrac{\frac{1}{2}(\frac{x}{2})^{n/2-1}e^{-x/2}}{\Gamma(\frac{n}{2})}$	n	$2n$	$\mathbb{E}(e^{tX})=(1-2t)^{-n/2}$	$\chi^2_n\sim\Gamma(\frac{n}{2},\frac{1}{2})$
Beta	$\mathrm{Beta}(r,s)$	$[0,1]$	$f(x)=\dfrac{x^{r-1}(1-x)^{s-1}}{\beta(r,s)}$	$\dfrac{r}{r+s}$	$\dfrac{r(r-1)}{r+s}$	$\mathbb{E}(e^{tX})$ has no closed form	$\beta(r,s)=\dfrac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}$
Cauchy	$\mathrm{Cauchy}(a)$	$\mathbb{R}$	$f(x)=\dfrac{a}{a^2+x^2}$	undefined	undefined	$\mathbb{E}(e^{tX})=\infty$ for $t\neq 0$	

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