Learning Guide & Examples – Continuous Mathematics

Part IB Lecture Course, Michaelmas Term

Aims

The aims of this course are to review some key concepts and operations defined in continuous mathematics involving real- and complex-valued functions of real variables. Focus is on the use and implementation of these notions in the discrete spaces we enter when computing. Topics include: expansions and basis functions; orthogonality and projections; differential equations and their computational solution; linear operators and their eigenfunctions; wavelets and Fourier analysis.

Lectures

- **Review of analysis.** Real and complex-valued functions of a real variable. Power series and transcendental functions. Expansions and basis functions. Smoothness, continuity, limits.
- **Linear vector spaces and decompositions.** Orthogonality, independence, and orthonormality. Linear combinations. Projections, inner products and completeness. Linear subspaces. Useful expansion bases for continuous functions.
- **Differential and integral operators in computation.** The infinitesimal calculus. Taylor series. Numerical integration. Differential equations and computational ways to solve them. Complex exponentials. Introduction to Fourier analysis in one and two dimensions; useful theorems. Convolution and filtering.
- **Signals and systems.** Eigenfunctions of linear operators. Fourier analysis and series; continuous Fourier Transforms and their inverses. Representation in non-orthogonal functions, and wavelets. The degrees-of-freedom in a signal. Sampling theorem. How to operate on continuous signals computationally in order to extract their information.

Objectives

At the end of the course students should:

- Understand how data or functions can be represented in terms of their projections onto other groups of functions.
- Be fluent in the use of, and properties of, complex variables.
- Be able to implement and use, in discrete computational form, such continuous notions as differentiation, integration, and convolution.
- Grasp key properties and uses of Fourier analysis, transforms, and wavelets.

Reference books

Kaplan, W. (1992). *Advanced Calculus.* Addison-Wesley (4th ed.).
Oppenheim, A.V. & Willsky, A.S. (1984). *Signals and Systems.* Prentice-Hall.

Worked Examples

Continuous Mathematics: Example Problem Set 1

Many important problems in mathematical modeling and scientific computing require the use of complex variables. Unfortunately, popular programming languages like C do not even have a complex variable type, and so we must construct them from reals. Assuming that the quantities a, b, c, d are all reals and $i = \sqrt{-1}$, resolve the following expressions, or explain the following operations, involving complex variables $\mathcal{Z}_1 = a + ib$ and $\mathcal{Z}_2 = c + id$:

1. Let $\mathcal{Z}_3 = \mathcal{Z}_1\mathcal{Z}_2$. What is the real part of $\mathcal{Z}_3$, and what is its imaginary part?

2. What is $\|\mathcal{Z}_1\|$, the modulus of $\mathcal{Z}_1$, and what is $\|\mathcal{Z}_3\|$, the modulus of $\mathcal{Z}_3 = \mathcal{Z}_1\mathcal{Z}_2$?

3. What is $\angle\mathcal{Z}_2$, the angle of complex variable $\mathcal{Z}_2$?

4. Express $\mathcal{Z}_1$ in polar complex form, not using the quantities a or b but rather the modulus $\|\mathcal{Z}_1\|$ and angle $\angle\mathcal{Z}_1$.

5. Suppose that $\mathcal{Z}_1$ and $\mathcal{Z}_2$ both have a modulus of 1. Explain how their product $\mathcal{Z}_3 = \mathcal{Z}_1\mathcal{Z}_2$ amounts to a rotation in the complex plane, with the aid of a diagram. Why is the multiplication of these complex variables reduced now to addition? Without using the quantities a, b, c, d, what is the value of $\|\mathcal{Z}_3\|$?

6. Suppose that in polar complex form, $\mathcal{Z} = \exp(2\pi i/5)$. What do you get if $\mathcal{Z}$ is multiplied by itself 5 times? Give the simplest possible answer that you can.

7. Consider the complex exponential function $f(x) = \exp(2\pi i\omega x)$. What function is its real part? What function is its imaginary part?

8. If the above function $f(x)$ passes through a linear system, i.e. is operated upon by any conceivable linear differential or integral operator, what is the most dramatic way in which $f(x)$ can possibly be affected?

Model Answer – Example Problem Set 1

1. The real part of $\mathcal{Z}_3$ is $ac - bd$. Its imaginary part is $ad + bc$.

2. Modulus $\|\mathcal{Z}_1\| = \sqrt{a^2 + b^2}$.

   Modulus $\|\mathcal{Z}_3\| = \sqrt{(ac - bd)^2 + (ad + bc)^2}$, or equivalently, $\|\mathcal{Z}_3\| = \sqrt{(a^2 + b^2)(c^2 + d^2)}$.

3. Angle $\angle\mathcal{Z}_2 = \tan^{-1}(\frac{d}{c})$.

4. In complex polar form, $\mathcal{Z}_1 = \|\mathcal{Z}_1\| \exp(i\angle\mathcal{Z}_1)$.

5. As is clear from describing these variables in complex polar form,

$$\mathcal{Z}_3 = \|\mathcal{Z}_1\| \exp(i\angle\mathcal{Z}_1)\|\mathcal{Z}_2\| \exp(i\angle\mathcal{Z}_2) = \|\mathcal{Z}_1\|\|\mathcal{Z}_2\| \exp[i(\angle\mathcal{Z}_1 + \angle\mathcal{Z}_2)],$$

   and so the angles of the two complex variables just add. This is a rotation in the complex plane. Since the two moduli both happen to equal 1, the modulus of the product $\mathcal{Z}_1\mathcal{Z}_2$ is also $\|\mathcal{Z}_3\| = 1$.

6. If $\mathcal{Z} = \exp(2\pi i/5)$ is multiplied by itself five times, its angle is just added to itself five times, producing $\exp(2\pi i) = 1$.

7. The real part of $f(x) = \exp(2\pi i\omega x)$ is the function $\cos(2\pi\omega x)$. Its imaginary part is the function $\sin(2\pi\omega x)$.

8. If the complex exponential $f(x) = \exp(2\pi i\omega x)$ is operated upon by any linear operator, its functional form cannot change. The most dramatic thing that can happen to it is that it gets multiplied by a complex constant. This means that only its amplitude and phase can be affected. Complex exponentials are the <u>eigenfunctions</u> of linear systems.

Continuous Mathematics: Example Problem Set 2

We compute the representation of some continuous function $f(t)$ in a space spanned by an orthonormal family $\{\Psi_j(t)\}$ of continuous basis functions by projecting $f(t)$ onto them. We express these projections in bracket notation $< f(t), \Psi_j(t) >$ denoting $\int_{-\infty}^{\infty} f(t)\Psi_j^*(t)dt$, where the * denotes complex conjugate in the case that the $\{\Psi_j(t)\}$ are complex-valued; and $f(t)$ is assumed to be square-integrable.

1. Give an expression for computing $f(t)$ if we know only its projections $< f(t), \Psi_j(t) >$ onto this set of basis functions $\{\Psi_j(t)\}$. Explain what is happening.

2. Now give an expression for computing $f^{(n)}(t)$, the nth derivative of $f(t)$ with respect to t, in terms of the same projections and continuous basis set. (You may assume the existence of all derivatives.) Explain your answer.

3. Now consider a linear, time-invariant system with impulse-response function $h(t)$, having time-varying input $s(t)$ and time-varying output $r(t)$:

$$s(t) \longrightarrow \boxed{h(t)} \longrightarrow r(t)$$

In the case that the input is the complex exponential $s(t) = \exp(i\mu_j t)$ (where $i = \sqrt{-1}$ and μ_j is a constant), what can you say about the output $r(t)$ of such a system?

4. If the input $s(t)$ has been represented in terms of a set of complex exponentials $\Psi_j(t) = \exp(i\mu_j t)$ as described at the beginning of these problems, is it possible for <u>different</u> complex exponentials (not included in this set) to appear in the output $r(t)$ when it too is represented in terms of complex exponentials? Justify your answer.

Model Answer – Example Problem Set 2

1. $f(t) = \sum_i < f(t), \Psi_i(t) > \Psi_i(t)$
   ie, a linear combination of all the basis functions $\Psi_i(t)$, each weighted by a coefficient giving the projection (inner product) of $f(t)$ onto that basis function.

2. $f^{(n)}(t) = \sum_i < f(t), \Psi_i(t) > \Psi_i^{(n)}(t)$
   ie, a linear combination of the nth derivatives of the basis functions, $\Psi_i^{(n)}(t)$, each weighted by a coefficient giving the projection (inner product) of $f(t)$ onto the original non-differentiated basis function.

3. The output $r(t)$ must also be a complex exponential, differing from the input one only by some complex coefficient (say α): $r(t) = \alpha \exp(i\mu_j t)$

4. No. Complex exponentials are the eigenfunctions of linear time-invariant systems. Therefore such inputs are unchanged in their functional form when they appear as outputs as a result of being processed by the system.

Continuous Mathematics: Example Problem Set 3

1. Suppose we wish to compute a predicted value for some continuous, differentiable function $f(t)$ at future time $t = b$, based only upon knowledge of its value now at time $t = a$, which is $f(a)$, and its first three derivatives now, namely $f'(a)$, $f''(a)$, and $f'''(a)$. How would you compute such a prediction for the future value $f(b)$? Give a formula or explain the sorts of terms that it would include and why they appear.

2. In practical computing problems with numerical data that is discretely sampled at periodic points in time, derivatives defined in terms of continuous limits must be replaced by discrete estimates. Provide finite-difference expressions for each of the three derivatives $f'(t)$, $f''(t)$, and $f'''(t)$ on the discrete set of samples

$$..., f(t_i), f(t_{i+1}), f(t_{i+2}), f(t_{i+3}), ...$$

assuming that this sequence of samples are separated by a constant interval $\Delta t = t_{i+1} - t_i$.

3. If you were trying to compute numerically the solution to a continuous-time first-order differential equation using the Runge-Kutta method of integration, how would you expect the error of your solution in the worst case to depend upon the stepsize Δt?

4. Real-world signals are generally accompanied by noise. Using the Differentiation Theorem of the Fourier Transform, comment upon how noise is amplified as a function of its spectral composition when computing the first-order, second-order, and third-order derivatives of the signal.

Model Answer – Example Problem Set 3

1. Prediction is $f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \frac{f'''(a)}{3!}(b-a)^3$

(A Taylor series which in effect gives a polynomial expansion of $f(t)$ around $t = a$, in terms of derivatives of successive powers of t.)

2.

$$
\begin{aligned}
f'(t) &\approx \frac{f(t_{i+1}) - f(t_i)}{\Delta t} \\
f''(t) &\approx \frac{f(t_{i+2}) - 2f(t_{i+1}) + f(t_i)}{(\Delta t)^2} \\
f'''(t) &\approx \frac{f(t_{i+3}) - 3f(t_{i+2}) + 3f(t_{i+1}) - f(t_i)}{(\Delta t)^3}
\end{aligned}
$$

3. The error will depend upon the stepsize raised to the 4^{th} power: $\epsilon \approx \mathcal{O}(\Delta t^4)$

4. When computing its n^{th}-order derivative, the noise in the signal is amplified as the n^{th}-power of frequency. Thus for the cases given and for frequency ω, this noise problem will grow as ω^1, ω^2, and ω^3, respectively.

Continuous Mathematics: Example Problem Set 4

Suppose that some continuous function $f(x)$ has Fourier Transform $F(\mu)$. Now consider the consequences in the Fourier domain of each of the following operations upon $f(x)$:

1. What will be the Fourier Transform of the m^{th} derivative of $f(x)$ with respect to x: $\left(\frac{d}{dx}\right)^m f(x)$?

2. What will be the Fourier Transform after shifting $f(x)$ by a distance α: $f(x-\alpha)$?

3. What will be the Fourier Transform after dilating $f(x)$ by a factor α: $f(x/\alpha)$?

4. Suppose that $f(x)$ is convolved with another function $g(x)$ whose Fourier Transform is $G(\mu)$. What will be the Fourier Transform $H(\mu)$ of the convolution result $h(x)$ where

$$h(x) = f(x) * g(x) = \int_{-\infty}^{+\infty} f(\alpha) g(x-\alpha) d\alpha?$$

5. Suppose now that a two-dimensional continuous function $f(x,y)$ has a 2D Fourier Transform $F(\mu,\nu)$. Define the Laplacian operator ∇^2 and express the 2D Fourier Transform of $\nabla^2 f(x,y)$.

NOTE: In Lecture Notes and Tripos Questions prior to 1999-2000, the convention was followed of defining frequency in units of 2π radians rather than in cycles. This introduced extra factors of 2π in many of the relations describing properties of Fourier analysis. Major textbooks are divided about this convention; beginning with 1999-2000, I switched. These Example Problems are consistent with the current Lecture Notes, but differ in this regard from previous Notes and Tripos Questions.

Model Answer – Example Problem Set 4

1. The new Fourier Transform will be: $\left(\frac{d}{dx}\right)^m f(x) \overset{FT}{\Longrightarrow} (i\mu)^m F(\mu)$.

2. The new Fourier Transform will be: $F(\mu)\exp(-i\alpha\mu)$.

3. The new Fourier Transform will be: $|\alpha|F(\alpha\mu)$.

4. The Fourier Transform $H(\mu)$ of the convolution result $h(x)$ from convolving $f(x)$ with another function $g(x)$, having respective Fourier Transforms $F(\mu)$ and $G(\mu)$, is the product of the two Transforms: $H(\mu) = F(\mu)G(\mu)$.

5. The Laplacian operator ∇^2 is defined as:

$$\nabla^2 \equiv \left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right)$$

and the 2D Fourier Transform of $\nabla^2 f(x,y)$ is:

$$\nabla^2 f(x,y) \equiv \left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right) f(x,y) \overset{2DFT}{\Longrightarrow} -(\mu^2+\nu^2)F(\mu,\nu).$$

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Continuous Mathematics: Example Problem Set 5

1. Suppose we need to solve a linear 2nd-order differential equation with constant coefficients A and B in which the combined derivatives of the solution we seek, $f(x)$, are only known to be related to another function, $g(x)$:

$$A\frac{d^2f(x)}{dx^2} + B\frac{df(x)}{dx} = g(x)$$

   We know the function $g(x)$, and we can compute its Fourier Transform $G(\mu)$.

   How can we use the properties of the Fourier Transform and its inverse in order to compute the solution $f(x)$ of this differential equation? Provide an expression for $F(\mu)$, the Fourier Transform of $f(x)$, in terms of $G(\mu)$, frequency variable μ, and the coefficients in the differential equation.

   What final step is now required in order to compute the actual solution $f(x)$ of the differential equation, given your expression for $F(\mu)$?

2. In numerical computing, differential operators must always be represented by finite differences. Assume that a function has been sampled at uniform, closely spaced, intervals. How many consecutive sample points are necessary in order to compute the N^{th} derivative of the function at some point?

   To compute the 3rd derivative in a local region of a function, what set of weights would you use to multiply consecutive samples of the function?

3. What is the principal computational advantage of using orthogonal functions, over non-orthogonal ones, when representing a set of data as a linear combination of a universal set of basis functions?

   If $\Psi_k(x)$ belongs to a set of orthonormal basis functions, and $f(x)$ is a function or a set of data that we wish to represent in terms of these basis functions, what is the basic computational operation we need to perform involving $\Psi_k(x)$ and $f(x)$?

Model Answer – Example Problem Set 5

1. By taking the Fourier Transform of both sides of the differential equation, and by exploiting the fact that this is a linear operation and using the Differentiation Theorem of Fourier analysis, we arrive at the following equation in the frequency domain:

$$A(i\mu)^2 F(\mu) + Bi\mu F(\mu) = G(\mu)$$

from which it follows that the Fourier Transform of the solution we seek is simply

$$F(\mu) = \frac{G(\mu)}{iB\mu - A\mu^2}$$

2. Finally, we may use an FFT to compute the inverse Fourier Transform of $F(\mu)$, and this will give us the solution $f(x)$ that we seek for the differential equation.

3. It requires $N + 1$ consecutive sample points in order to calculate the N^{th} derivative of a function in a region.

The weights which need to be multiplied by consecutive samples of the function in order to calculate its 3rd derivative are: -1, +3, -3, +1.

4. The principal advantage is that if orthogonal functions are used, then the expansion coefficients which give the representation of the function (or data) in terms of the universal basis functions are obtained merely by inner product projection. (A normalization factor would also be required if the basis functions were orthogonal but not orthonormal.) If we used functions that were non-orthogonal, then a great deal more linear algebra would be required in order to obtain their desired coefficients.

We must compute the inner product or projection of $\Psi_k(x)$ and $f(x)$, which will give us each basis coefficient a_k:

$$a_k \;=\; < \Psi_k(x), f(x) > \;=\; \int_{-\infty}^{\infty} \Psi_k^*(x) f(x) dx$$

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Continuous Mathematics: Example Problem Set 6

1. Define Gabor wavelets in one dimension in three parameter form, and explain the meaning of each of the three parameters.

2. Explain the Gabor-Heisenberg-Weyl "Uncertainty Principle". Express it as an inequality and define the quantities involved. Explain what is special about Gabor wavelets in terms of this principle.

3. If you construct an Information Diagram whose axes are time and frequency, how small an area in this plane can be occupied? Illustrate several differently parameterized Gabor wavelets in this plane, and explain what property they all share. How will they compare with all other possible functions in this plane?

4. Explain in what sense Gabor wavelets unify both the time domain and the frequency domain, constructing a continuous deformation between the two. Explain how the Fourier Transform is just a special case of the Gabor Transform, and what price is paid for this unification in terms of the computational difficulty of obtaining the Gabor Transform coefficients.

Model Answer – Example Problem Set 6

1. Gabor wavelets are defined as the following 3-parameter functions:

$$f(x) = e^{-i\mu_0 x} e^{-(x-x_0)^2/a^2}$$

The meanings of the three parameters are: localization at "epoch" x_0, modulation by frequency μ_0, and wavelet size or spread constant a.

2. Gabor-Heisenberg-Weyl Uncertainty Relation.
If we define the "effective support" of a function $f(x)$ by its normalized variance, or the normalized second-moment

$$(\Delta x)^2 = \frac{\int_{-\infty}^{+\infty} f(x)f^*(x)(x-x_0)^2 dx}{\int_{-\infty}^{+\infty} f(x)f^*(x)dx} \tag{1}$$

where x_0 is the mean value, or first-moment, of the function

$$x_0 = \frac{\int_{-\infty}^{+\infty} xf(x)f^*(x)dx}{\int_{-\infty}^{+\infty} f(x)f^*(x)dx} \tag{2}$$

and if we similarly define the effective support of the Fourier Transform $F(\mu)$ of the function by its normalized variance in the Fourier domain

$$(\Delta\mu)^2 = \frac{\int_{-\infty}^{+\infty} F(\mu)F^*(\mu)(\mu-\mu_0)^2 d\mu}{\int_{-\infty}^{+\infty} F(\mu)F^*(\mu)d\mu} \tag{3}$$

where μ_0 is the mean value, or first-moment, of the Fourier transform $F(\mu)$

$$\mu_0 = \frac{\int_{-\infty}^{+\infty} \mu F(\mu)F^*(\mu)d\mu}{\int_{-\infty}^{+\infty} F(\mu)F^*(\mu)d\mu} \tag{4}$$

then it can be proven (by Schwartz Inequality arguments) that there exists a fundamental lower bound on the product of these two "spreads," regardless of the function $f(x)$

$$\boxed{(\Delta x)(\Delta\mu) \geq \frac{1}{4\pi}} \tag{5}$$

This is the Gabor-Heisenberg-Weyl Uncertainty Principle, and the unique family of signals that actually achieve its lower bound are the complex exponentials multiplied by Gaussians. These are the "Gabor wavelets."

3. The smallest possible area that any function can occupy in the Information Diagram is: $(\Delta x)(\Delta\mu) = \frac{1}{4\pi}$. Gabor wavelets, regardless of their parameter values, always achieve this lower bound on area. Their shapes may of course differ, as the diagram shows, but their jointly occupied areas are always equal. All other functions will occupy areas larger than this in the Information Diagram, and thus have less sharp information resolution.

4. To compute the representation of a signal or of data in the Gabor domain, we find its expansion in terms of elementary functions having the form

$$f(x) = e^{-i\mu_0 x} e^{-(x-x_0)^2/a^2} \tag{6}$$

The single parameter a (the space-constant in the Gaussian term) actually builds a continuous bridge between the time domain and the Fourier domain: if the parameter a is made very large, then the second exponential above approaches 1.0, and so in the limit our expansion basis becomes

$$\lim_{a \to \infty} f(x) = e^{-i\mu_0 x} \tag{7}$$

the ordinary Fourier basis. If the parameter a is instead made very small, the Gaussian term becomes the approximation to a delta function at location x_o, and so the expansion basis implements (if we set μ_0 to 0) pure space-domain sampling:

$$\lim_{\mu_0, a \to 0} f(x) = \delta(x - x_0) \tag{8}$$

Hence the Gabor expansion basis "contains" both domains at once. It allows us to make a continuous deformation that selects a representation lying anywhere on a one-parameter continuum between two domains that were hitherto distinct and mutually unapproachable. The price paid for this unification of the two domains is that the new expansion basis is, in general, non-orthogonal. This makes it much more difficult to obtain the coefficients for the expansion.

Continuous Mathematics: Example Problem Set 7

1. Show that for all families of functions which are "self-Fourier" (i.e. equivalent in functional form to their own Fourier transforms), closure of a family under multiplication entails also their closure under convolution, and *vice versa.*

(Hint: closure of a set of functions under an operation means that applying that operation to any member of the set creates a function which is also a member of the set.)

2. A periodic square wave, which alternates between the constants $+\pi/4$ and $-\pi/4$ with period 2π has the following Fourier series, using just all odd integers n:

$$f(x) = \sum_{\text{odd } n=1}^{\infty} \frac{1}{n} \sin(nx)$$

Derive from this the Fourier series for a periodic triangular wave, which ramps up and down with slopes $+\pi/4$ and $-\pi/4$ and with period 2π.

3. Any real-valued function $f(x)$ can be represented as the sum of one function $f_e(x)$ that has even symmetry (it is unchanged after being flipped around the origin $x = 0$) so that $f_e(x) = f_e(-x)$, plus one function $f_o(x)$ that has odd symmetry, so that $f_o(x) = -f_o(-x)$. Such a decomposition of any function $f(x)$ into $f_e(x) + f_o(x)$ is illustrated by

$$f_e(x) = \frac{1}{2}f(x) + \frac{1}{2}f(-x)$$

$$f_o(x) = \frac{1}{2}f(x) - \frac{1}{2}f(-x)$$

Use this type of decomposition to explain why the Fourier transform of any real-valued function has *Hermitian symmetry:* its real-part has even symmetry, and its imaginary-part has odd symmetry.

Comment on how this redundancy can be exploited to simplify computation of Fourier transforms of real-valued, as opposed to complex-valued, data.

Model Answer – Example Problem Set 7

1. The duality of convolution and multiplication implies that if two functions are convolved, the Fourier transform of their result is the product of their respective Fourier transforms:

$$h(x) = f(x) * g(x) \stackrel{FT}{\Longrightarrow} H(\mu) = F(\mu)G(\mu)$$

where * denotes convolution, and $F(\mu)$, $G(\mu)$, and $H(\mu)$ are the respective Fourier transforms of $f(x)$, $g(x)$, and $h(x)$.

Similarly, if two functions are multiplied together, the Fourier transform of their product is the convolution of their respective Fourier transforms:

$$h(x) = f(x)g(x) \stackrel{FT}{\Longrightarrow} H(\mu) = F(\mu) * G(\mu)$$

Closure of the family of functions under multiplication means that $h(x) = f(x)g(x)$ will have the same functional form as $f(x)$ and $g(x)$. Therefore $H(\mu) = F(\mu) * G(\mu)$ will also have the same functional form as $F(\mu)$ and $G(\mu)$. Since these functions are self-Fourier, it follows that closure under multiplication entails closure under convolution. The same argument can be applied in the reverse direction.

2. Since the square wave is just the first derivative of the triangular wave, we can expand the differentiation across all the terms in the expansion series and demand that the Fourier series for the square wave consists of term-wise derivatives of the Fourier series for the triangular wave. This tells us that the triangular wave Fourier series must be:

$$f(x) = \sum_{\text{odd } n=1}^{\infty} -\frac{1}{n^2}\cos(nx)$$

3. Additivity of Fourier transforms tells us that the FT of $f(x)$ equals the FT of $f_e(x)$ plus the FT of $f_o(x)$.

Since $f_e(x)$ has even symmetry, its FT is purely real (i.e. the imaginary part of all Fourier coefficients is 0), because the integral of the product of an even function times an odd function is 0. Similarly, the FT of $f_o(x)$ is purely imaginary, because (again) the integral of the product of any odd function times an even function is 0, so the real-part is 0.

Thus a real-valued function has an FT whose real part has even symmetry, and whose imaginary part has odd symmetry. This is Hermitian symmetry.

The computational advantage which can be exploited is that for real-valued data, we need only compute the Fourier transform over the positive frequencies. The coefficients we get will automatically tell us the corresponding coefficients for the negative frequencies. So we need only compute half as many numbers as we would have needed to compute if the data had been complex rather than real.

Continuous Mathematics: Example Problem Set 8

Consider the family of 1D Gabor wavelets, parameterized for location x_0, size α, and frequency μ_0. Their functional form is:

$$f(x) = e^{-i\mu_0 x} e^{-(x-x_0)^2/\alpha^2}$$

and their Fourier Transform is:

$$F(\mu) = e^{-ix_0\mu} e^{-(\mu-\mu_0)^2\alpha^2}$$

1. Are such families of functions "self-Fourier?" Why or why not?

2. Explain the dualities of: (i) modulation and shifting; and (ii) similarity (reciprocal scaling), in terms of the behaviour of the parameters in the expressions above for $f(x)$ and $F(\mu)$.

3. What can you say about the Fourier transform of the *sum* of any two Gabor wavelets?

4. What can you say about the Fourier transform of the *product* of any two Gabor wavelets, and why?

5. What is the Fourier transform of $f^{(n)}(x)$, the n^{th}-derivative of a Gabor wavelet?

6. Show that the set of all Gabor wavelets is closed under convolution: *i.e.*, that the convolution of any two Gabor wavelets is itself a single Gabor wavelet.

Model Answer – Example Problem Set 8

1. Member of this family of wavelets are self-Fourier, because they have exactly the same functional form as their Fourier Transforms. Just their parameters are interchanged or inverted in the functional form in the two domains.

2. The modulation parameter μ_0 in $f(x)$ becomes the location parameter of the wavelet $F(\mu)$ which is its Fourier transform. The position parameter x_0 of wavelet $f(x)$ becomes the modulation parameter of the wavelet $F(\mu)$ which is its Fourier transform. Hence modulation and shifting are dual. Finally, the scale parameter α sets the size of the wavelet $f(x)$, and the reciprocal of α is the size of the wavelet $F(\mu)$ which is the Fourier transform of $f(x)$; this expresses the duality of scaling ("similarity").

3. The Fourier transform of the sum of any two Gabor wavelets is itself the sum of two Gabor wavelets.

4. The Fourier transform of the product of any two Gabor wavelets is a single Gabor wavelet. This is because the product of any two Gabor wavelets is itself just a new Gabor wavelet (the arguments of the exponentials simply add within each exponential), and the conclusion then follows from (1).

5. The Fourier transform of $f^{(n)}(x)$, the n^{th}-derivative of a Gabor wavelet, is:

$$F(\mu) = (i\mu)^n e^{-ix_0\mu} e^{-(\mu-\mu_0)^2\alpha^2}$$

6. The duality of convolution and multiplication implies that the convolution of any two Gabor wavelets has a Fourier transform which is the product of two Gabor wavelets. But we noted above in (4) that such a product is just a single Gabor wavelet. Combining this result with the self-Fourier property, we conclude that the convolution of any two Gabor wavelets is itself, yet again, just a single Gabor wavelet. Hence this family of functions is closed under convolution, just as it is under multiplication.

Continuous Mathematics: Example Problem Set 9

(A)
When numerically computing the solution to an ordinary differential equation (ODE) that involves higher-than first-order derivatives,

(i) What is to be done about the higher-than first-order terms, and how can this be accomplished?

(ii) Illustrate this step for the following ODE, in which functions $r(x)$ and $q(x)$ are known and we seek to compute the solution $y(x)$:

$$\frac{d^2y}{dx^2} + q(x)\frac{dy}{dx} = r(x)$$

(B)
(i) State the incrementing rule for the Euler method of numerical integration, in terms of:

- $f(x_n)$, the estimate of the solution $f(x)$ at the current point x_n
- $f(x_{n+1})$, the new estimate of $f(x)$ for the next point x_{n+1}
- the integration stepsize h, which is the interval $(x_{n+1} - x_n)$
- $f'(x_n)$, the expression given by the ODE for the derivative of the desired solution $f(x)$ at the current point x_n

(ii) What might happen to your solution if the stepsize h is too large?

(iii) What might happen to your solution if the stepsize h is too small?

(iv) What is the primary advantage of the Runge-Kutta method over the Euler method for numerical integration of ODEs?

(v) Under what conditions might you wish to make the stepsize h adaptive rather than fixed? How should you adapt it?

Model Answer – Example Problem Set 9

(A1): When numerically computing the solution to an ODE that involves higher-than first-order derivatives, the higher-order terms must be reduced to 1st-order terms. This can be accomplished by introducing new substituted variables such as

$$z(x) = \frac{dy(x)}{dx}$$

which adds another equation into the coupled system to be solved, but now we have terms like $\frac{dz(x)}{dx}$ instead of $\frac{dy^2(x)}{dx^2}$.

(A2) Thus the example ODE,

$$\frac{d^2y}{dx^2} + q(x)\frac{dy}{dx} = r(x)$$

would be replaced by the following 1st-order system:

$$\frac{dy(x)}{dx} = z(x)$$

$$\frac{dz(x)}{dx} = r(x) - q(x)z(x)$$

(B1) The incrementing rule for the Euler method of numerical integration is:

$$f(x_{n+1}) = f(x_n) + hf'(x_n)$$

(B2) If the stepsize h is too large, this piecewise-linear approximation to the solution will be inaccurate where the true solution has significant second derivatives (which this method ignores). The solution may "blow up."

(B3) If the stepsize h is made too small, the computation may take too long and the round-off error (truncation of floating-point numbers depending on the word length of the machine) will accumulate too much, making the solution inaccurate or even unstable.

(B4) The primary advantage of the Runge-Kutta method over the Euler method for numerical integration of ODEs is that its accumulated error depends on the stepsize h as its 4th power, h^4, whereas in the Euler method the accumulated error is simply linear in h. Thus, for example, reducing the stepsize h by half would only reduce the accumulated error by half in the Euler method, but it would reduce it 16-fold in the Runge-Kutta method.

(B5) The stepsize h should be made adaptive if the solution varies greatly in its rates of change. When the rate of change is low, the stepsize can be made large, but when the rate of change is rapid, the stepsize should be made small. Thus h should be linked inversely to $f'(x_n)$.

Continuous Mathematics: Example Problem Set 10

(A)
Newton's definition of a derivative in his formulation of The Calculus only captured the notion of integer-order differentiation, *e.g.* the first or second derivative, etc. But in scientific computing we sometimes need a notion of fractional-order derivatives, as for example in fluid mechanics.

Explain how "Fractional Differentiation" (derivatives of non-integer order) can be given precise quantitative meaning through Fourier analysis.

Suppose that a continuous function $f(x)$ has Fourier Transform $F(\mu)$. Outline an algorithm (as a sequence of mathematical steps, not an actual program) for computing the 1.5^{th} derivative of some function $f(x)$

$$\frac{d^{(1.5)}f(x)}{dx^{(1.5)}}$$

(B)
For $i = \sqrt{-1}$, consider the quantity $\sqrt{i}$.

Express $\sqrt{i}$ as a complex exponential.
In which quadrant of the complex plane does it lie?
What is the real part of $\sqrt{i}$?
What is the imaginary part of $\sqrt{i}$?
What is the length (the modulus) of $\sqrt{i}$?

(C)
Initial-value problems described by ordinary differential equations have solutions that can be propagated forward using incrementing rules such as Euler or Runge-Kutta. But boundary-value problems specified by partial differential equations (PDEs) such as Poisson's Equation,

$$\frac{\partial^2 \mu(x,y)}{\partial x^2} + \frac{\partial^2 \mu(x,y)}{\partial y^2} = \rho(x,y)$$

cannot be solved by such propagation methods. Why not?

State the principle for one general class of numerical methods for solving such PDEs.

Model Answer – Example Problem Set 10

(A) By generalization of the Differentiation Theorem of Fourier Analysis, if a function $f(x)$ has Fourier Transform $F(\mu)$, then $f^{(\beta)}(x)$, the β^{th} derivative of $f(x)$ (where β is not necessarily an integer) has Fourier Transform: $(i\mu)^{\beta}F(\mu)$.

The inverse Fourier Transform of this expression then gives precise quantitative meaning to Fractional Differentiation.

Algorithm:

1. Go into the Fourier domain by taking the Fourier Transform of $f(x)$:
$$f(x) \stackrel{FT}{\Longrightarrow} F(\mu)$$
2. Multiply $F(\mu)$ by the desired β power of $(i\mu)$, in this case 1.5:
$$F(\mu) \stackrel{\beta}{\Longrightarrow} (i\mu)^{1.5}F(\mu)$$
which in this case amounts to a soft high-pass filtering plus a phase rotation.
3. Compute the inverse Fourier Transform of the result to obtain the desired new function:
$$(i\mu)^{1.5}F(\mu) \stackrel{FT^{-1}}{\Longrightarrow} \frac{d^{(1.5)}f(x)}{dx^{(1.5)}}$$

(B)
Complex exponential form: $\sqrt{i} = \exp(i\pi/4)$. (Negative root $\sqrt{i} = \exp(i5\pi/4)$ okay too.)
The positive root lies in the 1st quadrant (and the negative root in the 3rd).
The real part of $\sqrt{i}$ is $\sqrt{2}/2$ (or -$\sqrt{2}/2$ for the negative root).
The imaginary part of $\sqrt{i}$ is also $\sqrt{2}/2$ (or -$\sqrt{2}/2$ for the negative root).
The length of $\sqrt{i}$ is 1.

(C) For partial differential equations expressing boundary value problems, there is no unique "starting point" nor direction of solution propagation. Such equations require a solution to satisfy a constraint along a perimeter. Simply propagating a solution in different directions will not lead to a consistent solution over the whole domain.

One general class of numerical methods for solving PDEs such as Poisson's Equation are relaxation methods. They proceed iteratively, until successive changes become vanishingly small. Divide up the domain into finite elements; impose the boundary value constraint; calculate the differential forces across each pair of finite elements according to the PDEs; and iteratively relax their differences with decreasing stepsizes until a stable and consistent solution is reached over all the finite elements and the boundary values.