## Continuity of the fixpoint operator

Let $D$ be a domain.

By Tarski's Fixed Point Theorem we know that each continuous function $f \in (D \to D)$ possesses a least fixed point, $\mathit{fix}(f) \in D$.

**Proposition.** *The function*

$$\mathit{fix} : (D \to D) \to D$$

*is continuous.*

Proof just uses defining properties of fix — (lfp 1) & (lfp 2) rather than the explicit construction $\mathit{fix}(f) = \bigsqcup_{n \geq 0} f^n(\perp)$.

## Pre-fixed points

Let $D$ be a poset and $f : D \to D$ be a function.

An element $d \in D$ is a pre-fixed point of $f$ if it satisfies $f(d) \sqsubseteq d$.

The *least pre-fixed point* of $f$, if it exists, will be written

$$\boxed{fix(f)}$$

It is thus (uniquely) specified by the two properties:

$$f(fix(f)) \sqsubseteq fix(f) \qquad \text{(lfp1)}$$

$$\forall d \in D.\ f(d) \sqsubseteq d \ \Rightarrow\ fix(f) \sqsubseteq d. \qquad \text{(lfp2)}$$

$$\boxed{\mathrm{fix} : (D \to D) \to D}$$

is monotone: if $f \sqsubseteq f'$ in $D \to D$, then

$$f(\mathrm{fix}\, f') \sqsubseteq f'(\mathrm{fix}\, f') \sqsubseteq \mathrm{fix}\, f'$$

$$\text{fix} : (D \to D) \to D$$

is <u>monotone</u>: if $f \sqsubseteq f'$ in $D \to D$, then

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(lfp1)

so $\text{fix}\, f'$ is a pre-fixed point of $f$

So by (lfp2) $\quad \text{fix}\, f \sqsubseteq \text{fix}\, f'$

$$\mathrm{fix} : (D \to D) \to D$$

is <u>continuous</u> : given $f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \cdots$ in $D \to D$

want to show $\mathrm{fix}(\bigsqcup_{n\geq 0} f_n) \sqsubseteq \bigsqcup_{n\geq 0} \mathrm{fix}(f_n)$

By (lfp2), enough to show

$(\bigsqcup_{n\geq 0} f_n)(d) \sqsubseteq d$ for $d = \bigsqcup_{n\geq 0} \mathrm{fix}(f_n)$

$$\mathrm{fix} : (D \to D) \to D$$

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By (lfp2), enough to show

$$\left(\bigsqcup_{n\geq 0} f_n\right)(d) \sqsubseteq d \quad \text{for} \quad d = \bigsqcup_{n\geq 0} \mathrm{fix}(f_n)$$

But

$$\begin{aligned}
\left(\bigsqcup_{n\geq 0} f_n\right)(d) &= \left(\bigsqcup_{n\geq 0} f_n\right)\left(\bigsqcup_{m\geq 0} \mathrm{fix}(f_m)\right) \\
&= \bigsqcup_{n\geq 0} \bigsqcup_{m\geq 0} f_n(\mathrm{fix}(f_m)) \\
&= \bigsqcup_{k\geq 0} f_k(\mathrm{fix}(f_k)) \\
&\sqsubseteq \bigsqcup_{k\geq 0} \mathrm{fix}(f_k) \\
&= d
\end{aligned}$$

(lfp1) for each $f_k$

# *Topic 4*

Scott Induction

## Chain-closed and admissible subsets

Let $D$ be a cpo. A subset $S \subseteq D$ is called chain-closed iff for all chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ in $D$

$$(\forall n \geq 0 \,.\, d_n \in S) \;\Rightarrow\; \left( \bigsqcup_{n \geq 0} d_n \right) \in S$$

If $D$ is a domain, $S \subseteq D$ is called admissible iff it is a chain-closed subset of $D$ and $\perp \in S$.

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If $D$ is a domain, $S \subseteq D$ is called admissible iff it is a chain-closed subset of $D$ and $\bot \in S$.

A property $\Phi(d)$ of elements $d \in D$ is called *chain-closed* (resp. *admissible*) iff $\{d \in D \mid \Phi(d)\}$ is a *chain-closed* (resp. *admissible*) subset of $D$.

## Scott's Fixed Point Induction Principle

Let $f : D \to D$ be a continuous function on a domain $D$.

For any admissible subset $S \subseteq D$, to prove that the least fixed point of $f$ is in $S$, *i.e.* that

$$\mathit{fix}(f) \in S \ ,$$

it suffices to prove

$$\forall d \in D \ (d \in S \ \Rightarrow \ f(d) \in S) \ .$$

## Tarski's Fixed Point Theorem

Let $f : D \to D$ be a continuous function on a domain $D$. Then

- $f$ possesses a least pre-fixed point, given by

$$\mathit{fix}(f) = \bigsqcup_{n \geq 0} f^n(\bot).$$

- Moreover, $\mathit{fix}(f)$ is a fixed point of $f$, *i.e.* satisfies $f\big(\mathit{fix}(f)\big) = \mathit{fix}(f)$, and hence is the least fixed point of $f$.

where $\begin{cases} f^0(\bot) \triangleq \bot \\ f^{n+1}(\bot) \triangleq f(f^n(\bot)) \end{cases}$

# Proof of the Scott Induction Principle

If we know $\forall d \in D.\ d \in S \Rightarrow f(d) \in S$, then

$\perp \in S$ since $S$ is admissible

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$f^n(\perp) \in S$ for all $n \in \mathbb{N}$

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If we know $\forall d \in D.\ d \in S \Rightarrow f(d) \in S$, then

$\perp \in S$ since $S$ is admissible

so $f(\perp) \in S$

so $f(f(\perp)) \in S$

....

$f^n(\perp) \in S$ for all $n \in \mathbb{N}$

Hence $\bigsqcup_{n \geq 0} f^n(\perp) \in S$ since $S$ is admissible

that is, $\boxed{fix(f) \in S}$

Q.E.D.

# Example 4.2.1

Given $\begin{cases} \text{domain } D \\ \text{continuous function } f : D \times D \times D \to D \end{cases}$

then $\begin{cases} g : D \times D \to D \times D & \text{is continuous.} \\ g(d_1, d_2) = (f(d_1, d_1, d_2), f(d_1, d_2, d_2)) \end{cases}$

So by Tarski's FPT we get $\mathrm{fix}(g) \in D \times D$.

$\boxed{\text{Claim}: \quad u_1 = u_2, \text{ where } (u_1, u_2) = \mathrm{fix}(g)}$

Proof : by Scott Induction...

# Example 4.2.1

$$\begin{cases} g : D \times D \to D \times D \\ g(d_1, d_2) = (f(d_1, d_1, d_2), f(d_1, d_2, d_2)) \end{cases}$$

$\boxed{\text{Claim: } u_1 = u_2, \text{ where } (u_1, u_2) = fix(g)}$

<u>Proof</u> $\Delta \triangleq \{(d,d) \mid d \in D\}$ is an admissible subset of $D \times D$ because

- $(\perp, \perp) \in \Delta$
- $(d_0, d_0') \sqsubseteq (d_1, d_1') \sqsubseteq \cdots$ & $\forall n. (d_n, d_n') \in \Delta$ implies $\bigsqcup_{n \geq 0} (d_n, d_n') = (\bigsqcup_{n \geq 0} d_n, \bigsqcup_{n \geq 0} d_n') = (\bigsqcup_{n \geq 0} d_n, \bigsqcup_{n \geq 0} d_n) \in \Delta$

# Example 4.2.1

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$\boxed{\text{Claim: } u_1 = u_2 \text{, where } (u_1, u_2) = fix(g)}$

<u>Proof</u> $\Delta = \{(d, d) \mid d \in D\}$ admissible

and $\forall (d, d') \in D \times D.\ (d, d') \in \Delta \Rightarrow g(d, d') \in \Delta$

because

$$(d, d') \in \Delta \Rightarrow d = d'$$

$$\Rightarrow g(d, d') = (f(d, d, d), f(d, d, d)) \in \Delta$$

# Example 4.2.1

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<u>Proof</u> $\Delta = \{(d, d) \mid d \in D\}$ admissible

and $\forall (d, d') \in D \times D.\ (d, d') \in \Delta \Rightarrow g(d, d') \in \Delta$

So by Scott Induction

$fix(g) \in \Delta$

QED

## Example (III): Partial correctness

Let $\mathcal{F} : State \rightharpoonup State$ be the denotation of

$$\textbf{while } X > 0 \textbf{ do } (Y := X * Y; X := X - 1) \ .$$

For all $x, y \geq 0$,

$$\mathcal{F}[X \mapsto x, Y \mapsto y] \downarrow$$
$$\implies \mathcal{F}[X \mapsto x, Y \mapsto y] = [X \mapsto 0, Y \mapsto !x \cdot y].$$

Recall that $\mathcal{F} = \mathrm{fix}(f)$ where

$$f : (\mathrm{State} \rightharpoonup \mathrm{State}) \to (\mathrm{State} \rightharpoonup \mathrm{State})$$

is given by

$$\boxed{f(w)[X \mapsto x, Y \mapsto y] = \begin{cases} [X \mapsto x, Y \mapsto y] & \text{if } x \leq 0 \\ \\ w[X \mapsto x-1, Y \mapsto xy] & \text{if } x > 0 \end{cases}}$$

for all $w \in \mathrm{State} \rightharpoonup \mathrm{State}$ & $x, y \in \mathbb{Z}$

*Proof by Scott induction.*

We consider the admissible subset of $(State \rightharpoonup State)$ given by

$$S = \left\{ w \;\middle|\; \begin{array}{l} \forall x, y \geq 0. \\ \quad w[X \mapsto x, Y \mapsto y]\!\downarrow \\ \qquad \Rightarrow w[X \mapsto x, Y \mapsto y] = [X \mapsto 0, Y \mapsto !x \cdot y] \end{array} \right\}$$

and show that

$$w \in S \implies f(w) \in S \; .$$

From now on, let's just write $[X \mapsto x, Y \mapsto y]$
as $(x, y)$
(i.e. identify State with $\mathbb{Z} \times \mathbb{Z}$)

Suppose $w \in S$. Want to show $f(w) \in S$, i.e.

$$x, y \geq 0 \ \& \ f(w)(x,y)\downarrow \Rightarrow f(w)(x,y) = (0, !x \cdot y)$$

So suppose $x, y \geq 0 \ \& \ f(w)(x,y)\downarrow$

Case $x = 0$ :

$$f(w)(x,y) \underset{\substack{\uparrow \\ \text{by def.}\\ \text{of } f}}{=} (x,y) \underset{\substack{\uparrow \\ \text{Since}\\ x=0}}{=} (0,y) \underset{\substack{\uparrow \\ \text{Since}\\ 0!=1}}{=} (0, !0 \cdot y) \underset{\substack{\uparrow \\ \text{since}\\ x=0}}{=} (0, !x \cdot y) \quad \checkmark$$

Suppose $w \in S$. Want to show $f(w) \in S$, i.e.

$x, y \geq 0$ & $f(w)(x,y)\downarrow \Rightarrow f(w)(x,y) = (0, !x \cdot y)$

So suppose $x, y \geq 0$ & $f(w)(x,y)\downarrow$

Case $x > 0$ : Since $f(w)(x,y)\downarrow$

get $w(x-1, x \cdot y)\downarrow$ by definition of $f$

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Case $x > 0$ : Since $f(w)(x,y)\downarrow$

get $w(x-1, x \cdot y)\downarrow$ by definition of f

But $x-1, x \cdot y \geq 0$ & $w \in S$

so $w(x-1, x\cdot y) = (0, !(x-1)\cdot(x \cdot y))$ by def^n of S

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But $x-1, x \cdot y \geq 0$ & $w \in S$

so $w(x-1, x \cdot y) = (0, !(x-1) \cdot (x \cdot y))$ by def^n of S

$$= (0, !x \cdot y)$$

so $f(w)(x,y) = w(x-1, x \cdot y) = (0, !x \cdot y)$ ✓

↑ def^n of f

## Building chain-closed subsets (I)

Let $D, E$ be cpos.

**Basic relations:**

- For every $d \in D$, the subset

$$\downarrow(d) \stackrel{\text{def}}{=} \{ x \in D \mid x \sqsubseteq d \}$$

  of $D$ is chain-closed.

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**Basic relations:**

- For every $d \in D$, the subset

$$\downarrow(d) \stackrel{\text{def}}{=} \{\, x \in D \mid x \sqsubseteq d \,\}$$

  of $D$ is chain-closed.

- The subsets

$$\text{and} \quad \begin{array}{l} \{(x, y) \in D \times D \mid x \sqsubseteq y\} \\ \{(x, y) \in D \times D \mid x = y\} \end{array}$$

  of $D \times D$ are chain-closed.

## Building chain-closed subsets (II)

**Inverse image:**

Let $f : D \to E$ be a continuous function.

If $S$ is a chain-closed subset of $E$ then the inverse image

$$f^{-1}S = \{x \in D \mid f(x) \in S\}$$

is a chain-closed subset of $D$.

## Building chain-closed subsets (II)

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Proof: if $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ in $D$ & $\forall n.\ d_n \in f^{-1}S$

then $\forall n.\ f(d_n) \in S$, so $\bigsqcup_{n \geq 0} f(d_n) \in S$ ('cos $S$ ch.-cl.)

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Proof: if $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ in $D$ & $\forall n.\ d_n \in f^{-1}S$

then $\forall n.\ f(d_n) \in S$, so $\bigsqcup_{n \geq 0} f(d_n) \in S$ ('cos $S$ ch.-cl.)

so $f(\bigsqcup_{n \geq 0} d_n) \in S$ ('cos $f$ cts.)

so $\bigsqcup_{n \geq 0} d_n \in f^{-1}S$

## Example (II)

Let $D$ be a domain and let $f, g : D \to D$ be continuous functions such that $f \circ g \sqsubseteq g \circ f$. Then,

$$f(\perp) \sqsubseteq g(\perp) \implies \mathit{fix}(f) \sqsubseteq \mathit{fix}(g) \ .$$

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Let $D$ be a domain and let $f, g : D \to D$ be continuous functions such that $f \circ g \sqsubseteq g \circ f$. Then,

$$f(\bot) \sqsubseteq g(\bot) \implies \mathit{fix}(f) \sqsubseteq \mathit{fix}(g) \ .$$

*Proof by Scott induction.*

Consider the admissible property $\Phi(x) \equiv \big(f(x) \sqsubseteq g(x)\big)$ of $D$.

Since

$$f(x) \sqsubseteq g(x) \Rightarrow g(f(x)) \sqsubseteq g(g(x)) \Rightarrow f(g(x)) \sqsubseteq g(g(x))$$

we have that

$$f(\mathit{fix}(g)) \sqsubseteq g(\mathit{fix}(g)) \ .$$

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Since

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we have that

$$f(\mathit{fix}(g)) \sqsubseteq g(\mathit{fix}(g)) \sqsubseteq fix(g) \text{ by (lfp1) for } g$$

so by (lfp2) for $f$, we have

$$fix(f) \sqsubseteq fix(g)$$

Q.E.D.

# Building chain-closed subsets (III)

**Logical operations:**

- If $S, T \subseteq D$ are chain-closed subsets of $D$ then

  $$S \cup T \qquad \text{and} \qquad S \cap T$$

  are chain-closed subsets of $D$.

- If $\{ S_i \}_{i \in I}$ is a family of chain-closed subsets of $D$ indexed by a set $I$, then $\bigcap_{i \in I} S_i$ is a chain-closed subset of $D$.

- If a property $P(x, y)$ determines a chain-closed subset of $D \times E$, then the property $\forall x \in D.\ P(x, y)$ determines a chain-closed subset of $E$.

# $S, T$ chain-closed $\Rightarrow S \cup T$ chain-closed

Suppose $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \cdots$ in $D$ & $\forall n.\ d_n \in S \cup T$

If $\bigsqcup_{n \geq 0} d_n \in S$, we are done.

So suppose $\bigsqcup_{n \geq 0} d_n \notin S$

For each $m \geq 0$,

$$(\forall n \geq m.\ d_n \in S) \Rightarrow \bigsqcup_{n \geq 0} d_n = \bigsqcup_{n \geq m} d_n \in S$$

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So $\neg(\forall n \geq m.\ d_n \in S)$

i.e. $\exists n \geq m.\ d_n \in T$ since

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# S, T chain-closed ⇒ S ∪ T chain-closed

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If $\bigsqcup_{n \geq 0} d_n \in S$, we are done.

So suppose $\bigsqcup_{n \geq 0} d_n \notin S$

For each $m \geq 0$, $\exists n \geq m.\ d_n \in T$

So we can choose $n_0 \leq n_1 \leq n_2 \leq \cdots$ satisfying

$\forall m.\ m \leq n_m$ & $d_{n_m} \in T$.

So $\bigsqcup_{n \geq 0} d_n = \bigsqcup_{m \geq 0} d_{n_m} \in T$

Q.E.D.

## Building chain-closed subsets (III)

**Logical operations:**

- If $S, T \subseteq D$ are chain-closed subsets of $D$ then

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- If $\{ S_i \}_{i \in I}$ is a family of chain-closed subsets of $D$ indexed by a set $I$, then $\bigcap_{i \in I} S_i$ is a chain-closed subset of $D$.

$= \{ d \in D \mid \forall i.\ d \in S_i \}$

- If a property $P(x, y)$ determines a chain-closed subset of $D \times E$, then the property $\forall x \in D.\ P(x, y)$ determines a chain-closed subset of $E$.

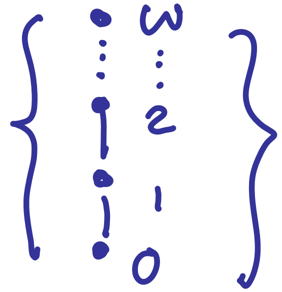
N.B. in general $\bigcup_{i \in I} S_i = \{ d \mid \exists i.\ d \in S_i \}$ & $D - S$ need not be chain-closed.

for each $i \in \mathbb{N}$

$S_i = \{0,1,2,\dots,i\}$ is chain-closed subset of the domain $\Omega = \left\{ \begin{array}{c} \omega \\ \vdots \\ 2 \\ 1 \\ 0 \end{array} \right\}$

but

$\bigcup_{i \in \mathbb{N}} S_i = \mathbb{N}$ is not a chain-closed subset of the domain $\Omega = \left\{ \begin{array}{c} \omega \\ \vdots \\ 2 \\ 1 \\ 0 \end{array} \right\}$

$S = \{0,2,4,\ldots\} \cup \{\omega\}$ is chain-closed subset of the domain $\Omega =$ 


but

$D - S = \{1,3,5,\ldots\}$

is <u>not</u> a chain-closed subset of the domain $\Omega =$