Cpo's and domains

A chain complete poset, or cpo for short, is a poset $(D, \sqsubseteq)$ in which all countable increasing chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ have least upper bounds, $\bigsqcup_{n \geq 0} d_n$:

$$\forall m \geq 0 \,.\, d_m \sqsubseteq \bigsqcup_{n \geq 0} d_n \tag{lub1}$$

$$\forall d \in D \,.\, (\forall m \geq 0 \,.\, d_m \sqsubseteq d) \;\Rightarrow\; \bigsqcup_{n \geq 0} d_n \sqsubseteq d. \tag{lub2}$$

A domain is a cpo that possesses a least element, $\perp$:

$$\forall d \in D \,.\, \perp \sqsubseteq d.$$

"lub" = least upper bound

Continuity and strictness

- If D and E are cpo's, the function f is continuous iff

  1. it is monotone, and

  2. it preserves lubs of chains, *i.e.* for all chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$ in D, it is the case that

$$f(\bigsqcup_{n \geq 0} d_n) = \bigsqcup_{n \geq 0} f(d_n) \quad \text{in } E.$$

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NB
$$f(d_0) \subseteq f(d_1) \subseteq f(d_2) \subseteq \ldots$$
'cos f monotone

NB $\forall i.\ d_i \sqsubseteq \bigsqcup_{n \geq 0} d_n \xRightarrow{\text{monotonicity}} \forall i.\ f(d_i) \sqsubseteq f(\bigsqcup_{n \geq 0} d_n)$

$\Longrightarrow \bigsqcup_{i \geq 0} f(d_i) \sqsubseteq f(\bigsqcup_{n \geq 0} d_n)$

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$\boxed{\text{So given 1, for 2 just need } f(\bigsqcup_{n\geq 0} d_n) \sqsubseteq \bigsqcup_{n\geq 0} f(d_n)}$

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- If D and E have least elements, then the function f is strict iff $f(\perp) = \perp$.

Tarski's Fixed Point Theorem

Let $f : D \to D$ be a continuous function on a domain D. Then

- f possesses a least pre-fixed point, given by

$$fix(f) = \bigsqcup_{n \geq 0} f^n(\perp).$$

- Moreover, $fix(f)$ is a fixed point of f, *i.e.* satisfies $f\big(fix(f)\big) = fix(f)$, and hence is the least fixed point of f.

where $\begin{cases} f^0(\perp) \triangleq \perp \\ f^{n+1}(\perp) \triangleq f(f^n(\perp)) \end{cases}$

Pre-fixed points

Let D be a poset and $f : D \to D$ be a function.

An element $d \in D$ is a pre-fixed point of f if it satisfies $f(d) \sqsubseteq d$.

The *least* *pre-fixed point* of f, if it exists, will be written

$$\boxed{\mathit{fix}(f)}$$

It is thus (uniquely) specified by the two properties:

$$f(\mathit{fix}(f)) \sqsubseteq \mathit{fix}(f) \tag{lfp1}$$

$$\forall d \in D.\ f(d) \sqsubseteq d \ \Rightarrow\ \mathit{fix}(f) \sqsubseteq d. \tag{lfp2}$$

Proof of Tarski's Theorem

$\perp \sqsubseteq f(\perp)$ because $\perp$ is least elt of D

Proof of Tarski's Theorem

$\bot \sqsubseteq f(\bot)$ because $\bot$ is least elt of D

so $f(\bot) \sqsubseteq f(f(\bot)) \triangleq f^2(\bot)$ by monotonicity of f

so $f^2(\bot) \sqsubseteq f(f^2(\bot)) = f^3(\bot)$

etc.

Proof of Tarski's Theorem

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etc.

We get a chain $\perp \sqsubseteq f(\perp) \sqsubseteq f^2(\perp) \sqsubseteq f^3(\perp) \sqsubseteq \cdots$
and can form its lub $\bigsqcup_{n \geq 0} f^n(\perp)$

Proof of Tarski's Theorem

Applying f to $\perp \sqsubseteq f(\perp) \sqsubseteq f^2(\perp) \sqsubseteq \cdots \sqsubseteq \bigsqcup_{n\geq 0} f^n(\perp)$

we get $\underbrace{f(\perp) \sqsubseteq f(f(\perp)) \sqsubseteq f(f^2(\perp)) \sqsubseteq \cdots \sqsubseteq f\left(\bigsqcup_{n\geq 0} f^n(\perp)\right)}_{\text{by monotonicity of } f}$

Proof of Tarski's Theorem

Applying f to $\bot \sqsubseteq f(\bot) \sqsubseteq f^2(\bot) \sqsubseteq \cdots \sqsubseteq \bigsqcup_{n\geq 0} f^n(\bot)$

we get $f(\bot) \sqsubseteq f(f(\bot)) \sqsubseteq f(f^2(\bot)) \sqsubseteq \cdots \sqsubseteq f(\bigsqcup_{n\geq 0} f^n(\bot))$

by continuity of f

$$\| $$

$$\bigsqcup_{n\geq 0} f(f^n(\bot))$$

$$\| $$

$$\bigsqcup_{n\geq 0} f^{n+1}(\bot)$$

$$\| $$

$$\bigsqcup_{m\geq 1} f^m(\bot)$$

Proof of Tarski's Theorem

$\boxed{\text{So } \bigsqcup_{n\geq 0} f^n(\bot) \text{ is a (pre-)fixed point for } f}$

$$f\left(\bigsqcup_{n\geq 0} f^n(\bot)\right)$$

$$\|$$

$$\bigsqcup_{n\geq 0} f(f^n(\bot))$$

$$\|$$

$$\bigsqcup_{n\geq 0} f^{n+1}(\bot)$$

$$\|$$

$$\bigsqcup_{m\geq 0} f^m(\bot) = \bigsqcup_{m\geq 1} f^m(\bot)$$

Proof of Tarski's Theorem

For any pre-fixed point $f(d) \sqsubseteq d$ we have

$\perp \sqsubseteq d$ because $\perp$ is least elt of D

Proof of Tarski's Theorem

For any pre-fixed point $f(d) \sqsubseteq d$ we have

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etc.

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etc.

We get $f^n(\perp) \sqsubseteq d$ for all $n \geq 0$

So $\bigsqcup_{n \geq 0} f^n(\perp) \sqsubseteq d$

Proof of Tarski's Theorem

For any pre-fixed point $f(d) \sqsubseteq d$ we have

$\boxed{\text{So } \bigsqcup_{n \geq 0} f^n(\bot) \text{ is a \underline{least} pre-fixed point}}$

QED

We get

$$\bigsqcup_{n \geq 0} f^n(\bot) \sqsubseteq d$$

Fixed point property of $[\![\textbf{while } B \textbf{ do } C]\!]$

$$[\![\textbf{while } B \textbf{ do } C]\!] = f_{[\![B]\!],[\![C]\!]}([\![\textbf{while } B \textbf{ do } C]\!])$$

where, for each $b : State \to \{true, false\}$ and $c : State \rightharpoonup State$, we define

we now know this is a domain

as $$f_{b,c} : (State \rightharpoonup State) \to (State \rightharpoonup State)$$

$$f_{b,c} = \lambda w \in (State \rightharpoonup State).\, \lambda s \in State.$$
$$if\,\big(b(s), w(c(s)), s\big).$$

- Why does $w = f_{[\![B]\!],[\![C]\!]}(w)$ have a solution?
- What if it has several solutions—which one do we take to be $[\![\textbf{while } B \textbf{ do } C]\!]$?

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- Why does $w = f_{[\![B]\!],[\![C]\!]}(w)$ have a solution? Tarski's Theorem (need to show $f_{b,c}$ is continuous)
- What if it has several solutions—which one do we take to be $[\![\mathbf{while}\ B\ \mathbf{do}\ C]\!]$? least (pre-) fixed point

Continuity of $f_{b,c}$

Suppose $c_0 \subseteq c_1 \subseteq c_2 \subseteq \cdots$ in $State \rightharpoonup State$

$$f_{b,c}\left(\bigcup_{n\geq 0} c_n\right) = \lambda s \in State.\ if\left(b(s), \left(\bigcup_{n\geq 0} c_n\right)(c(s)), s\right)$$

that is

$$f_{b,c}\left(\bigcup_{n\geq 0} c_n\right) = \left\{ (s,s') \;\middle|\; \begin{array}{l} b(s) = true \wedge \exists s''.\ c(s) = s'' \wedge \\ \qquad\qquad \left(\bigcup_{n\geq 0} c_n\right)(s'') = s' \\ \vee \\ b(s) = false \wedge s = s' \end{array} \right\}$$

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$$= \bigcup_{n\geq 0} \{(s,s') \mid if(b(s), c_n(c(s)), s) = s'\}$$

$$= \bigcup_{n\geq 0} f_{b,c}(c_n)$$

QED

$[\![\textbf{while } B \textbf{ do } C]\!]$

$[\![\textbf{while } B \textbf{ do } C]\!]$

$= \mathit{fix}(f_{[\![B]\!],[\![C]\!]})$ Tarski Theorem

$\doteq \bigsqcup_{n \geq 0} {f_{[\![B]\!],[\![C]\!]}}^{n}(\bot)$ requires proof...

$\doteq \lambda s \in \mathit{State}.$

$$\begin{cases} [\![C]\!]^k(s) & \text{if } k \geq 0 \text{ is such that } [\![B]\!]([\![C]\!]^k(s)) = \mathit{false} \\ & \text{and } [\![B]\!]([\![C]\!]^i(s)) = \mathit{true} \text{ for all } 0 \leq i < k \\ \text{undefined} & \text{if } [\![B]\!]([\![C]\!]^i(s)) = \mathit{true} \text{ for all } i \geq 0 \end{cases}$$

Example

Domain $D = (\mathcal{P}(\mathbb{N}), \subseteq)$ (same as $\mathbb{N} \to 1$)

Function $f : D \to D$

$$\boxed{f(S) \triangleq \{0\} \cup \{x+2 \mid x \in S\}}$$

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$S \in D$ is a pre-fixed point of f if

$$f(S) \subseteq S$$

i.e. $0 \in S$ & $x+2 \in S$ for all $x \in S$

i.e. S is closed under the rules $\dfrac{}{0 \in S}$ & $\dfrac{x \in S}{x+2 \in S}$

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$$\boxed{\text{So expect least pre-fixed point of } f \text{ to be } \mathit{Even} = \{2x \mid x \in \mathbb{N}\}}$$

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f is montone : $S \subseteq S' \Rightarrow f(S) \subseteq f(S')$ ✓

f is continuous : $f(\bigcup_{n\geq 0} S_n) = \{0\} \cup \{x+2 \mid x \in \bigcup_{n\geq 0} S_n\}$

$= \{0\} \cup \bigcup_{n\geq 0} \{x+2 \mid x \in S_n\}$

$= \bigcup_{n\geq 0} f(S_n)$ ✓

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Tarski Theorem applies:

$$fix(f) = \bigcup_{n \geq 0} f^n(\emptyset)$$

$$f(\emptyset) = \{0\}$$
$$f^2(\emptyset) = \{0\} \cup \{0+2\}$$
$$f^3(\emptyset) = \{0, 2, 4\}$$
$$\vdots$$
$$f^n(\emptyset) = \{0, 2, 4, \ldots 2(n-1)\}$$

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Tarski Theorem applies:

$$fix(f) = \bigcup_{n \geq 0} f^n(\emptyset) = \{0, 2, 4, 8, \ldots\}$$

$$= \{2x \mid x \in \mathbb{N}\}$$

(as expected).

$f(\emptyset) = \{0\}$

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$\vdots$

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