Least Fixed Points

Thesis

All domains of computation are partial orders with a least element.

All computable functions are mononotic.

"domain theory" = mathematics underpinning denotational semantics of PLs

$$\frac{}{x \sqsubseteq x}$$ reflexive

$$\frac{x \sqsubseteq y \qquad y \sqsubseteq z}{x \sqsubseteq z}$$ transitive

$$\frac{x \sqsubseteq y \qquad y \sqsubseteq x}{x = y}$$ anti-symmetric

Partially ordered sets

A binary relation $\sqsubseteq$ on a set D is a partial order iff it is

reflexive: $\forall d \in D.\ d \sqsubseteq d$

transitive: $\forall d, d', d'' \in D.\ d \sqsubseteq d' \sqsubseteq d'' \Rightarrow d \sqsubseteq d''$

anti-symmetric: $\forall d, d' \in D.\ d \sqsubseteq d' \sqsubseteq d \Rightarrow d = d'$.

Such a pair $(D, \sqsubseteq)$ is called a partially ordered set, or poset.

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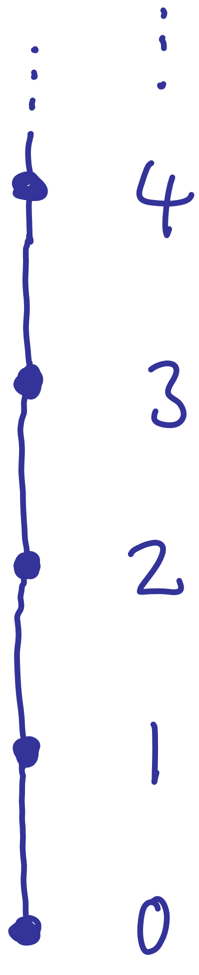
NB we often refer to "$(D, \sqsubseteq)$" just as "D", leaving $\sqsubseteq$ implicit.

Examples of posets

$\mathbb{N} = \{0, 1, 2, 3, \ldots\} + \leqslant$



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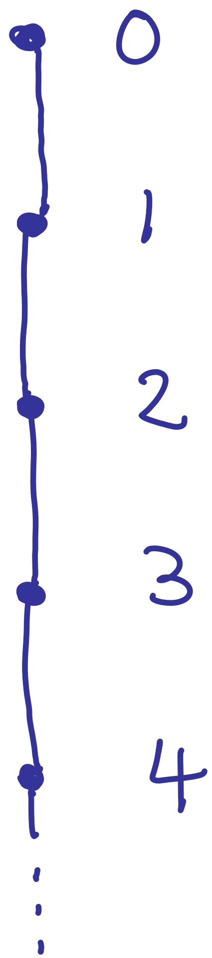

Examples of posets

Given a set X,
powerset $PX = \{S \mid S \subseteq X\}$ (all subsets of X)
+ $\subseteq$ (subset inclusion)

E.g. when $X = \{0,1\}$, $(PX, \subseteq)$ looks like:

$$\begin{tikzcd}[column sep=small]
 & \{0,1\} \arrow[dl, no head] \arrow[dr, no head] & \\
\{0\} \arrow[dr, no head] & & \{1\} \arrow[dl, no head] \\
 & \varnothing &
\end{tikzcd}$$

Domain of partial functions, $X \rightharpoonup Y$

Underlying set: all partial functions, f, with domain of definition $dom(f) \subseteq X$ and taking values in Y.

Partial functions

Notation:

- "$f(x) = y$" means $(x, y) \in f$
- "$f(x)\downarrow$" means $\exists y \in Y \, (f(x) = y)$
- "$f(x)\uparrow$" means $\neg\exists y \in Y \, (f(x) = y)$ "f(x) is undefined"
- $X \rightharpoonup Y$ = set of all partial functions from X to Y

Definition. A partial function from a set X to a set Y is specified by any subset $f \subseteq X \times Y$ satisfying

$$(x, y) \in f \wedge (x, y') \in f \rightarrow y = y'$$

for all $x \in X$ and $y, y' \in Y$.

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- $\mathrm{dom}(f) = \{x \in X \mid f(x)\downarrow\}$

  $\mathrm{graph}(f) = \{(x,y) \in X \times Y \mid f(x) = y\} = f$

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& \forall x \in dom(f).\ f(x) = g(x) \\
\text{iff} \quad & graph(f) \subseteq graph(g) \\
\text{iff} \quad & f \subseteq g
\end{aligned}
$$

(we identify partial functions with their graphs)

Monotonicity

- A function $f : D \to E$ between posets is monotone iff

$$\forall d, d' \in D.\ d \sqsubseteq d' \Rightarrow f(d) \sqsubseteq f(d').$$

$$\frac{x \sqsubseteq y}{f(x) \sqsubseteq f(y)} \quad (f \text{ monotone})$$

Least Elements

Suppose that D is a poset and that S is a subset of D.

An element $d \in S$ is the *least* element of S if it satisfies

$$\forall x \in S.\ d \sqsubseteq x \ .$$

- Note that because $\sqsubseteq$ is anti-symmetric, S has at most one least element.
- Note also that a poset may not have least element.

Pre-fixed points

Let D be a poset and $f : D \to D$ be a function.

An element $d \in D$ is a pre-fixed point of f if it satisfies $f(d) \sqsubseteq d$.

The *least pre-fixed point* of f, if it exists, will be written

$$\boxed{\mathit{fix}(f)}$$

It is thus (uniquely) specified by the two properties:

$$f(\mathit{fix}(f)) \sqsubseteq \mathit{fix}(f) \qquad \text{(lfp1)}$$

$$\forall d \in D.\ f(d) \sqsubseteq d \ \Rightarrow\ \mathit{fix}(f) \sqsubseteq d. \qquad \text{(lfp2)}$$

Proof principle

2. Let D be a poset and let $f : D \to D$ be a function with a least pre-fixed point $\mathit{fix}(f) \in D$.

   For all $x \in D$, to prove that $\mathit{fix}(f) \sqsubseteq x$ it is enough to establish that $f(x) \sqsubseteq x$.

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$$\frac{f(x) \sqsubseteq x}{\mathit{fix}(f) \sqsubseteq x}$$

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Least pre-fixed points are fixed points

If it exists, the least pre-fixed point of a monotone function on a partial order is necessarily a fixed point.

i.e. $fix(f)$ satisfies $f(fix(f)) = fix(f)$

$f(fix(f)) \sqsubseteq fix(f)$ by (lfp1)

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anti-symmetry

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i.e. $\mathit{fix}(f)$ satisfies $f(\mathit{fix}(f)) = \mathit{fix}(f)$

eg. $\lambda x . x+1 : \mathbb{N} \to \mathbb{N}$ is monotone (for $\leq$)

but has no (least) fixed point

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i.e. $fix(f)$ satisfies $f(fix(f)) = fix(f)$

$$\boxed{\text{Seek a notion of "domain" where least fixed points always exist for "good" functions}}$$

Cpo's and domains

A chain complete poset, or cpo for short, is a poset $(D, \sqsubseteq)$ in which all countable increasing chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ have least upper bounds, $\bigsqcup_{n\geq 0} d_n$:

$$\forall m \geq 0 \,.\, d_m \sqsubseteq \bigsqcup_{n\geq 0} d_n \tag{lub1}$$

$$\forall d \in D \,.\, (\forall m \geq 0 \,.\, d_m \sqsubseteq d) \;\Rightarrow\; \bigsqcup_{n\geq 0} d_n \sqsubseteq d. \tag{lub2}$$

A domain is a cpo that possesses a least element, $\bot$:

$$\forall d \in D \,.\, \bot \sqsubseteq d.$$

$$\frac{}{\perp \sqsubseteq x}$$

$$\frac{}{x_i \sqsubseteq \bigsqcup_{n \geq 0} x_n} \quad (i \geq 0 \text{ and } \langle x_n \rangle \text{ a chain})$$

lub 1

$$\frac{\forall n \geq 0 \,.\, x_n \sqsubseteq x}{\bigsqcup_{n \geq 0} x_n \sqsubseteq x} \quad (\langle x_i \rangle \text{ a chain})$$

lub 2

Thesis*

All domains of computation are
complete partial orders with a least element.

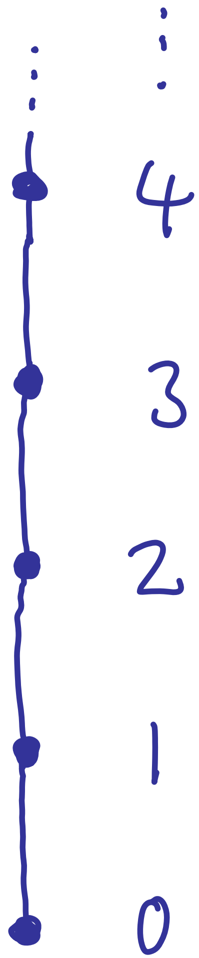
All computable functions are
continuous.

next lecture

(guarantees that least fixed points always exist)

Non-Example of cpo

$\mathbb{N} = \{0, 1, 2, 3, \ldots\} + \leqslant$

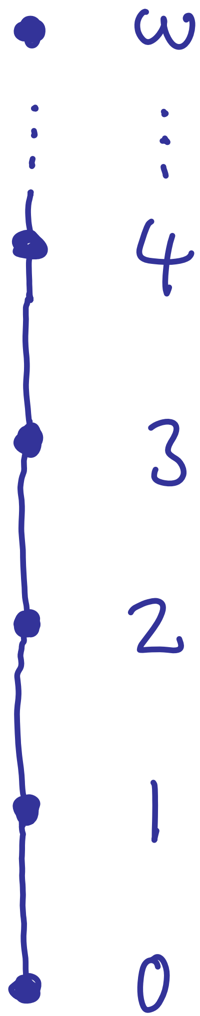


$0 \sqsubseteq 1 \sqsubseteq 2 \sqsubseteq 3 \sqsubseteq \cdots$

has no upper bound in $\mathbb{N}$

Example of cpo

$$\Omega = \{0, 1, 2, 3, \ldots\} \cup \{\omega\}$$



$(n \sqsubseteq \omega, \text{ all } n \in \mathbb{N})$

why does every chain in Ω have a lub?

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Lub of chain $f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \ldots$ is the partial function f with $dom(f) = \bigcup_{n \geq 0} dom(f_n)$ and

$$
f(x) = \begin{cases} f_n(x) & \text{if } x \in dom(f_n), \text{ some } n \\ \text{undefined} & \text{otherwise} \end{cases}
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Least element $\perp$ is the totally undefined partial function.

Some properties of lubs of chains

Let D be a cpo.

1. For $d \in D$, $\bigsqcup_n d = d$.
2. For every chain $d_0 \sqsubseteq d_1 \sqsubseteq \ldots \sqsubseteq d_n \sqsubseteq \ldots$ in D,

$$\bigsqcup_n d_n = \bigsqcup_n d_{N+n}$$

   for all $N \in \mathbb{N}$.

3. For every pair of chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots \sqsubseteq d_n \sqsubseteq \ldots$ and $e_0 \sqsubseteq e_1 \sqsubseteq \ldots \sqsubseteq e_n \sqsubseteq \ldots$ in D,

   if $d_n \sqsubseteq e_n$ for all $n \in \mathbb{N}$ then $\bigsqcup_n d_n \sqsubseteq \bigsqcup_n e_n$.

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if $d_n \sqsubseteq e_n$ for all $n \in \mathbb{N}$ then $\bigsqcup_n d_n \sqsubseteq \bigsqcup_n e_n$.

$$\frac{\forall n \geq 0 \,.\, x_n \sqsubseteq y_n}{\bigsqcup_n x_n \sqsubseteq \bigsqcup_n y_n} \quad (\langle x_n \rangle \text{ and } \langle y_n \rangle \text{ chains})$$

Diagonalising a double chain

Lemma. *Let D be a cpo. Suppose that the doubly-indexed family of elements $d_{m,n} \in D$ ($m, n \geq 0$) satisfies*

$$m \leq m' \;\&\; n \leq n' \;\Rightarrow\; d_{m,n} \sqsubseteq d_{m',n'}. \qquad (\dagger)$$

Then

$$\bigsqcup_{n \geq 0} d_{0,n} \;\sqsubseteq\; \bigsqcup_{n \geq 0} d_{1,n} \;\sqsubseteq\; \bigsqcup_{n \geq 0} d_{2,n} \;\sqsubseteq\; \ldots$$

and

$$\bigsqcup_{m \geq 0} d_{m,0} \;\sqsubseteq\; \bigsqcup_{m \geq 0} d_{m,1} \;\sqsubseteq\; \bigsqcup_{m \geq 0} d_{m,3} \;\sqsubseteq\; \ldots$$

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Moreover

$$\bigsqcup_{m \geq 0} \left(\bigsqcup_{n \geq 0} d_{m,n} \right) = \bigsqcup_{k \geq 0} d_{k,k} = \bigsqcup_{n \geq 0} \left(\bigsqcup_{m \geq 0} d_{m,n} \right) .$$