Lambda-Definable Functions

Encoding data in λ-calculus

Computation in λ-calculus is given by β-reduction. To relate this to register/Turing-machine computation, or to partial recursive functions, we first have to see how to encode numbers, pairs, lists, ... as λ-terms.

We will use the original encoding of numbers due to Church...

Church's numerals

Notation:

$$\begin{cases} M^0 N & \triangleq N \\ M^1 N & \triangleq M\, N \\ M^{n+1} N & \triangleq M(M^n N) \end{cases}$$

Church's numerals

$$
\begin{aligned}
\underline{0} &\triangleq \lambda f\,x.x \\
\underline{1} &\triangleq \lambda f\,x.f\,x \\
\underline{2} &\triangleq \lambda f\,x.f(f\,x) \\
&\vdots \\
\underline{n} &\triangleq \lambda f\,x.\underbrace{f(\cdots(f}_{n \text{ times}}\,x)\cdots)
\end{aligned}
$$

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so we can write $\underline{n}$ as $\lambda f\,x.f^n x$ and we have $\boxed{\underline{n}\,M\,N =_\beta M^n\,N}$.

Church's numerals

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$$\underline{2} \triangleq \lambda f\, x.f(f\, x)$$

N.B. not $f f x$, which stands for $(f f)x$

$$\vdots$$

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λ-Definable functions

Definition. $f \in \mathbb{N}^n \rightharpoonup \mathbb{N}$ is λ-definable if there is a closed λ-term F that represents it: for all $(x_1, \ldots, x_n) \in \mathbb{N}^n$ and $y \in \mathbb{N}$

- if $f(x_1, \ldots, x_n) = y$, then $F\, \underline{x_1} \cdots \underline{x_n} =_\beta \underline{y}$
- if $f(x_1, \ldots, x_n)\uparrow$, then $F\, \underline{x_1} \cdots \underline{x_n}$ has no β-nf.

For example, addition is λ-definable because it is represented by $P \triangleq \lambda x_1\, x_2. \lambda f\, x.\, x_1\, f\,(x_2\, f\, x)$:

$$\begin{aligned} P\, \underline{m}\, \underline{n} &=_\beta \lambda f\, x.\, \underline{m}\, f\,(\underline{n}\, f\, x) \\ &=_\beta \lambda f\, x.\, \underline{m}\, f\,(f^n x) \\ &=_\beta \lambda f\, x.\, f^m (f^n x) \\ &= \lambda f\, x. f^{m+n} x \\ &= \underline{m+n} \end{aligned}$$

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&=_\beta \lambda f\, x .\, f^m(f^n x)\\
&= \lambda f\, x . f^{m+n} x\\
&= \underline{m+n}
\end{aligned}
$$

can prove this equality by induction on n

Computable = λ-definable

Theorem. A partial function is computable if and only if it is λ-definable.

We already know that

$$
\begin{aligned}
& \text{Register Machine computable} \\
= \; & \text{Turing computable} \\
= \; & \text{partial recursive.}
\end{aligned}
$$

Using this, we break the theorem into two parts:

- every partial recursive function is λ-definable
- λ-definable functions are RM computable

λ-Definable functions

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- if $f(x_1, \ldots, x_n) = y$, then $F\,\underline{x_1} \cdots \underline{x_n} =_\beta \underline{y}$
- if $f(x_1, \ldots, x_n)\uparrow$, then $F\,\underline{x_1} \cdots \underline{x_n}$ has no β-nf.

This condition can make it quite tricky to find a λ-term representing a non-total function.

For now, we concentrate on total functions. First, let us see why the elements of **PRIM** (primitive recursive functions) are λ-definable.

Basic functions

- Projection functions, $\mathtt{proj}_i^n \in \mathbb{N}^n \to \mathbb{N}$:

$$\mathtt{proj}_i^n(x_1, \ldots, x_n) \triangleq x_i$$

- Constant functions with value $\mathbf{0}$, $\mathtt{zero}^n \in \mathbb{N}^n \to \mathbb{N}$:

$$\mathtt{zero}^n(x_1, \ldots, x_n) \triangleq \mathbf{0}$$

- Successor function, $\mathtt{succ} \in \mathbb{N} \to \mathbb{N}$:

$$\mathtt{succ}(x) \triangleq x + 1$$

Basic functions are representable

- $\mathtt{proj}_i^n \in \mathbb{N}^n \to \mathbb{N}$ is represented by $\lambda x_1 \ldots x_n . x_i$
- $\mathtt{zero}^n \in \mathbb{N}^n \to \mathbb{N}$ is represented by $\lambda x_1 \ldots x_n . \underline{0}$
- $\mathtt{succ} \in \mathbb{N} \to \mathbb{N}$ is represented by

$$\mathbf{Succ} \triangleq \lambda x_1\, f\, x . f\,(x_1\, f\, x)$$

since

$$\begin{aligned}\mathbf{Succ}\,\underline{n} &=_\beta \lambda f\, x .\, f\,(\underline{n}\, f\, x)\\ &=_\beta \lambda f\, x .\, f\,(f^n\, x)\\ &= \lambda f\, x .\, f^{n+1}\, x\\ &= \underline{n+1}\end{aligned}$$

Basic functions are representable

- $\mathtt{proj}_i^n \in \mathbb{N}^n \to \mathbb{N}$ is represented by $\lambda x_1 \ldots x_n . x_i$
- $\mathtt{zero}^n \in \mathbb{N}^n \to \mathbb{N}$ is represented by $\lambda x_1 \ldots x_n . \underline{0}$
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($\lambda x_1 f x . x_1 f (f x)$ also represents succ)

Representing composition

If total function $f \in \mathbb{N}^n \to \mathbb{N}$ is represented by F and total functions $g_1, \ldots, g_n \in \mathbb{N}^m \to \mathbb{N}$ are represented by $G_1, \ldots, G_n$, then their composition $f \circ (g_1, \ldots, g_n) \in \mathbb{N}^m \to \mathbb{N}$ is represented simply by

$$\lambda x_1 \ldots x_m. F\,(G_1\,x_1 \ldots x_m) \ldots (G_n\,x_1 \ldots x_m)$$

because

$$\begin{aligned} & F\,(G_1\,\underline{a_1} \ldots \underline{a_m}) \ldots (G_n\,\underline{a_1} \ldots \underline{a_m}) \\ =_\beta\; & F\,\underline{g_1(a_1, \ldots, a_m)} \ldots \underline{g_n(a_1, \ldots, a_m)} \\ =_\beta\; & \underline{f(g_1(a_1, \ldots, a_m), \ldots, g_n(a_1, \ldots, a_m))} \\ =\; & \underline{f \circ (g_1, \ldots, g_n)(a_1, \ldots, a_m)} \end{aligned}$$

.

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$$\lambda x_1 \ldots x_m . F\,(G_1\,x_1 \ldots x_m) \ldots (G_n\,x_1 \ldots x_m)$$

This does not necessarily work for partial functions. E.g. totally undefined function $u \in \mathbb{N} \rightharpoonup \mathbb{N}$ is represented by $U \triangleq \lambda x_1 . \Omega$ (why?) and $\mathtt{zero}^1 \in \mathbb{N} \to \mathbb{N}$ is represented by $Z \triangleq \lambda x_1 . \underline{0}$; but $\mathtt{zero}^1 \circ u$ is not represented by $\lambda x_1 . Z(U\,x_1)$, because $(\mathtt{zero}^1 \circ u)(n)\uparrow$ whereas $(\lambda x_1 . Z(U\,x_1))\,\underline{n} =_\beta Z\,\Omega =_\beta \underline{0}$. (What is $\mathtt{zero}^1 \circ u$ represented by?)

(see Ex. 12)

Primitive recursion

Theorem. Given $f \in \mathbb{N}^n \rightharpoonup \mathbb{N}$ and $g \in \mathbb{N}^{n+2} \rightharpoonup \mathbb{N}$, there is a unique $h \in \mathbb{N}^{n+1} \rightharpoonup \mathbb{N}$ satisfying

$$\begin{cases} h(\vec{x}, 0) & \equiv f(\vec{x}) \\ h(\vec{x}, x+1) & \equiv g(\vec{x}, x, h(\vec{x}, x)) \end{cases}$$

for all $\vec{x} \in \mathbb{N}^n$ and $x \in \mathbb{N}$.

We write $\rho^n(f, g)$ for h and call it the partial function defined by primitive recursion from f and g.

Representing primitive recursion

If $f \in \mathbb{N}^n \to \mathbb{N}$ is represented by a λ-term F and

$g \in \mathbb{N}^{n+2} \to \mathbb{N}$ is represented by a λ-term G,

we want to show λ-definability of the unique
$h \in \mathbb{N}^{n+1} \to \mathbb{N}$ satisfying

$$\begin{cases} h(\vec{a}, 0) & = f(\vec{a}) \\ h(\vec{a}, a+1) & = g(\vec{a}, a, h(\vec{a}, a)) \end{cases}$$

or equivalently

$$\begin{aligned} h(\vec{a}, a) = \mathit{if}\ a = 0\ \mathit{then}\ & f(\vec{a}) \\ \mathit{else}\ & g(\vec{a}, a-1, h(\vec{a}, a-1)) \end{aligned}$$

Representing primitive recursion

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where $\Phi_{f,g} \in (\mathbb{N}^{n+1} \to \mathbb{N}) \to (\mathbb{N}^{n+1} \to \mathbb{N})$ is given by

$$\begin{aligned}\Phi_{f,g}(h)(\vec{a}, a) \triangleq\ & \mathit{if}\ a = 0\ \mathit{then}\ f(\vec{a}) \\ & \mathit{else}\ g(\vec{a}, a-1, h(\vec{a}, a-1))\end{aligned}$$

Representing primitive recursion

If $f \in \mathbb{N}^n \to \mathbb{N}$ is represented by a λ-term F and

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Strategy:

- show that $\Phi_{f,g}$ is λ-definable;
- show that we can solve fixed point equations $\boxed{X = M\,X}$ up to β-conversion in the λ-calculus.

Representing booleans

$$\begin{aligned}\textbf{True} &\triangleq \lambda x\, y.\, x\\ \textbf{False} &\triangleq \lambda x\, y.\, y\\ \textbf{If} &\triangleq \lambda f\, x\, y.\, f\, x\, y\end{aligned}$$

satisfy

- $\textbf{If True}\ M\ N =_\beta \textbf{True}\ M\ N =_\beta M$
- $\textbf{If False}\ M\ N =_\beta \textbf{False}\ M\ N =_\beta N$

Representing test-for-zero

$$\mathsf{Eq}_0 \triangleq \lambda x.\, x(\lambda y.\, \mathsf{False})\, \mathsf{True}$$

satisfies

- $$\begin{aligned} \mathsf{Eq}_0\, \underline{0} &=_\beta \underline{0}\, (\lambda y.\, \mathsf{False})\, \mathsf{True} \\ &=_\beta \mathsf{True} \end{aligned}$$
- $$\begin{aligned} \mathsf{Eq}_0\, \underline{n+1} &=_\beta \underline{n+1}\, (\lambda y.\, \mathsf{False})\, \mathsf{True} \\ &=_\beta (\lambda y.\, \mathsf{False})^{n+1}\, \mathsf{True} \\ &=_\beta (\lambda y.\, \mathsf{False})((\lambda y.\, \mathsf{False})^n\, \mathsf{True}) \\ &=_\beta \mathsf{False} \end{aligned}$$

Representing predecessor

Want λ-term **Pred** satisfying

$$\begin{aligned} \mathbf{Pred}\,\underline{n+1} &=_\beta \underline{n} \\ \mathbf{Pred}\,\underline{0} &=_\beta \underline{0} \end{aligned}$$

Have to show how to reduce the "$n+1$-iterator" $\underline{n+1}$ to the "n-iterator" $\underline{n}$.

Idea: given f, iterating the function

$$g_f : (x, y) \mapsto (f(x), x)$$

$n+1$ times starting from (x, x) gives the pair $(f^{n+1}(x), f^n(x))$. So we can get $f^n(x)$ from $f^{n+1}(x)$ *parametrically in* f *and* x, by building g_f from f, iterating $n+1$ times from (x, x) and then taking the second component.

Hence...

Representing ordered pairs

$$\begin{aligned}
\textsf{Pair} &\triangleq \lambda x\,y\,f.\,f\,x\,y\\
\textsf{Fst} &\triangleq \lambda f.\,f\,\textsf{True}\\
\textsf{Snd} &\triangleq \lambda f.\,f\,\textsf{False}
\end{aligned}$$

satisfy

- $$\begin{aligned}
\textsf{Fst}(\textsf{Pair}\,M\,N) &=_\beta \textsf{Fst}(\lambda f.\,f\,M\,N)\\
&=_\beta (\lambda f.\,f\,M\,N)\,\textsf{True}\\
&=_\beta \textsf{True}\,M\,N\\
&=_\beta M
\end{aligned}$$
- $\textsf{Snd}(\textsf{Pair}\,M\,N) =_\beta \cdots =_\beta N$

Representing predecessor

Want λ-term **Pred** satisfying

$$\mathbf{Pred}\,\underline{n+1} =_\beta \underline{n}$$
$$\mathbf{Pred}\,\underline{0} =_\beta \underline{0}$$

$$\mathbf{Pred} \triangleq \lambda y\, f\, x.\,\mathbf{Snd}(y\,(G\,f)\,(\mathbf{Pair}\,x\,x))$$

where

$$G \triangleq \lambda f\, p.\,\mathbf{Pair}(f\,(\mathbf{Fst}\,p))\,(\mathbf{Fst}\,p)$$

has the required β-reduction properties.

Show

$(\forall n \in \mathbb{N})\ \underline{n+1}(Gf)(\text{Pair}\ x\,x) =_\beta \text{Pair}\,(\underline{n+1}\, f x)(\underline{n}\, f x)$

by induction on $n \in \mathbb{N}$:

Base case $n=0$:

$$\underline{1}(Gf)(\text{Pair}\,x\,x) =_\beta Gf(\text{Pair}\ x\,x)$$
$$=_\beta \text{Pair}\ (fx)\ x$$
$$=_\beta \text{Pair}\,(\underline{1}\, fx)(\underline{0}\, fx)$$

✓

Show

$$(\forall n \in \mathbb{N})\ \underline{n+1}\,(Gf)(\mathrm{Pair}\ x\,x) =_\beta \mathrm{Pair}\,(\underline{n+1}\,f x)(\underline{n}\,f x)$$

by induction on $n \in \mathbb{N}$:

Induction step:

$$\underline{n+2}\,(Gf)(\mathrm{Pair}\ x\,x) =_\beta (Gf)\ \underline{n+1}\,(Gf)(\mathrm{Pair}\ x\,x)$$

by ind.hyp.

$$=_\beta (Gf)\ \mathrm{Pair}\,(\underline{n+1}\,f x)(\underline{n}\,f x)$$

Show

$$(\forall n \in \mathbb{N})\ \underline{n+1}\,(Gf)(\mathrm{Pair}\ x\,x) =_\beta \mathrm{Pair}\,(\underline{n+1}\ f x)(\underline{n}\ f x)$$

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by ind. hyp.

$$=_\beta (Gf)\ \mathrm{Pair}\,(\underline{n+1}\ f x)(\underline{n}\ f x)$$

$$=_\beta \mathrm{Pair}\,(f(\underline{n+1} f x))(\underline{n+1}\ f x)$$

$$=_\beta \mathrm{Pair}\,(\underline{n+2}\ f x)(\underline{n+1}\ f x) \quad \checkmark$$

Show

$$(\forall n \in \mathbb{N})\ \underline{n+1}\,(Gf)(\mathrm{Pair}\ x\,x) =_\beta \mathrm{Pair}\,(\underline{n+1}\,f\,x)(\underline{n}\,f\,x)$$

So

$$\mathrm{Pred}\ \underline{n+1} =_\beta \lambda f x.\,\mathrm{Snd}(\underline{n+1}\,(Gf)(\mathrm{Pair}\,x\,x))$$

$$=_\beta \lambda f x.\,\mathrm{Snd}\,(\mathrm{Pair}\,(\underline{n+1}\,f\,x)(\underline{n}\,f\,x))$$

$$
\begin{aligned}
\text{Pred}\ \underline{n+1} &=_\beta \lambda fx.\,\text{Snd}(\underline{n+1}\,(G f)(\text{Pair}\,x\,x)) \\
&=_\beta \lambda fx.\,\text{Snd}\,(\text{Pair}\,(\underline{n+1}\,f x)(\underline{n}\,f x)) \\
&=_\beta \lambda fx.\,\underline{n}\,f x \\
&=_\beta \lambda fx.\,f^n x \\
&= \underline{n}
\end{aligned}
$$

Representing primitive recursion

If $f \in \mathbb{N}^n \to \mathbb{N}$ is represented by a λ-term F and

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where $\Phi_{f,g} \in (\mathbb{N}^{n+1} \to \mathbb{N}) \to (\mathbb{N}^{n+1} \to \mathbb{N})$ is given by...

Strategy:

- show that $\Phi_{f,g}$ is λ-definable;

$\lambda z \vec{x}\, x.\, \mathrm{If}(\mathrm{Eq}_0\, x)(F \vec{x})(G \vec{x}\, (\mathrm{Pred}\, x)(z \vec{x}\, (\mathrm{Pred}\, x)))$