Exercise Problems 9–12: Information Theory

Exercise 9

Y and Z are two continuous random variables.

Y has an exponential probability density distribution $p(x)$ over $x \in [0, \infty]$: $p(x) = e^{-x}$.
Note that

$$\int_0^\infty e^{-x}\, dx = \left[-e^{-x}\right]_0^\infty = 1\,.$$

Z has a uniform probability density distribution: $p(x) = 1/\alpha$ for $x \in [0, \alpha]$, else $p(x) = 0$.

Calculate the differential entropies $h(Y)$ and $h(Z)$ for these two continuous random variables, and find the value of α for which these differential entropies are the same. Sketch these distributions.

Exercise 10

(a) What does it mean for a function to be "self-Fourier?" Name three functions which are of importance in information theory and that have the self-Fourier property, and in each case mention a topic or a theorem exploiting it.

(b) Show that the set of all Gabor wavelets is closed under convolution, *i.e.* that the convolution of any two Gabor wavelets is just another Gabor wavelet. [HINT: This property relates to the fact that these wavelets are also closed under multiplication, and that they are also self-Fourier. You may address this question for just 1D wavelets if you wish.]

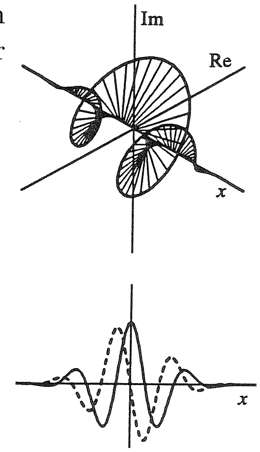
(c) Show that the family of sinc functions used in the Nyquist Sampling Theorem,

$$\text{sinc}(x) = \frac{\sin(\lambda x)}{\lambda x}$$

is closed under convolution. Show further that when two different sinc functions are convolved, the result is simply whichever one of them had the lower frequency, *i.e.* the smaller λ.

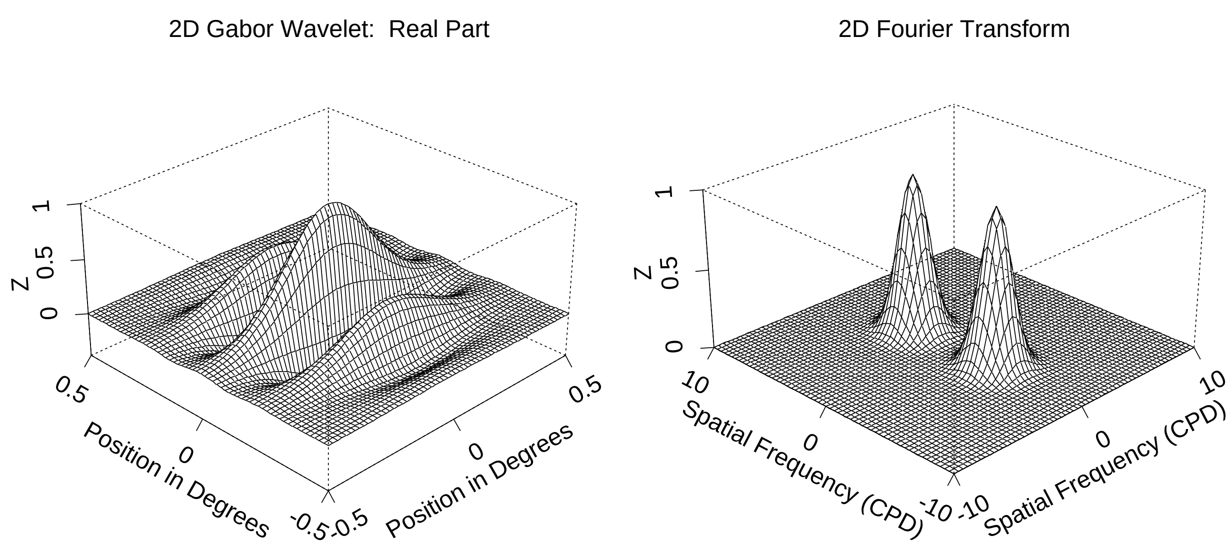
Exercise 11

(a) An important class of complex-valued functions for encoding information with maximal resolution simultaneously in the frequency domain and the signal domain are Gabor wavelets. Using an expression for their functional form, explain:



1. their spiral helical trajectory as phasors, shown here with projections of their real and imaginary parts;
2. the Uncertainty Principle under which they are optimal;
3. the spaces they occupy in the Information Diagram;
4. some of their uses in pattern encoding and recognition.

(b) Explain why the real-part of a 2D Gabor wavelet has a 2D Fourier transform with two peaks, not just one, as shown in the right panel of the Figure below.



Exercise 12

(a) Compare and contrast the compression strategies deployed in the JPEG and JPEG-2000 protocols. Include these topics: the underlying transforms used; their computational efficiency and ease of implementation; artefacts introduced in lossy mode; typical compression factors; and their relative performance when used to achieve severe compression rates.

(b) Define the Kolmogorov algorithmic complexity K of a string of data, and say whether or not it is computable. What relationship is to be expected between the Kolmogorov complexity K and the Shannon entropy H for a given set of data? Give a reasonable estimate of K for a fractal, and explain why it is reasonable. Discuss the following concepts in Kolmogorov's theory of pattern complexity: how writing a program that generates a pattern is a way of compressing it, and executing such a program decompresses it; Kolmogorov incompressibility, and patterns that are their own shortest possible description.