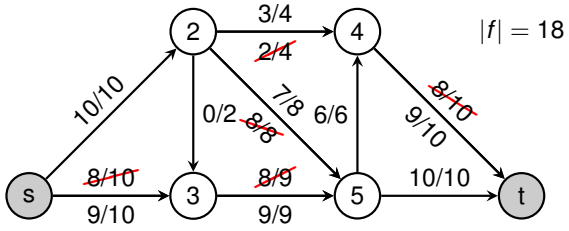
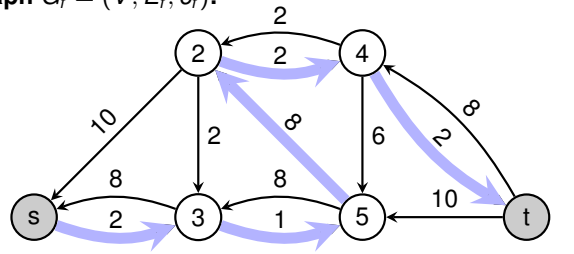
Graph $G = (V, E, c)$:



Residual Graph $G_f = (V, E_f, c_f)$:



6.6: Maximum flow

Frank Stajano

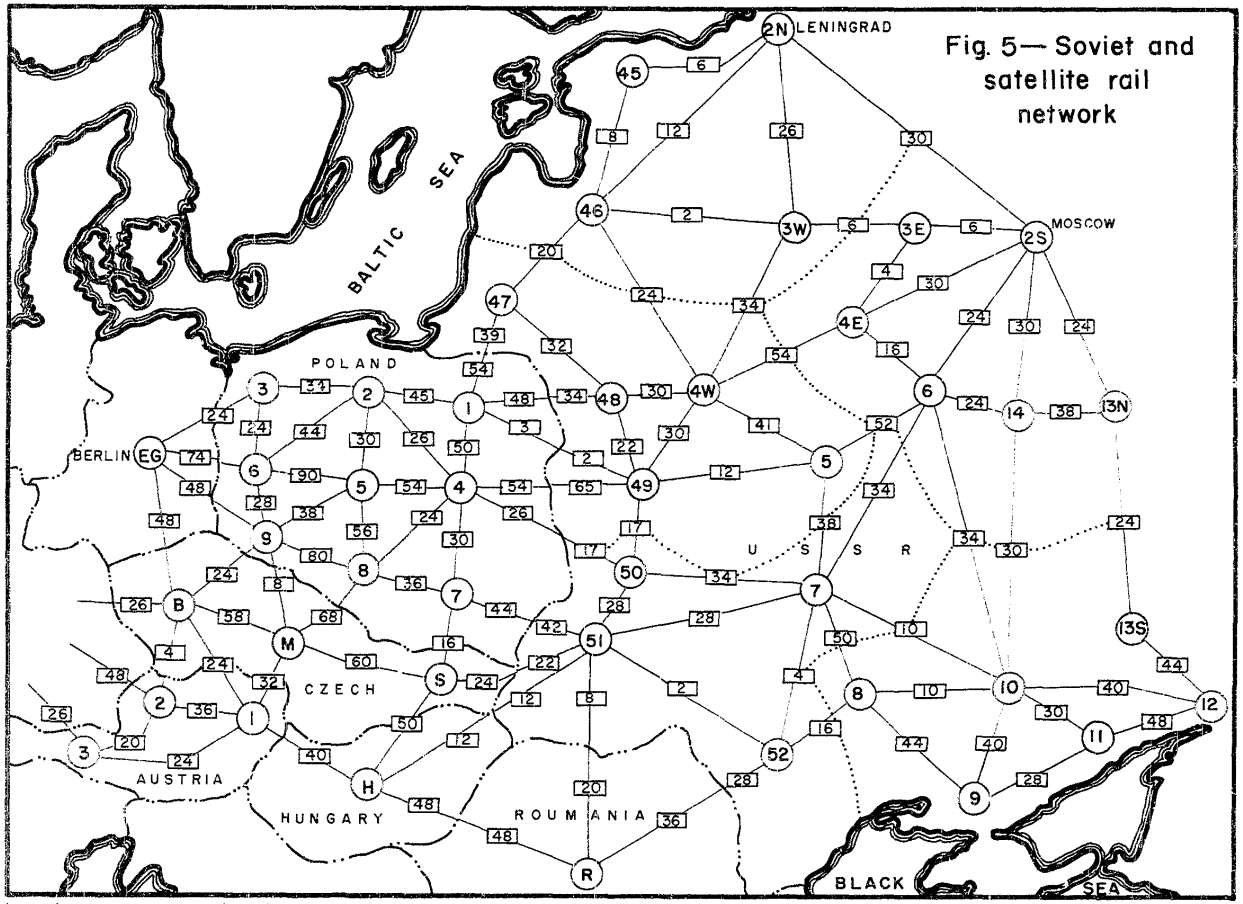
Thomas Sauerwald

Lent 2016

UNIVERSITY OF CAMBRIDGE

Outline

History of the Maximum Flow Problem [Harris, Ross (1955)]



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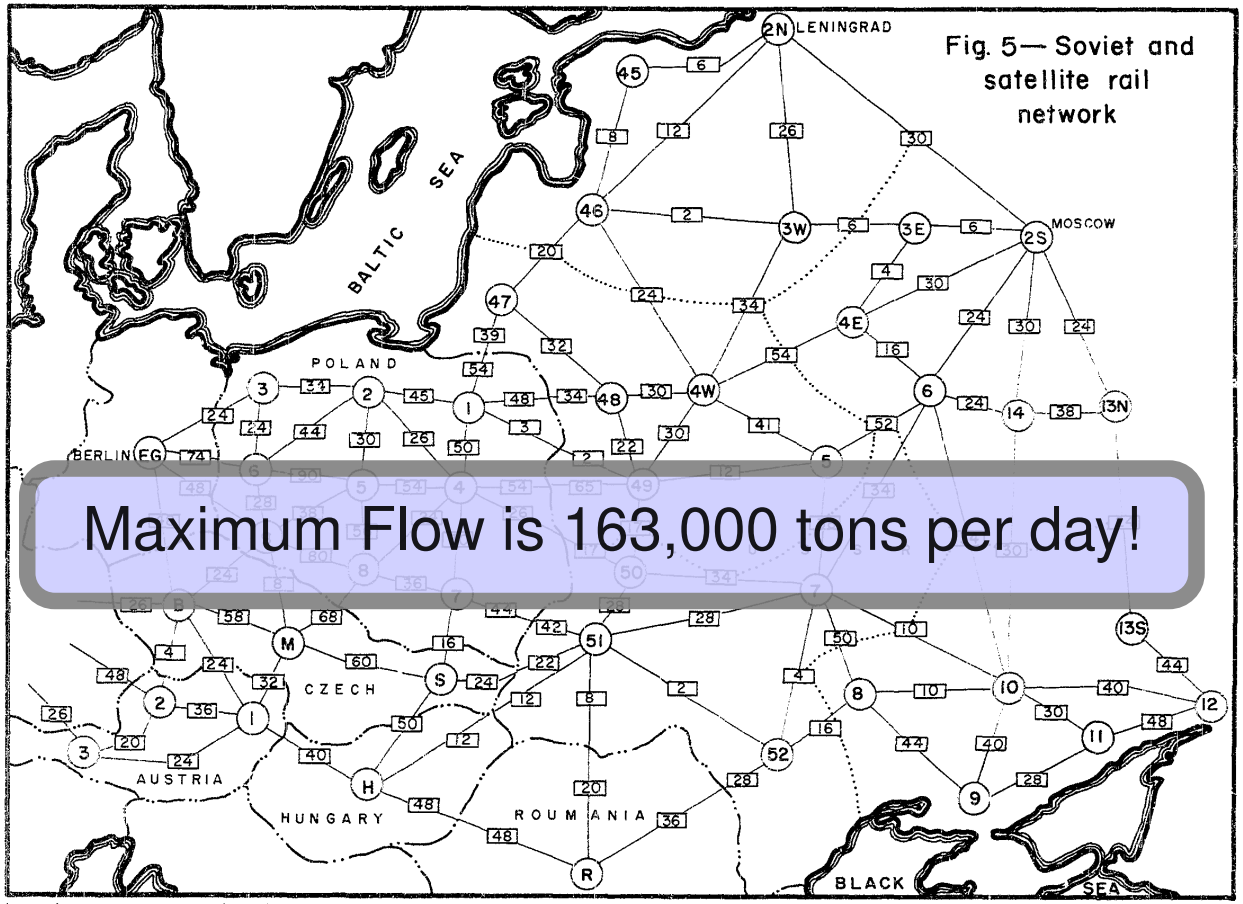
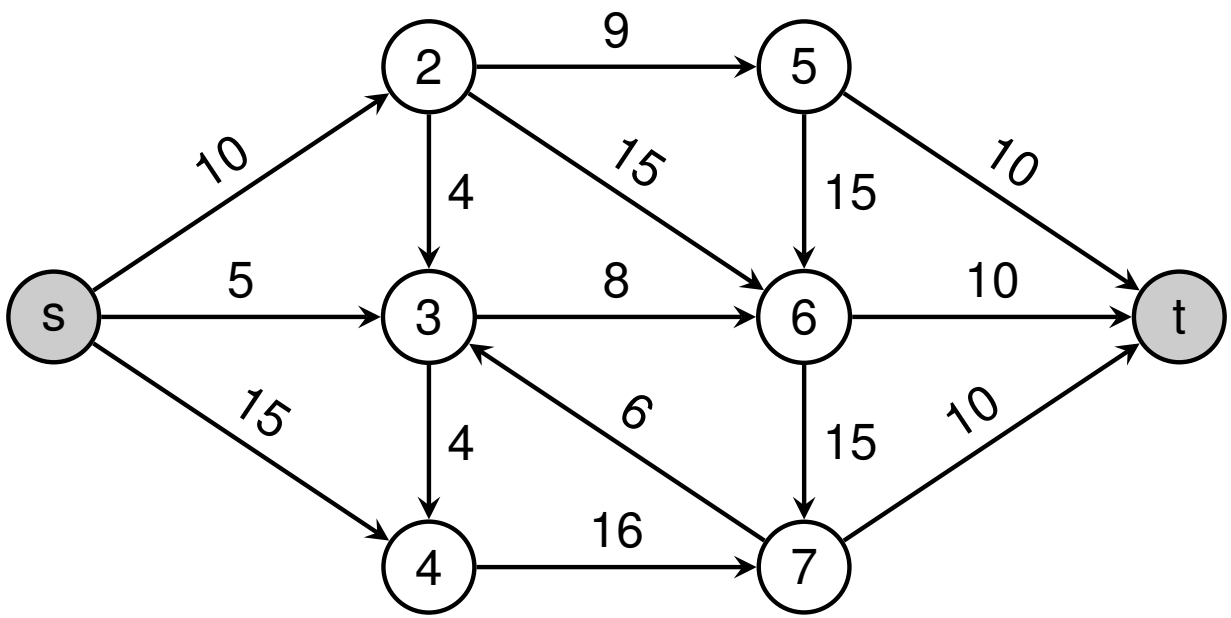


Flow Network

Flow Network

- Abstraction for material (one commodity!) flowing through the edges
- $G = (V, E)$ directed graph without parallel edges
- distinguished nodes: source s and sink t
- every edge e has a capacity $c(e)$



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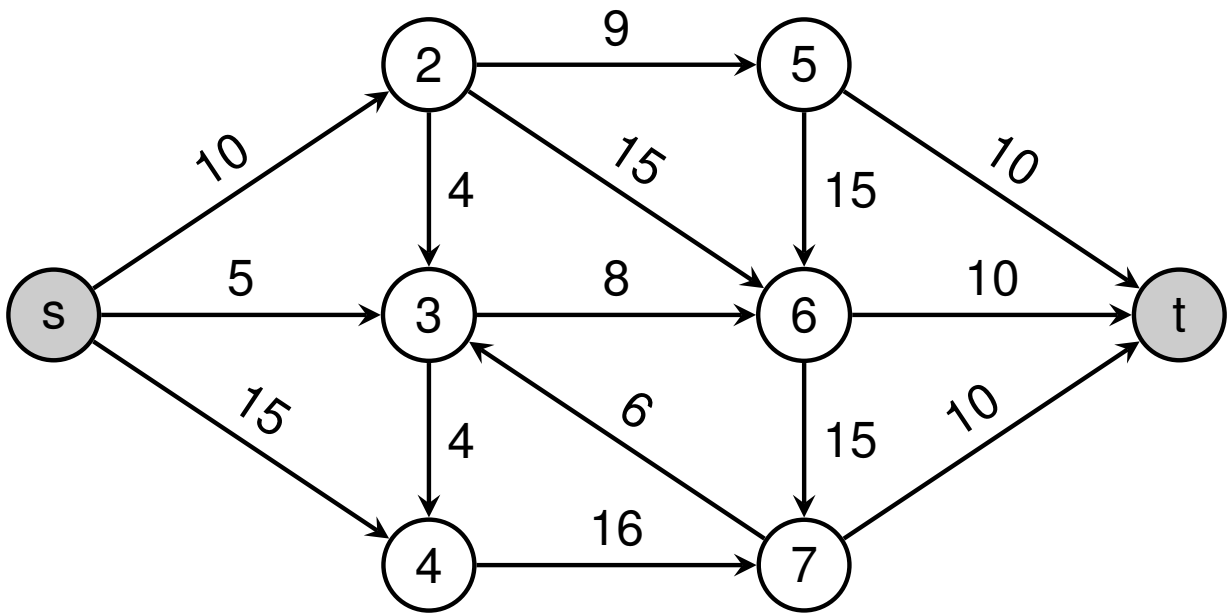

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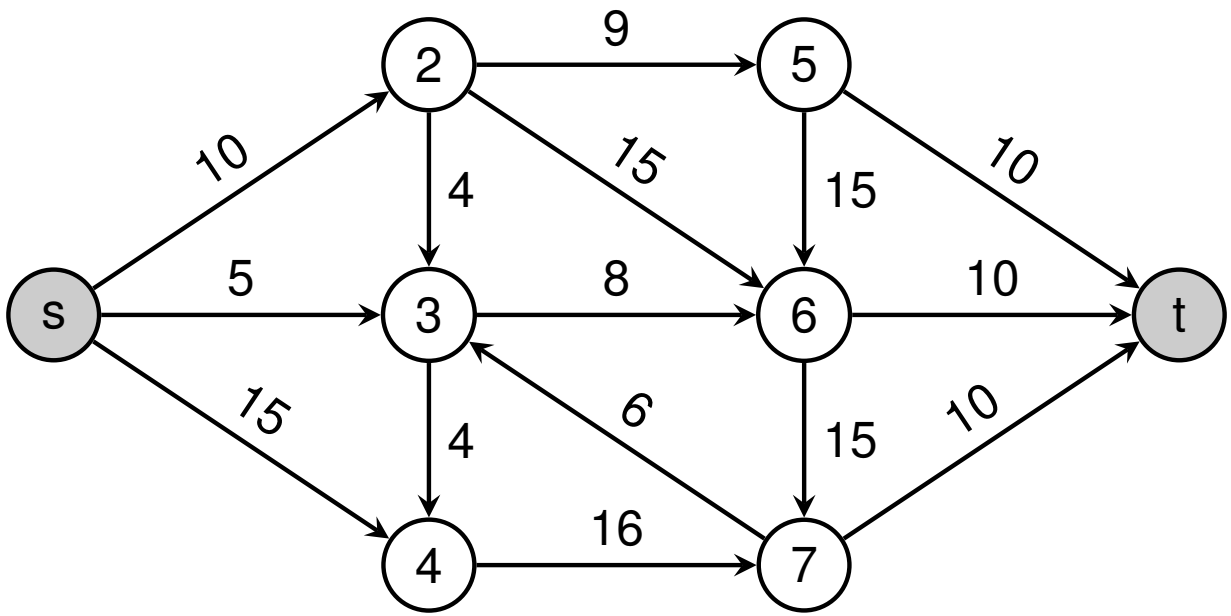
$c(u, v) = 0 \Leftrightarrow (u, v) \notin E$



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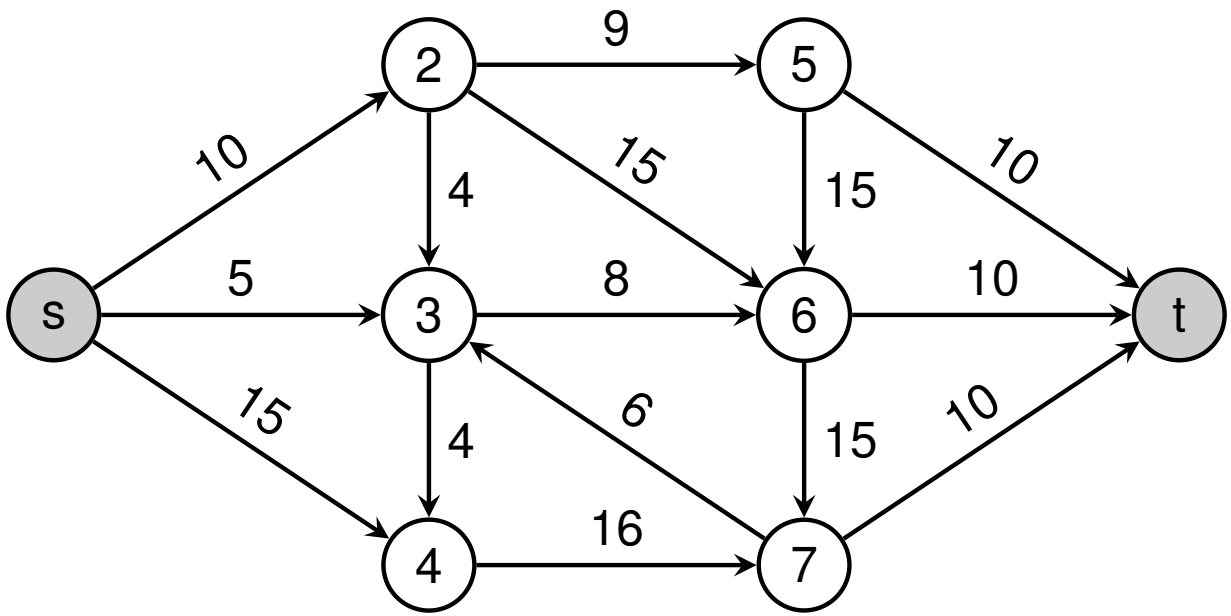

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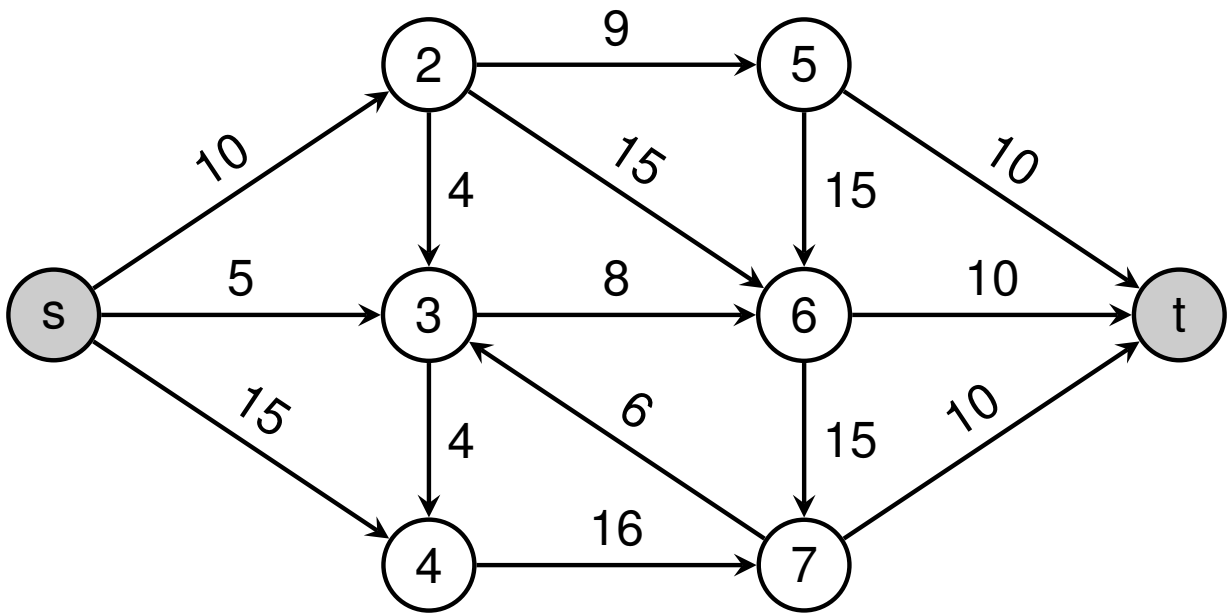

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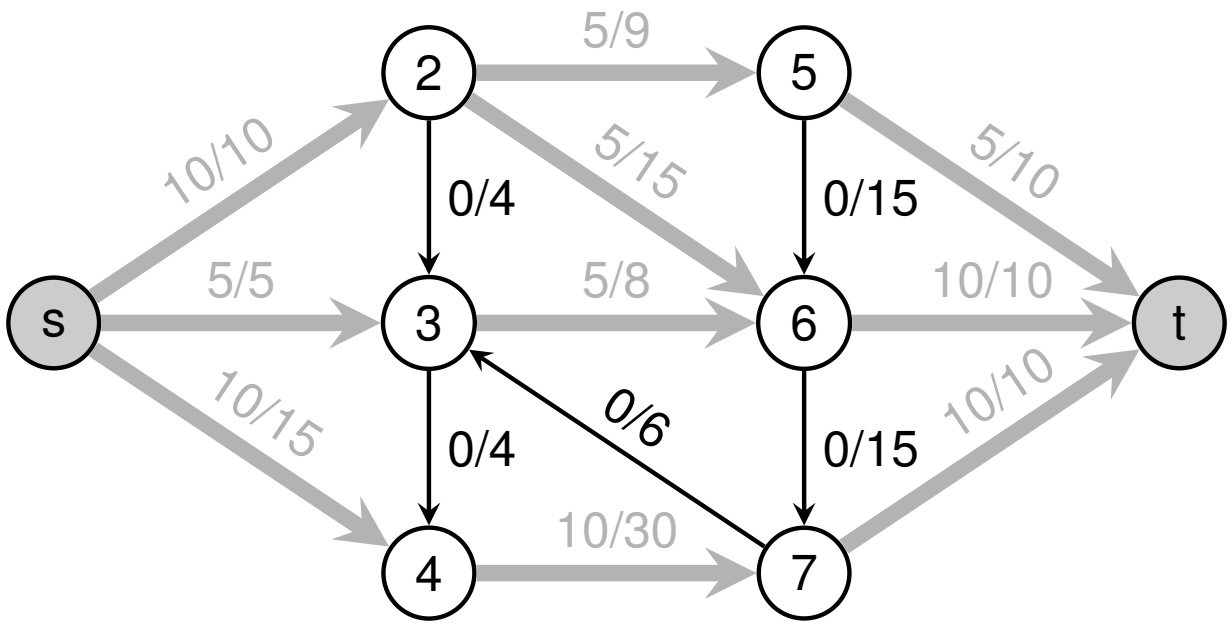

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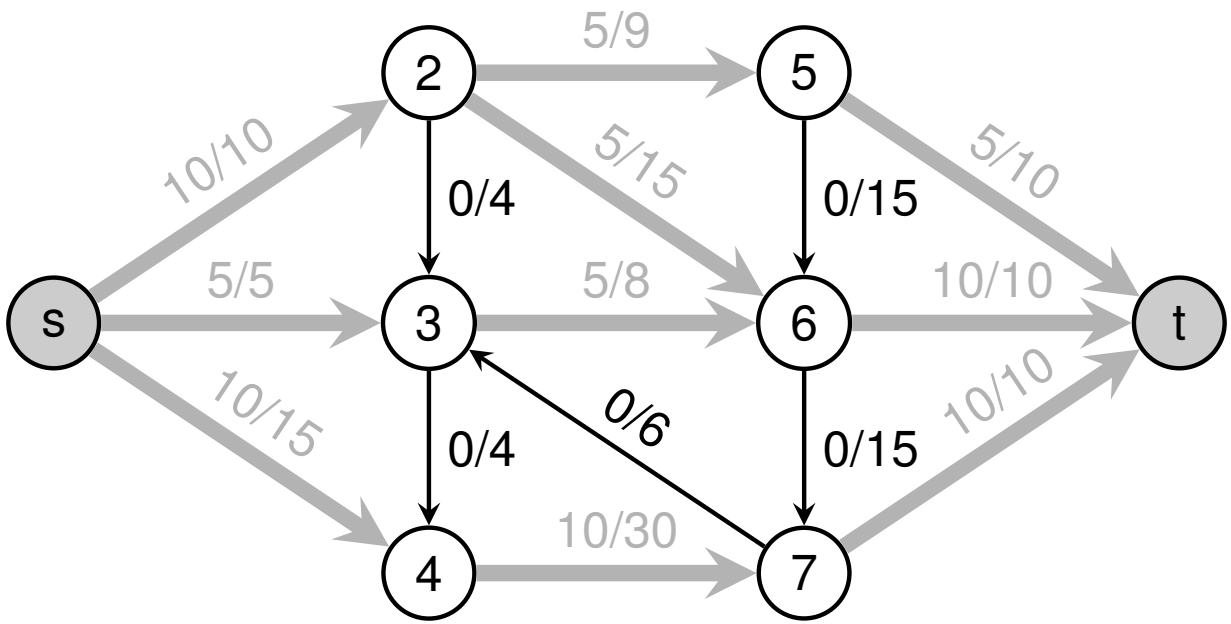

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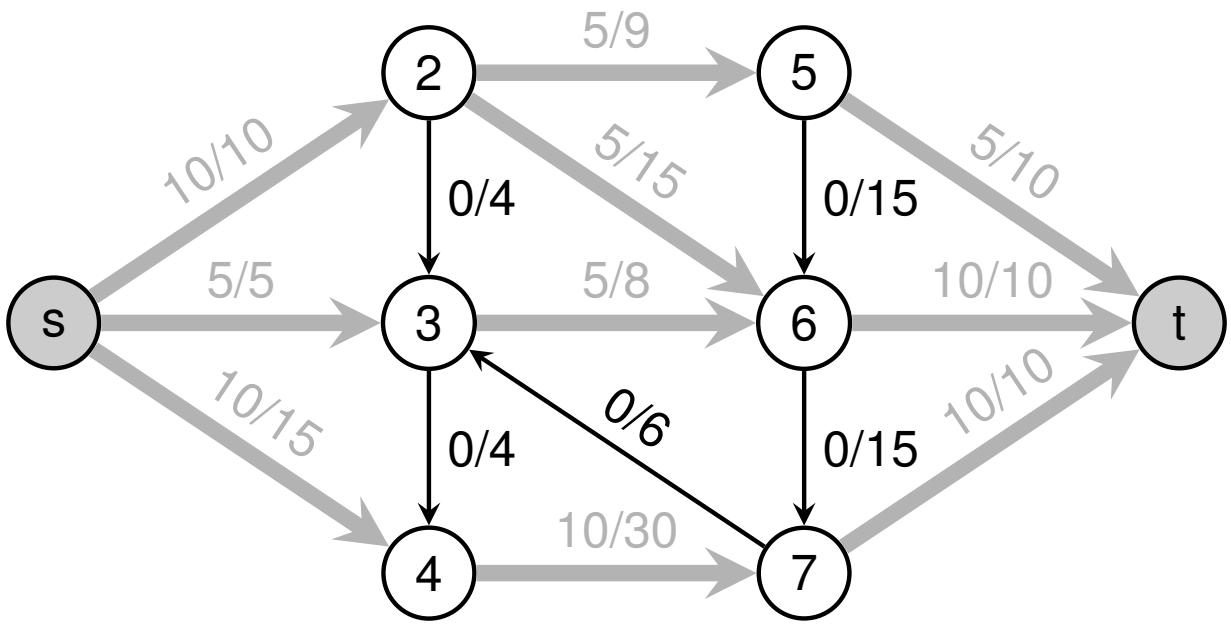

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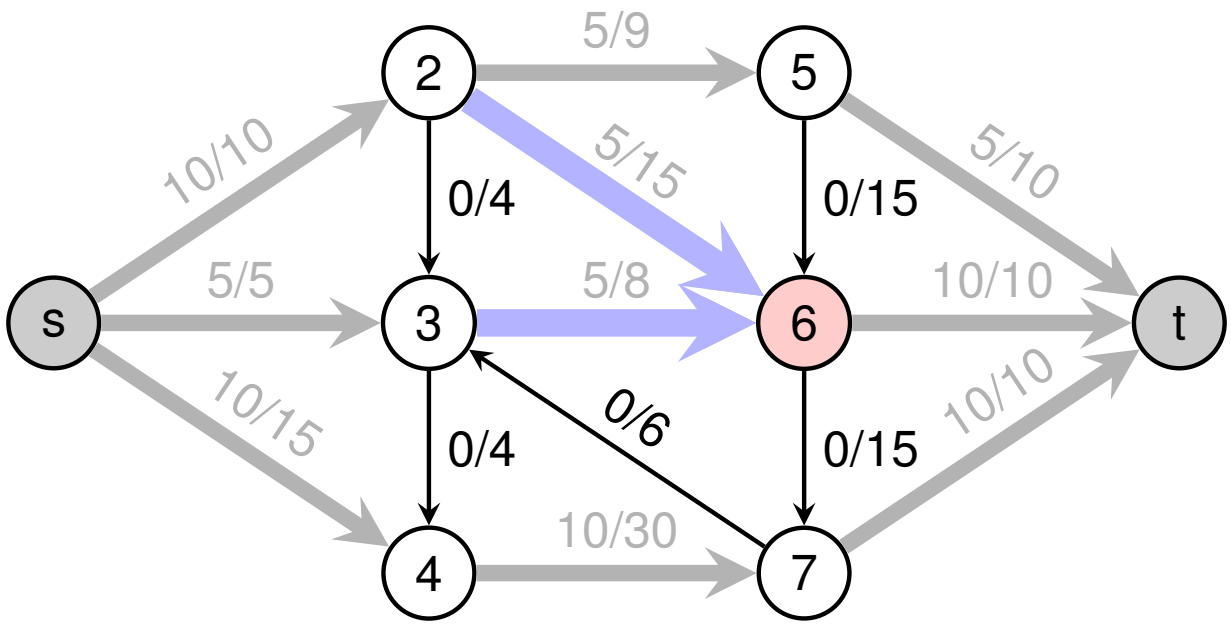
Flow Conservation



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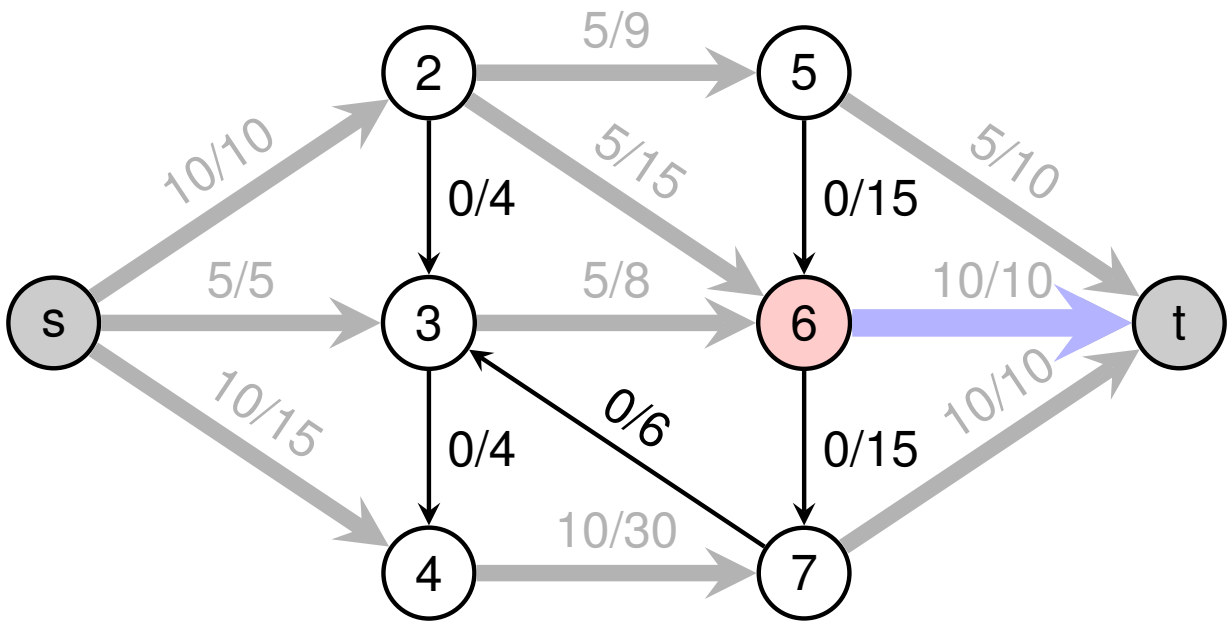

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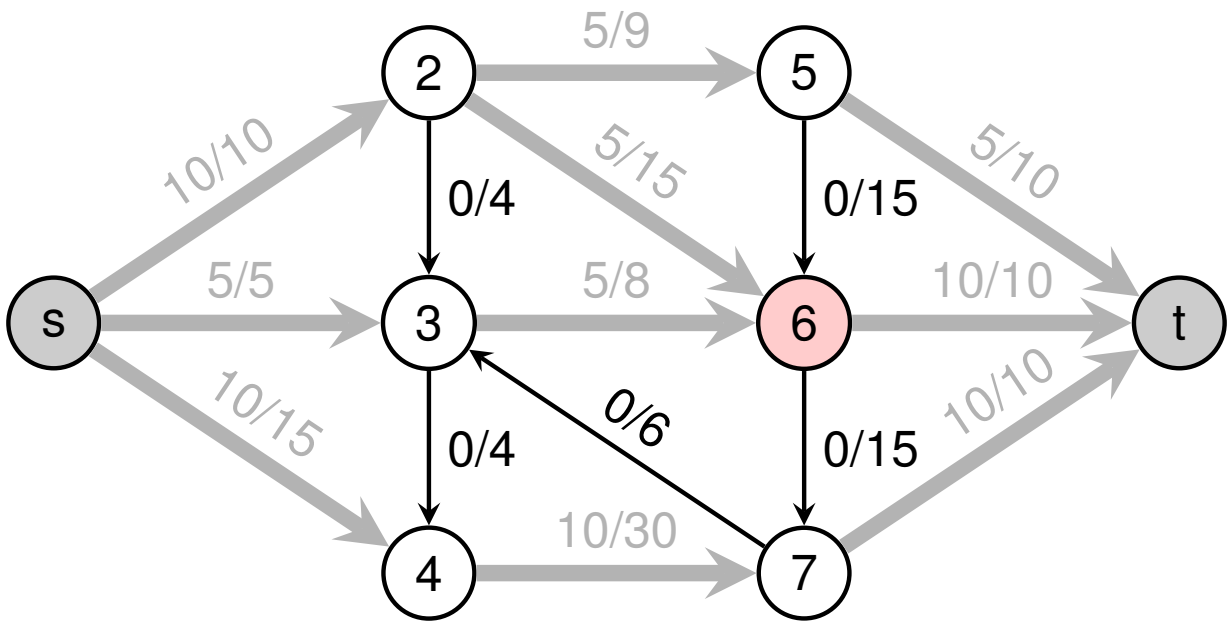

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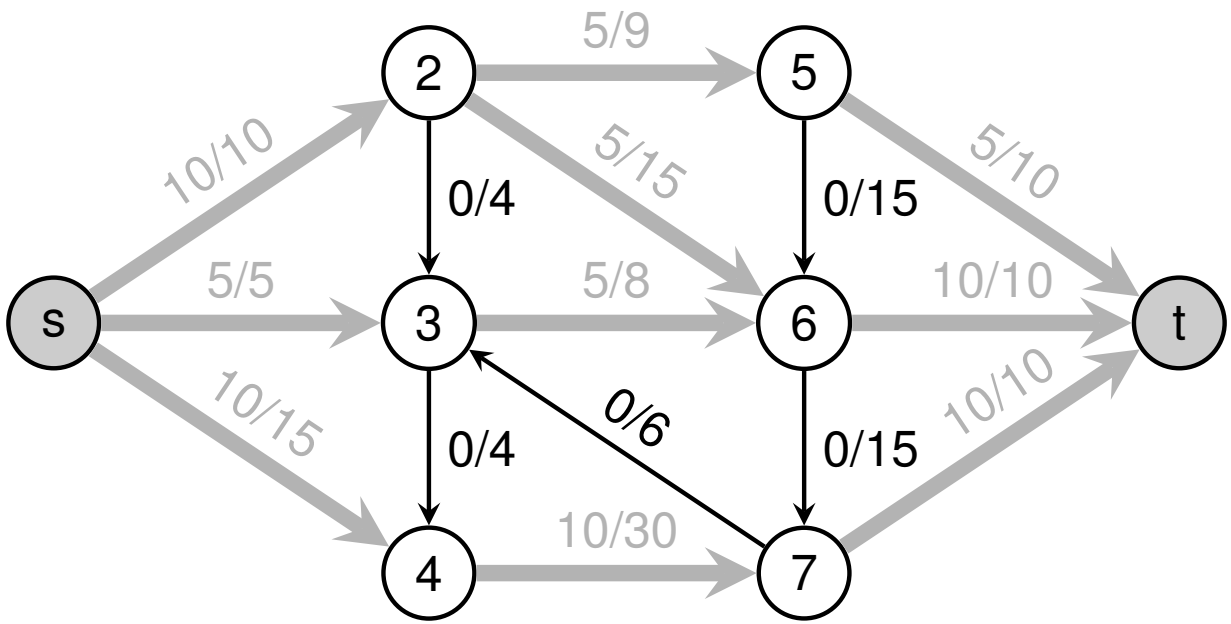

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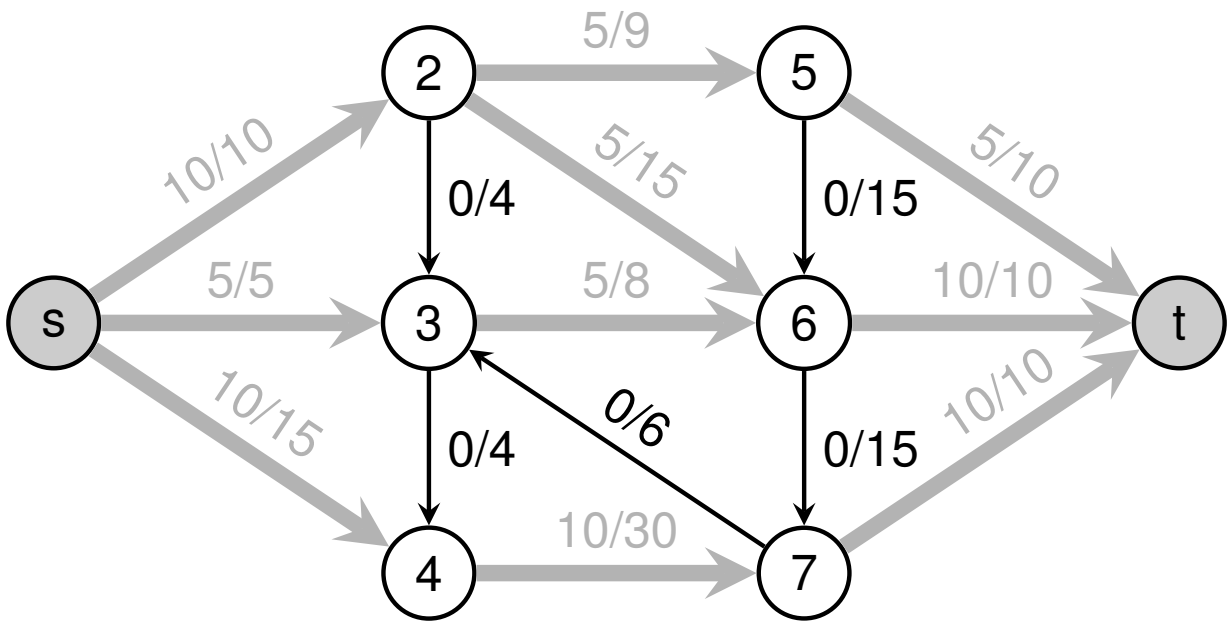

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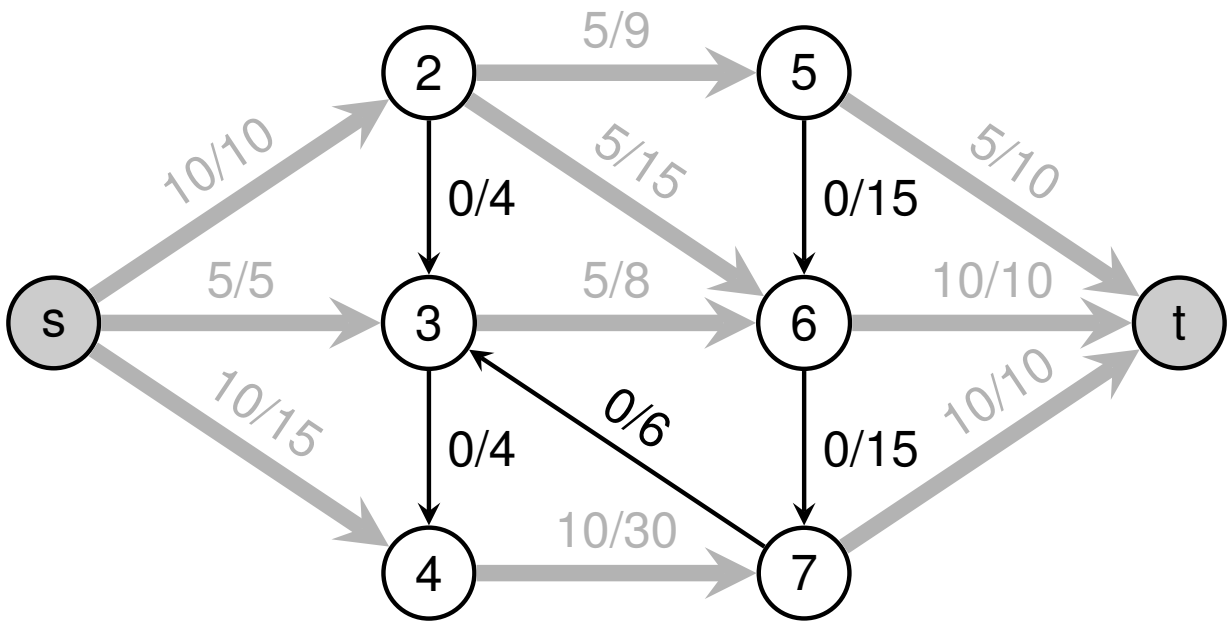

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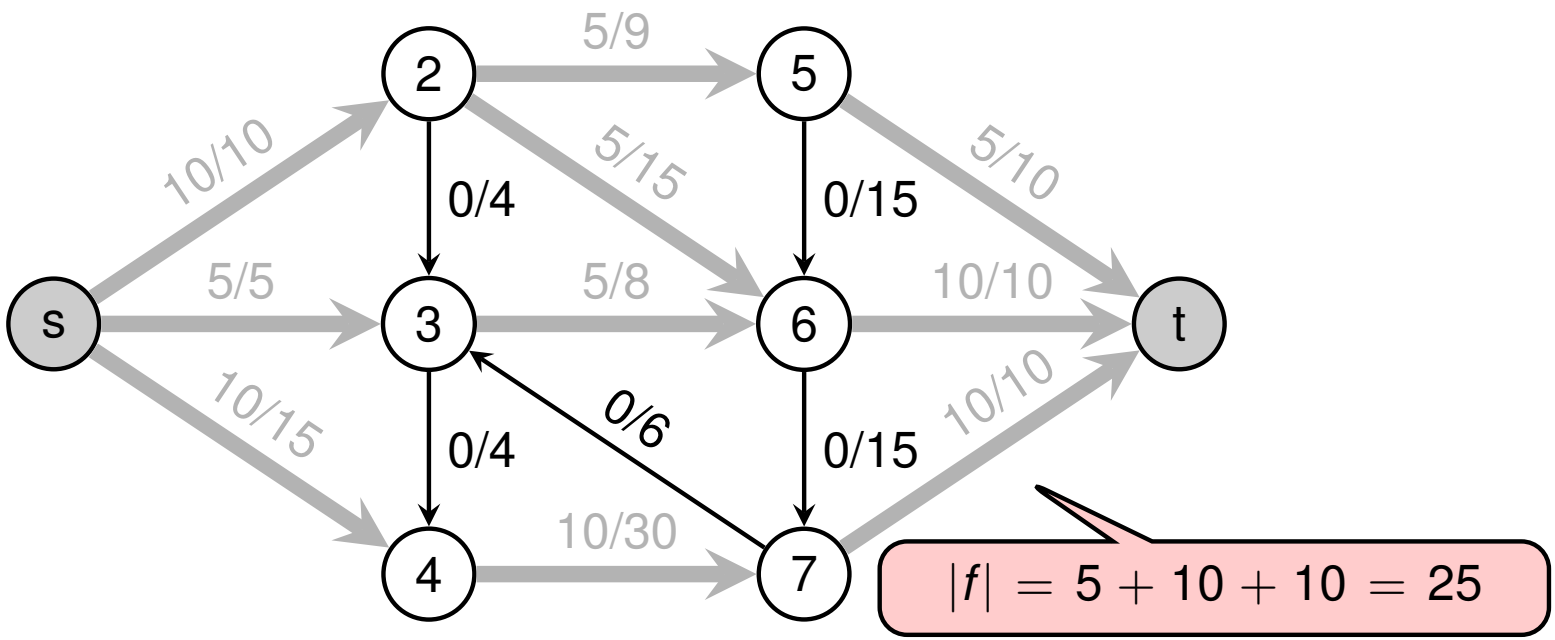

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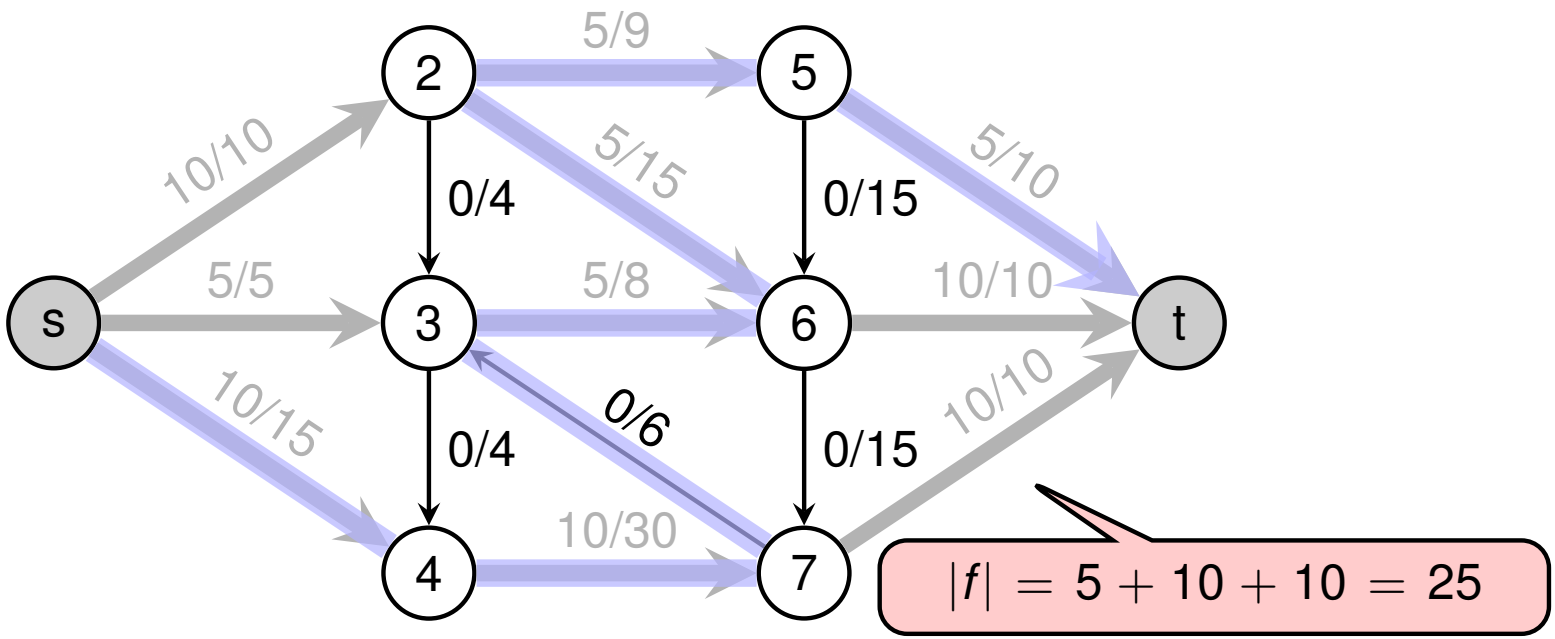

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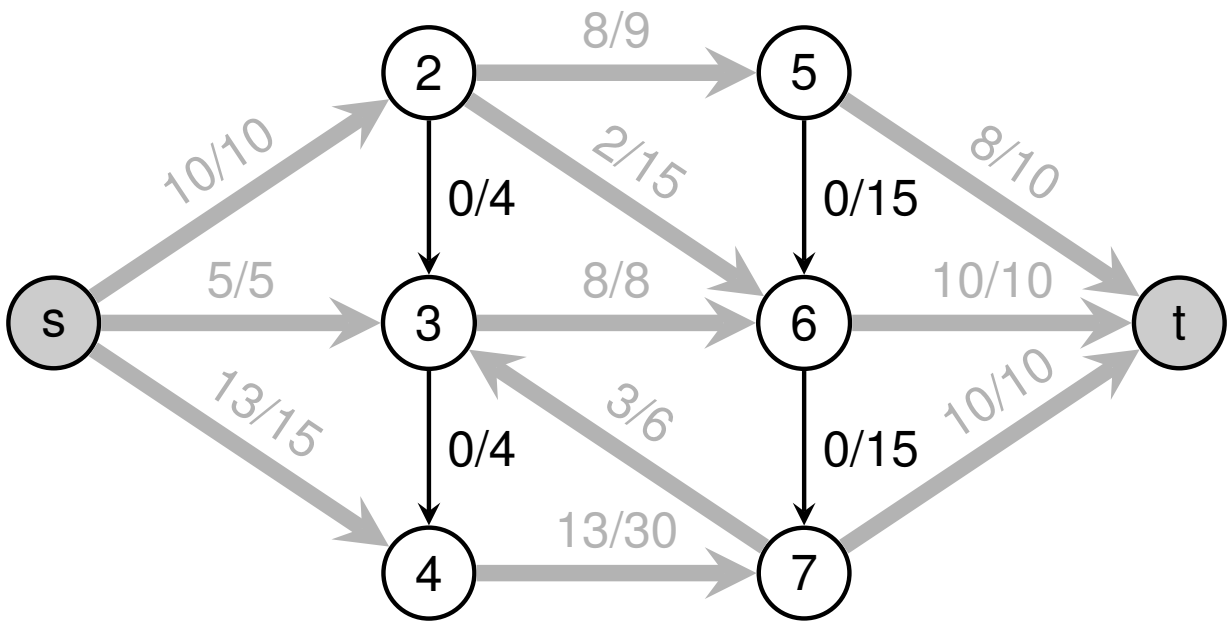

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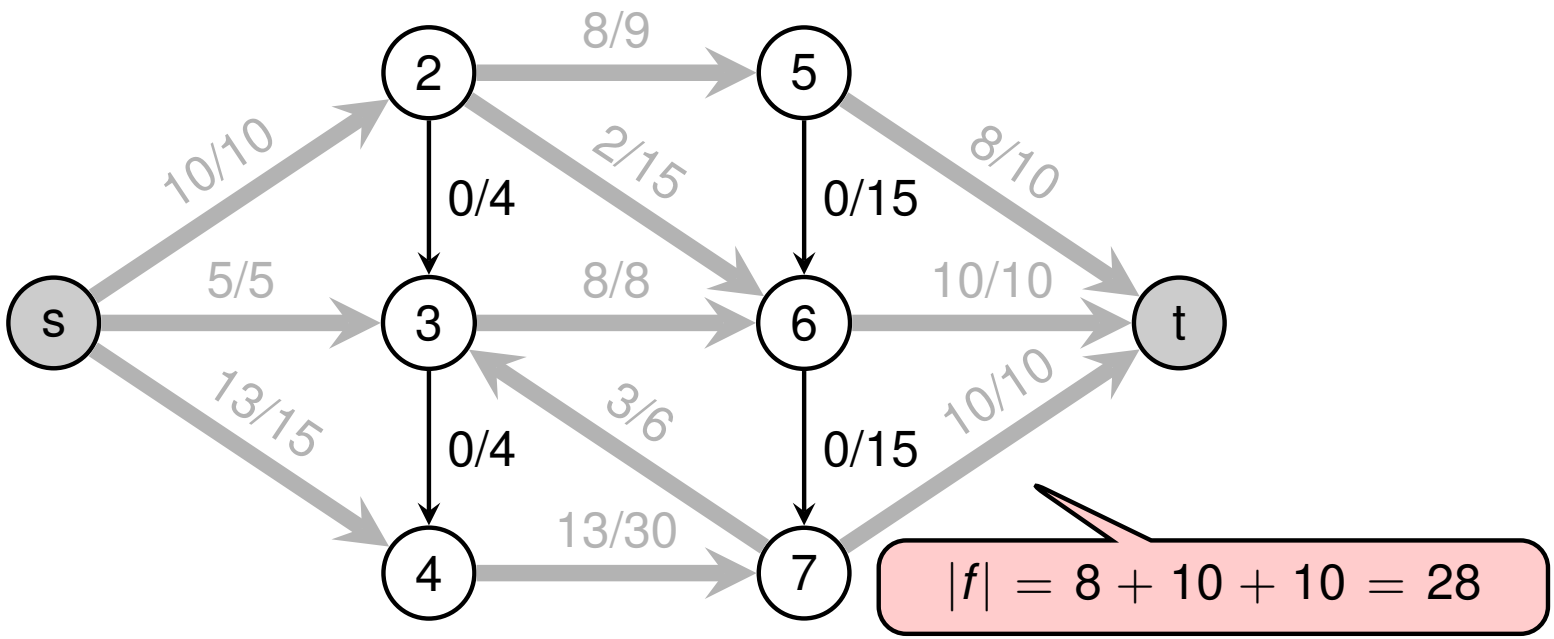

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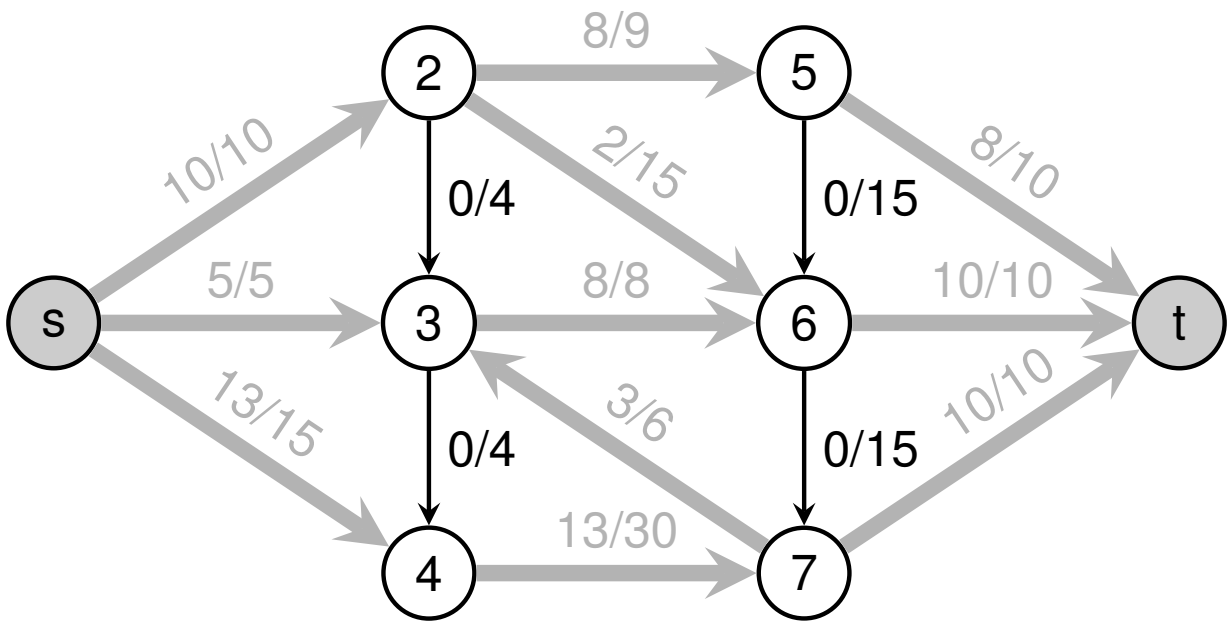
How to find a Maximum Flow?



A First Attempt

Greedy Algorithm

- Start with $f(u, v) = 0$ everywhere
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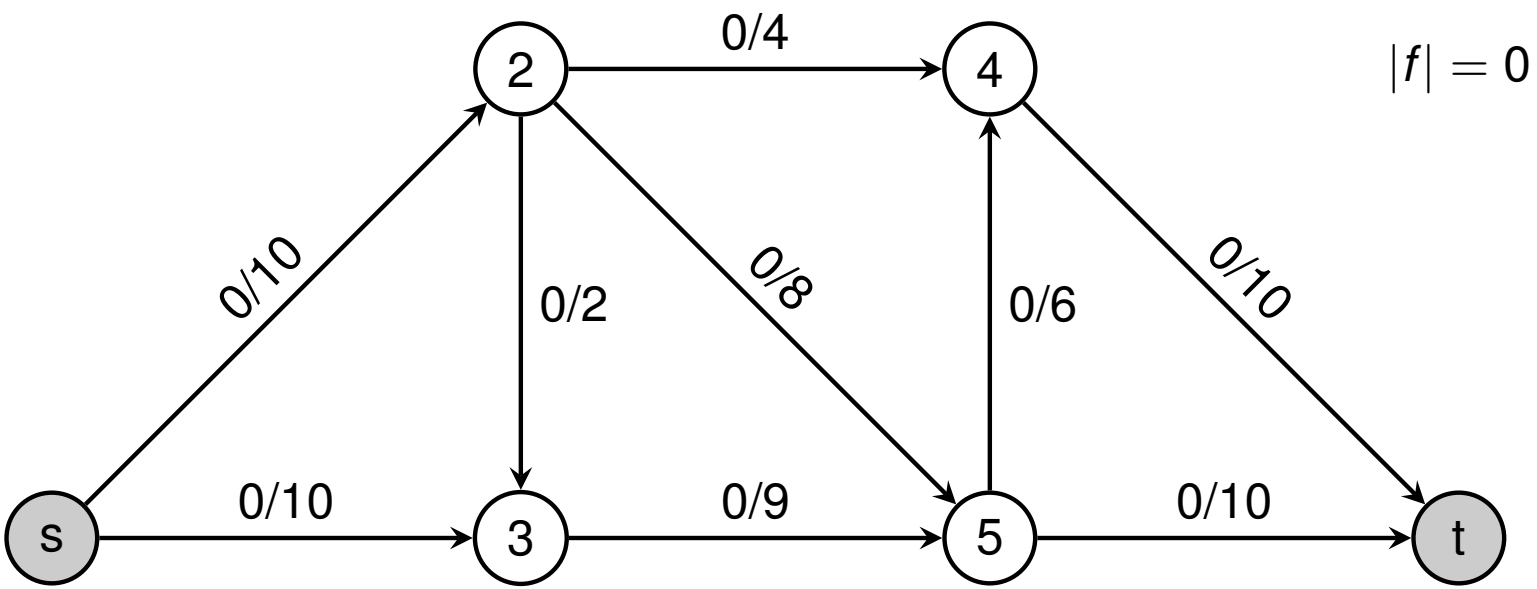

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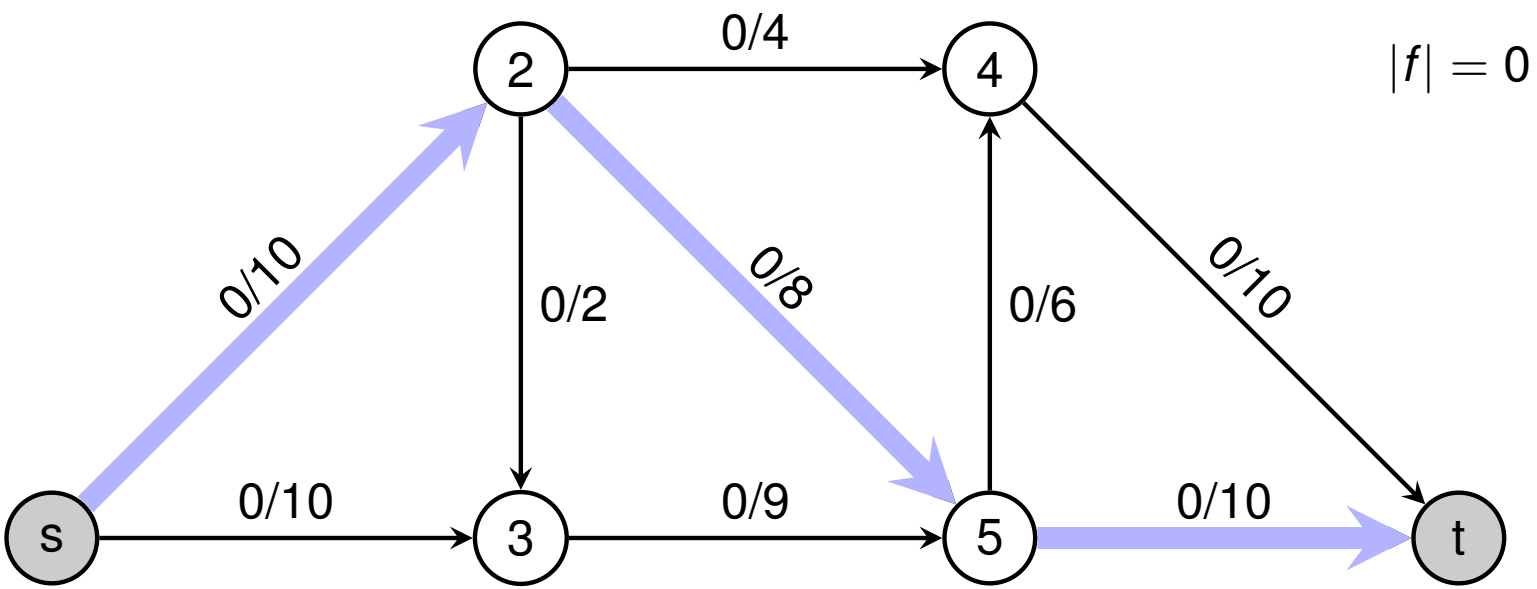

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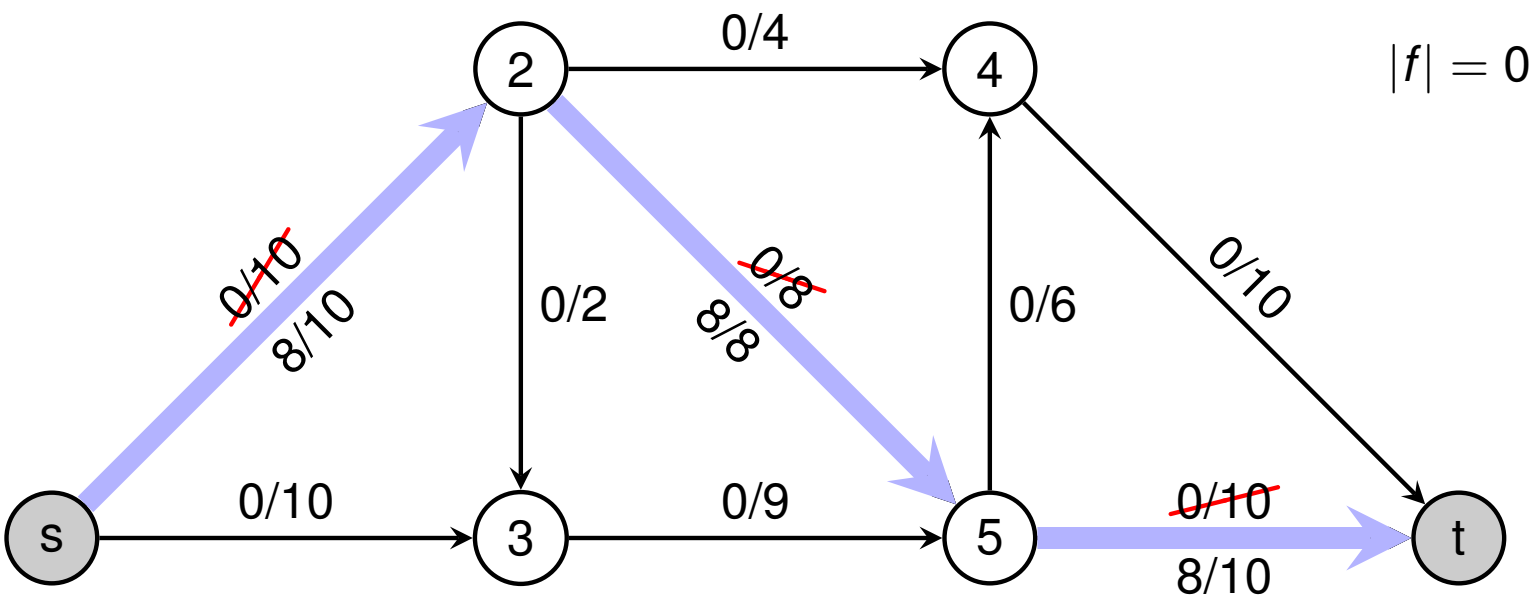

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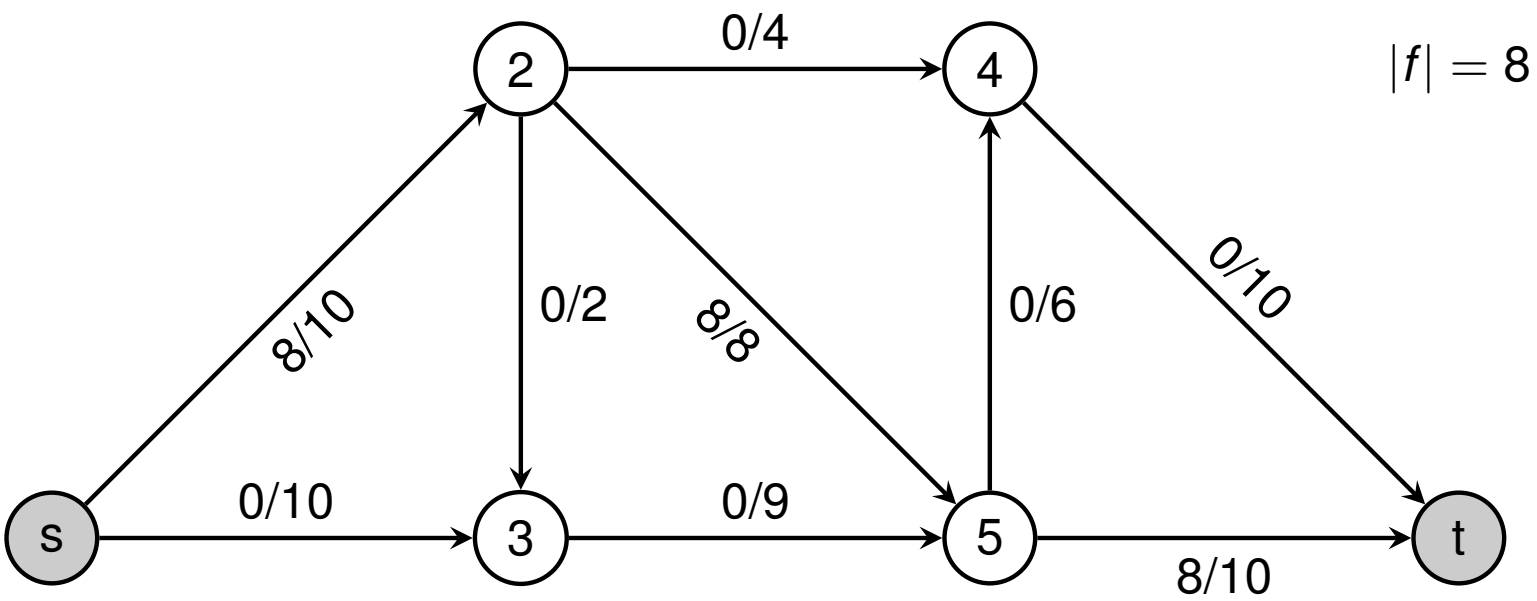

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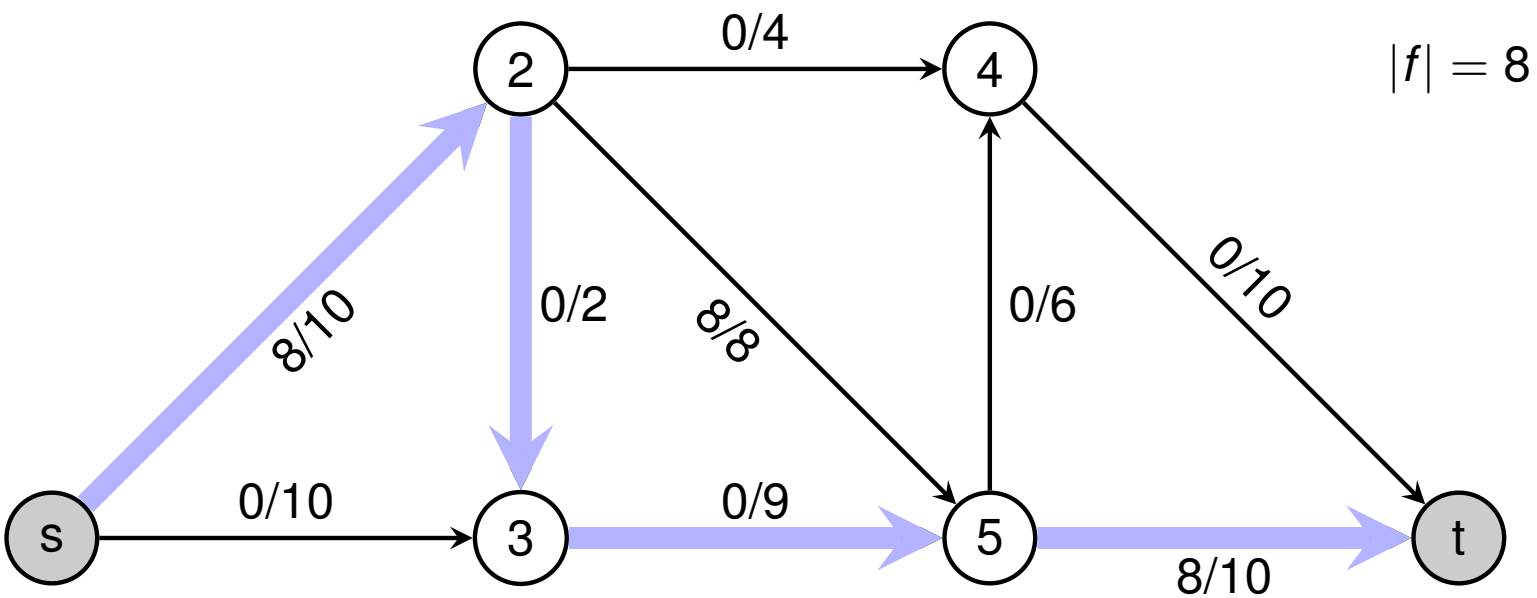

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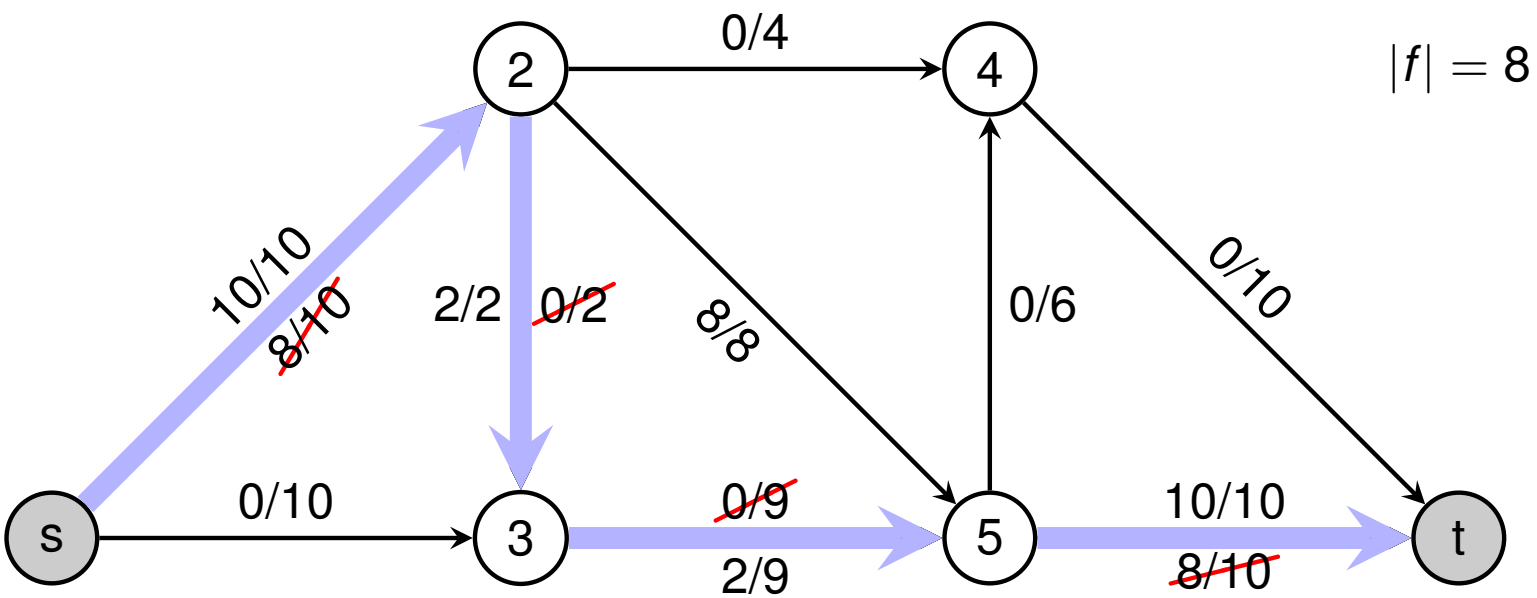

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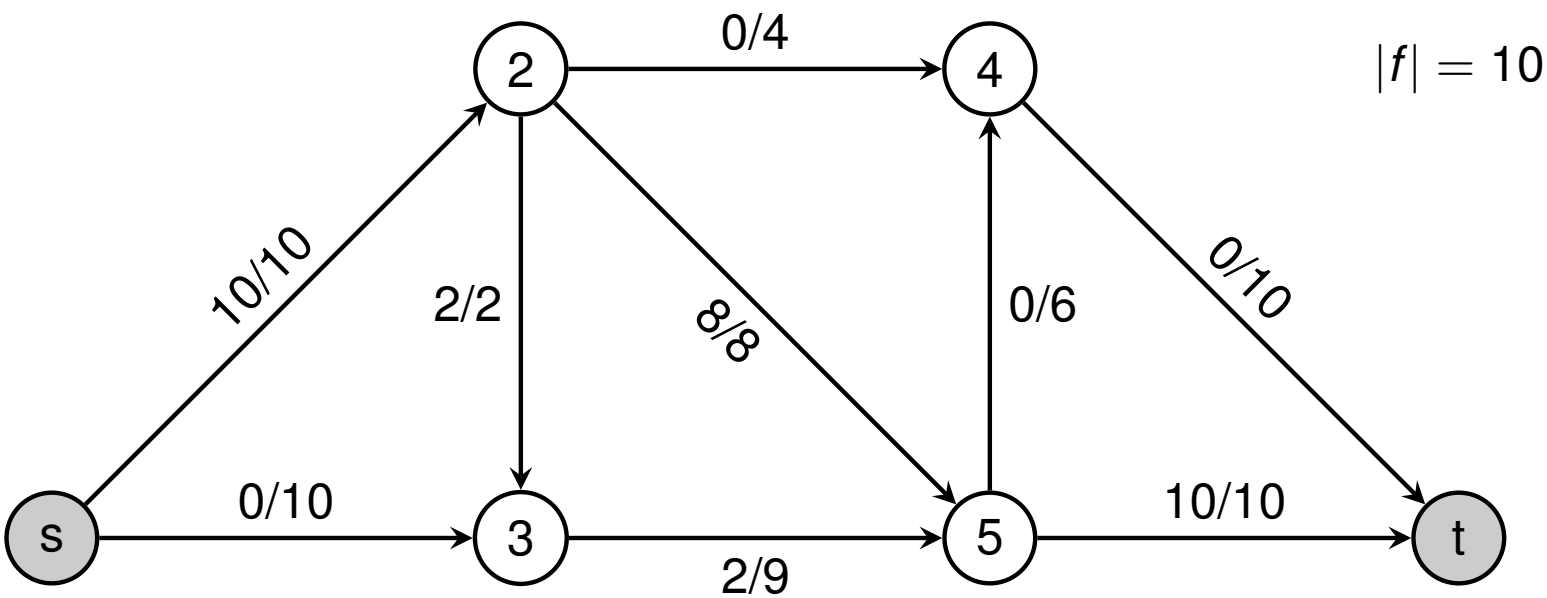

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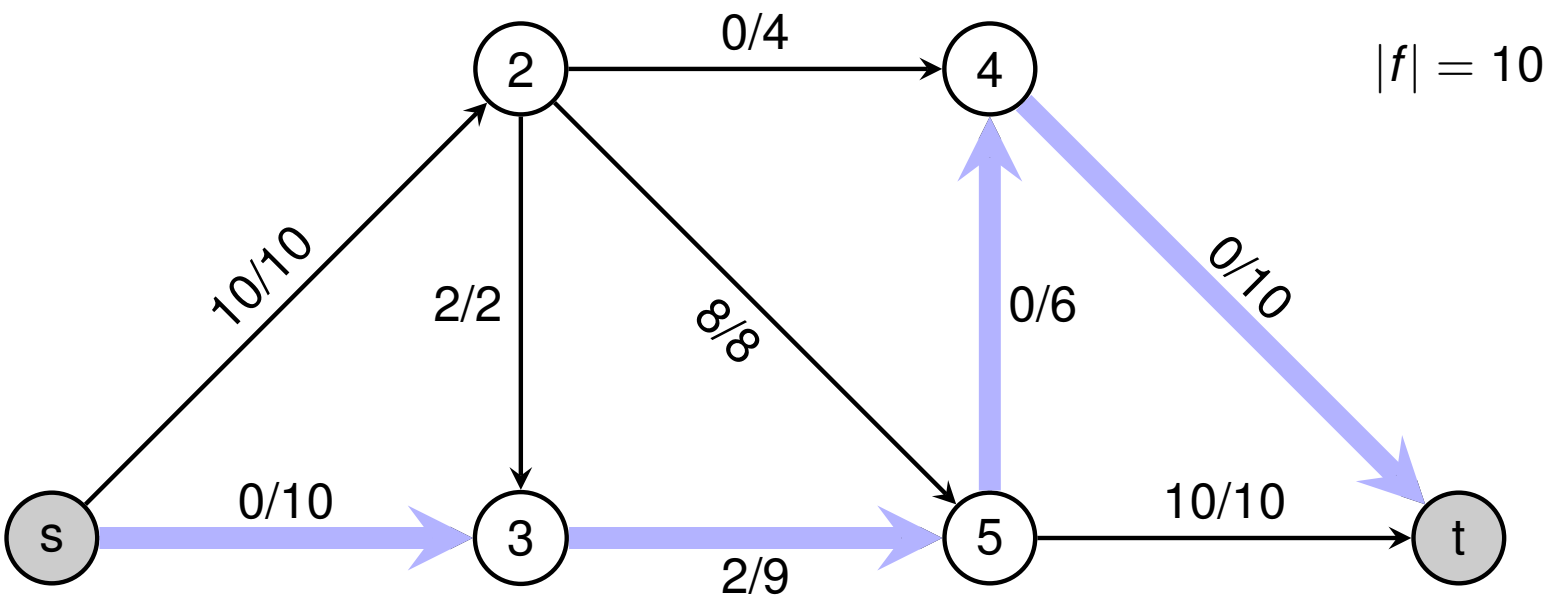

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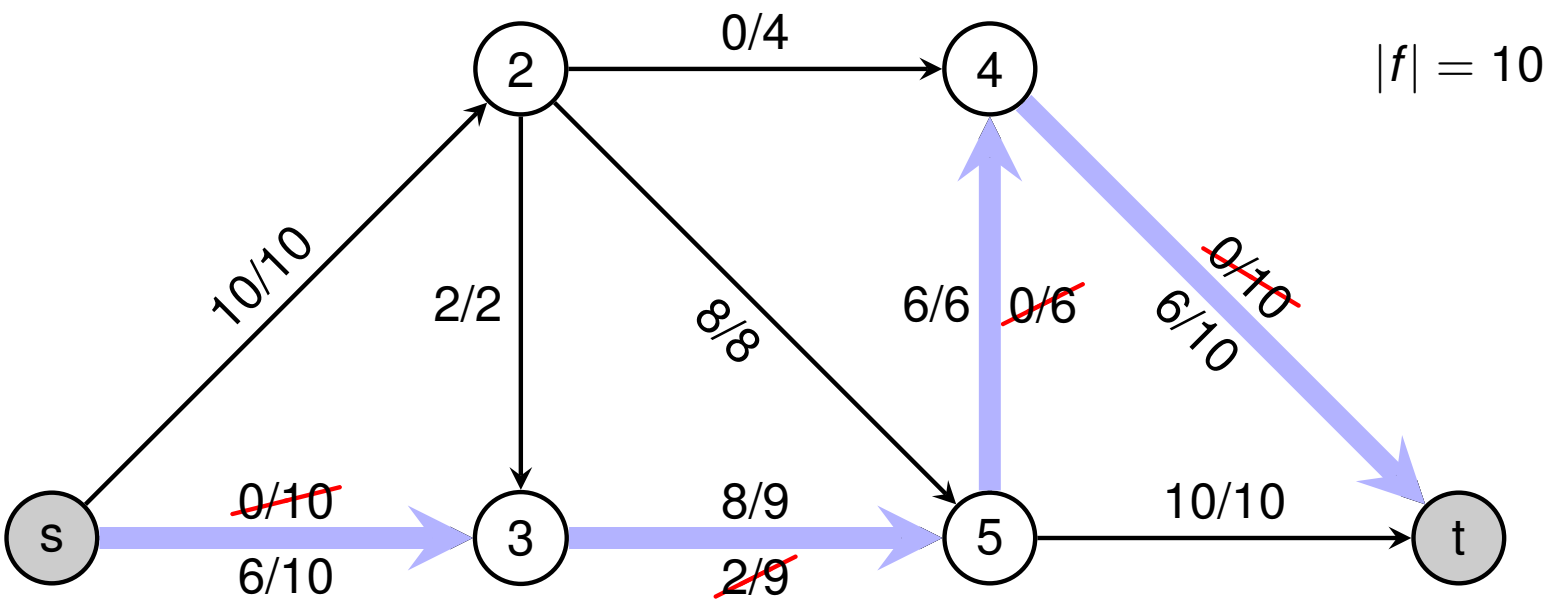

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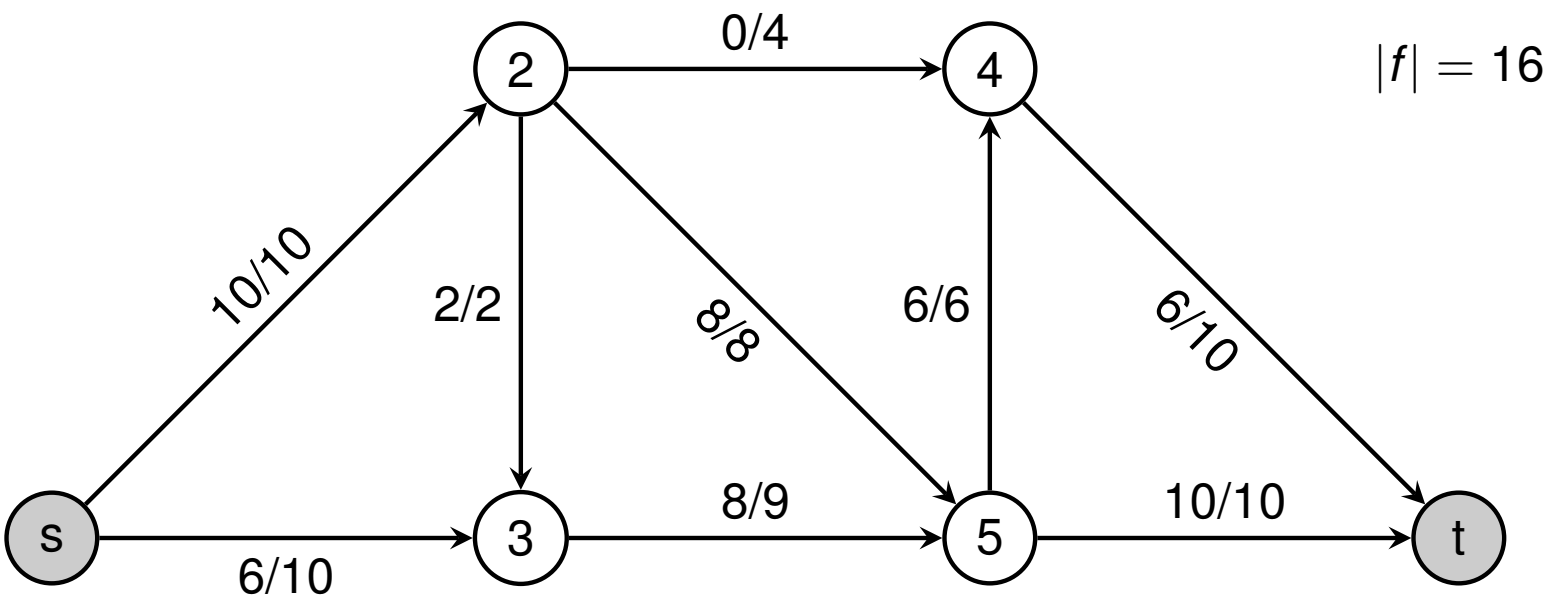

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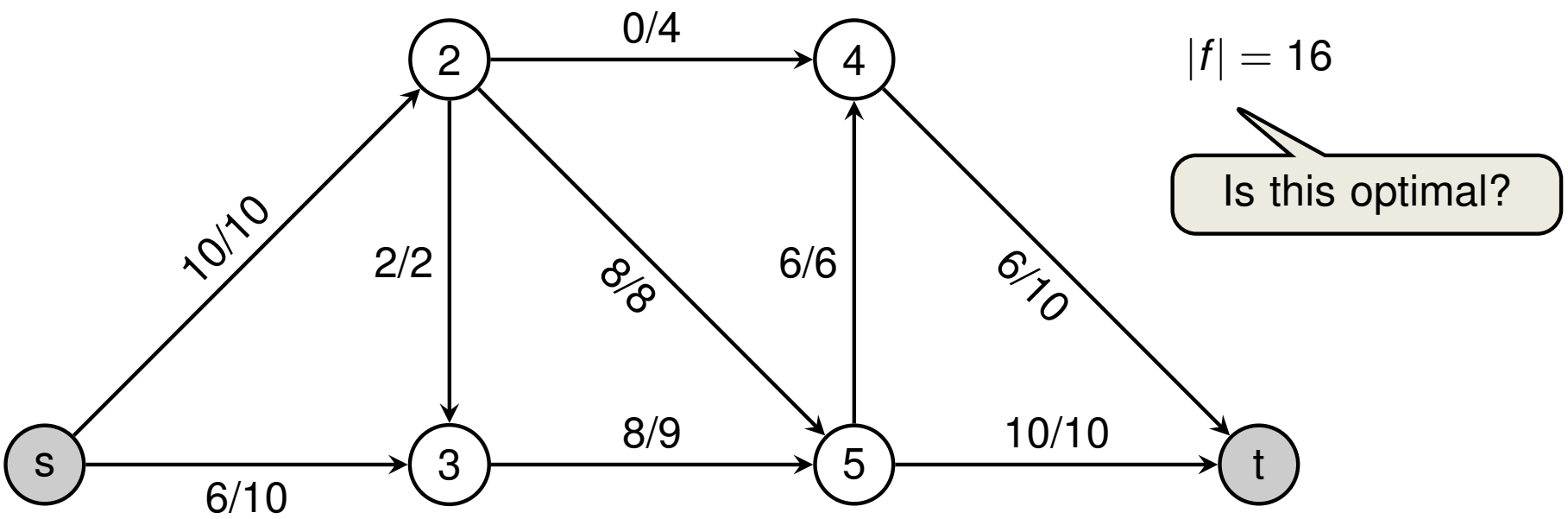

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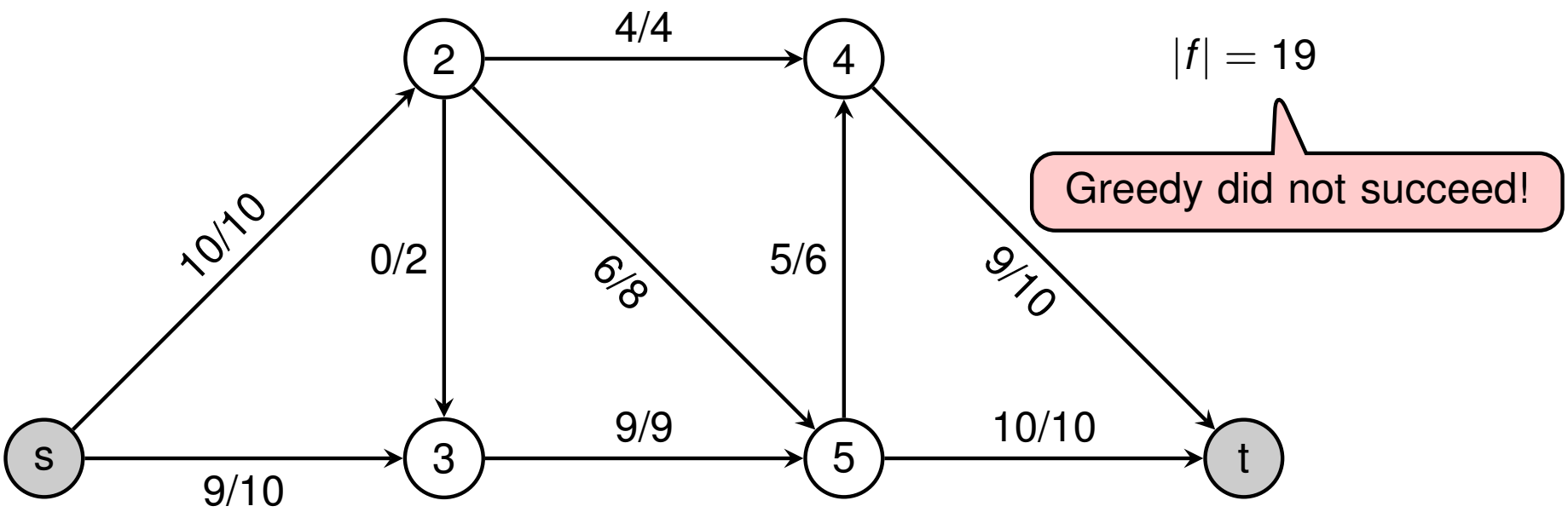

Outline

Residual Graph

Original Edge

Edge $e = (u, v) \in E$

- flow $f(u, v)$ and capacity $c(u, v)$

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Edge $e = (u, v) \in E$

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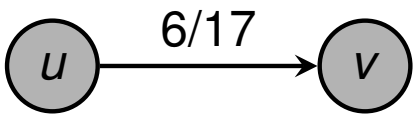


Residual Graph

Original Edge

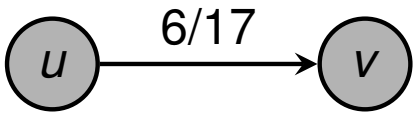
Edge $e = (u, v) \in E$

- flow $f(u, v)$ and capacity $c(u, v)$

Residual Capacity

$$c_f(u, v) = \begin{cases} c(u, v) - f(u, v) & \text{if } (u, v) \in E\,, \\ f(v, u) & \text{if } (v, u) \in E\,, \\ 0 & \text{otherwise.} \end{cases}$$

Graph G:



Residual Graph

Original Edge

Edge $e = (u, v) \in E$

- flow $f(u, v)$ and capacity $c(u, v)$

Residual Capacity

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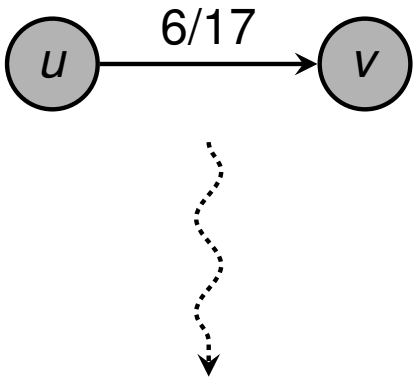

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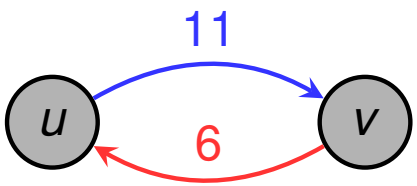

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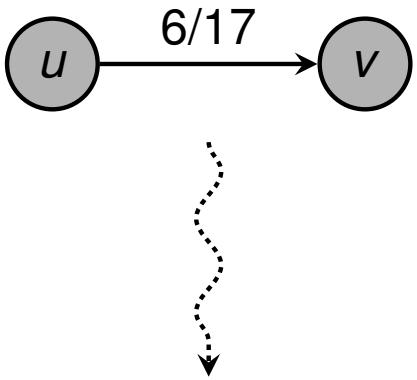

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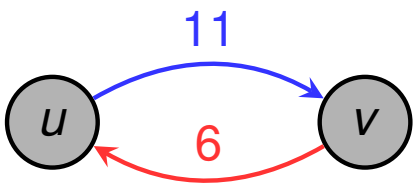

Residual Graph with anti-parallel edges

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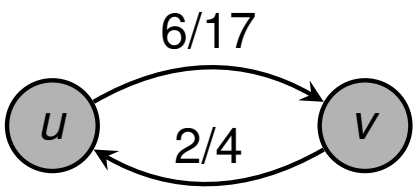
Graph G:



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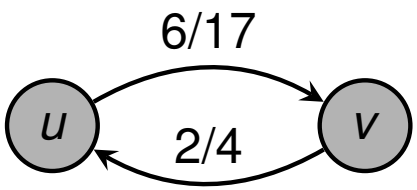

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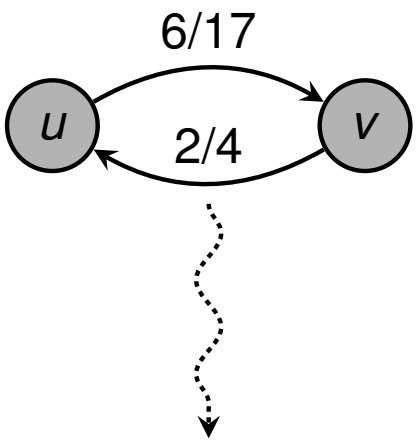

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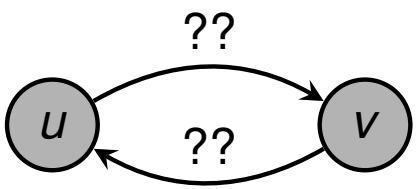

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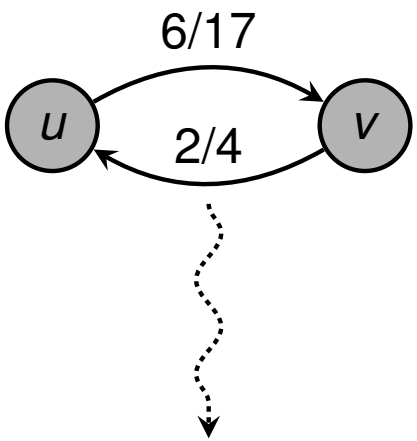

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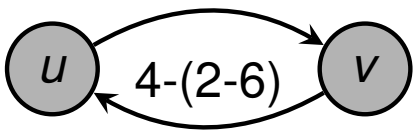

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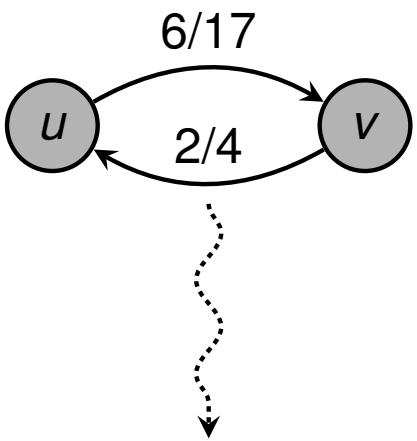

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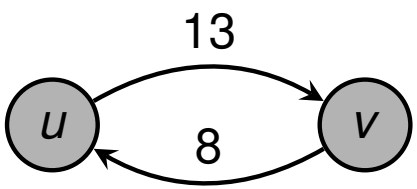

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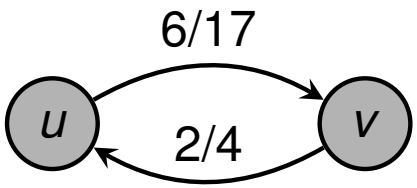

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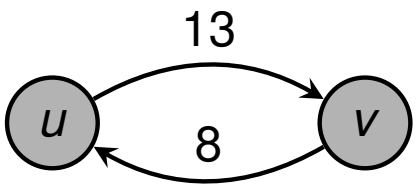
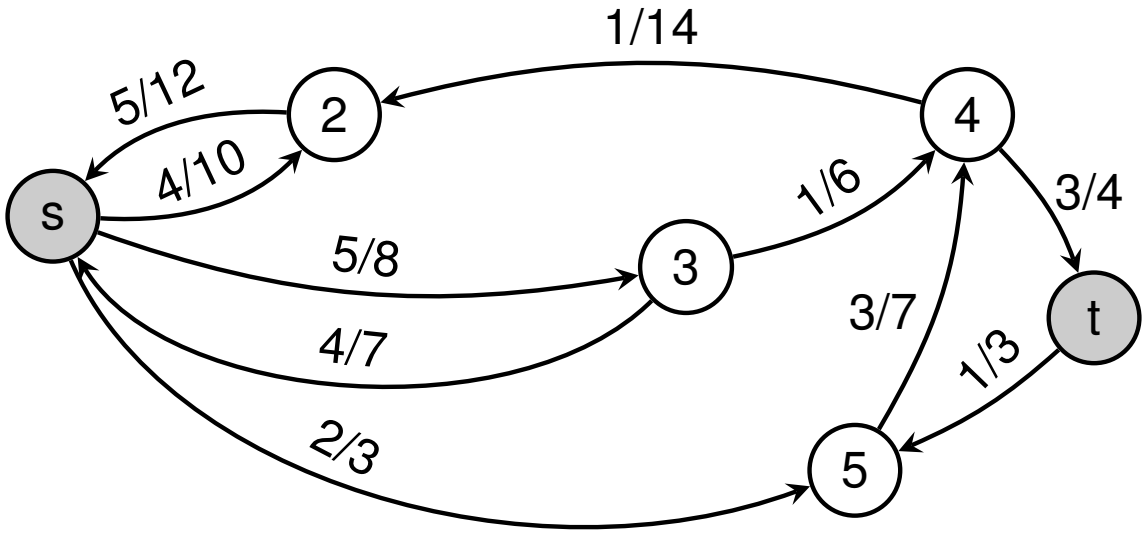
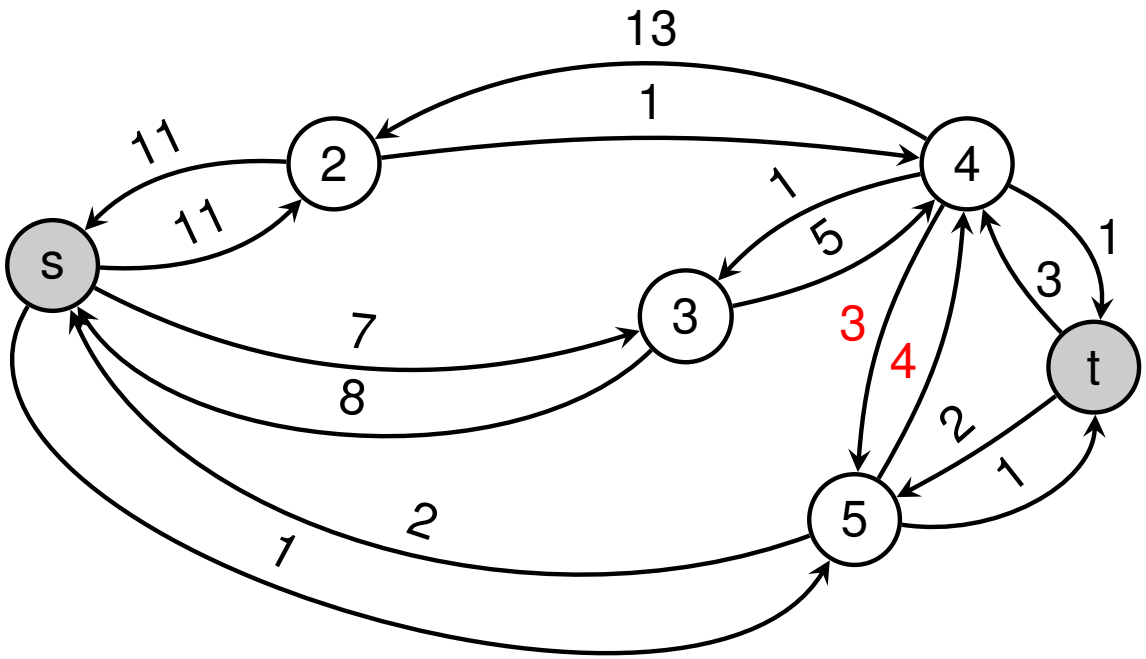


Example of a Residual Graph (Handout)

Flow network G



Residual Graph G_f



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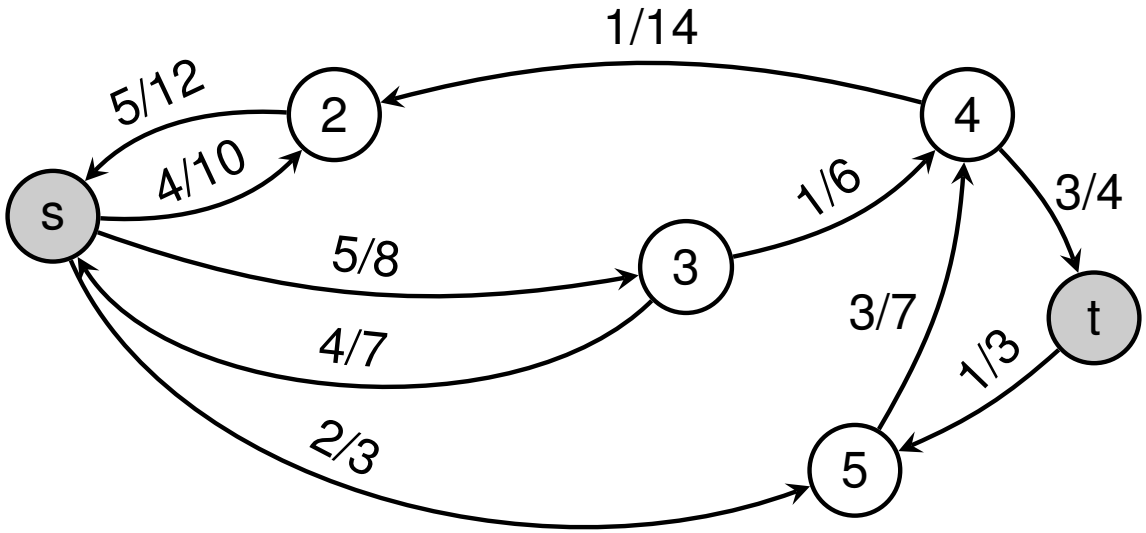

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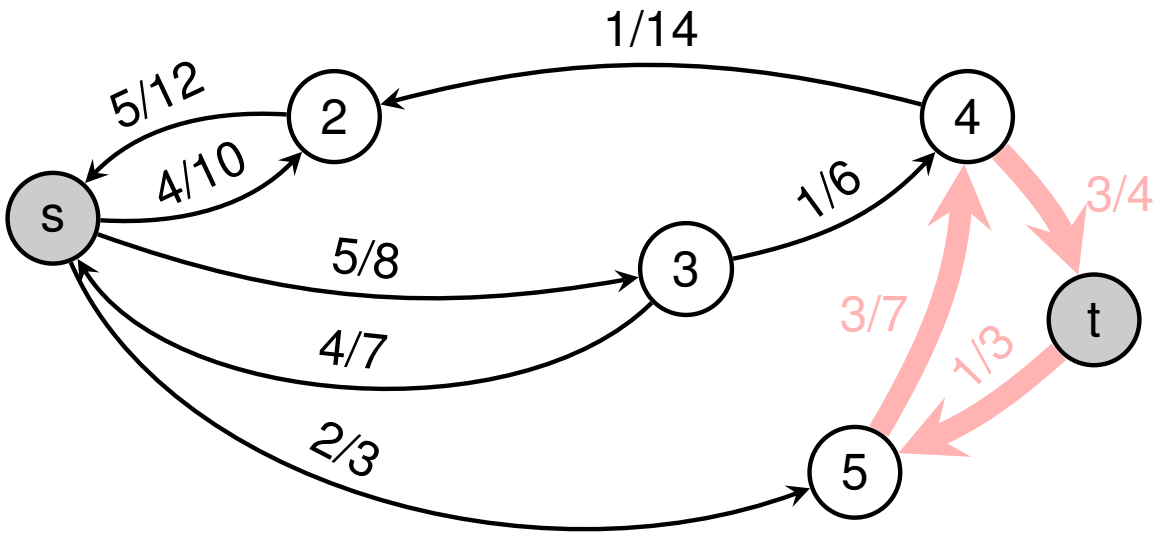

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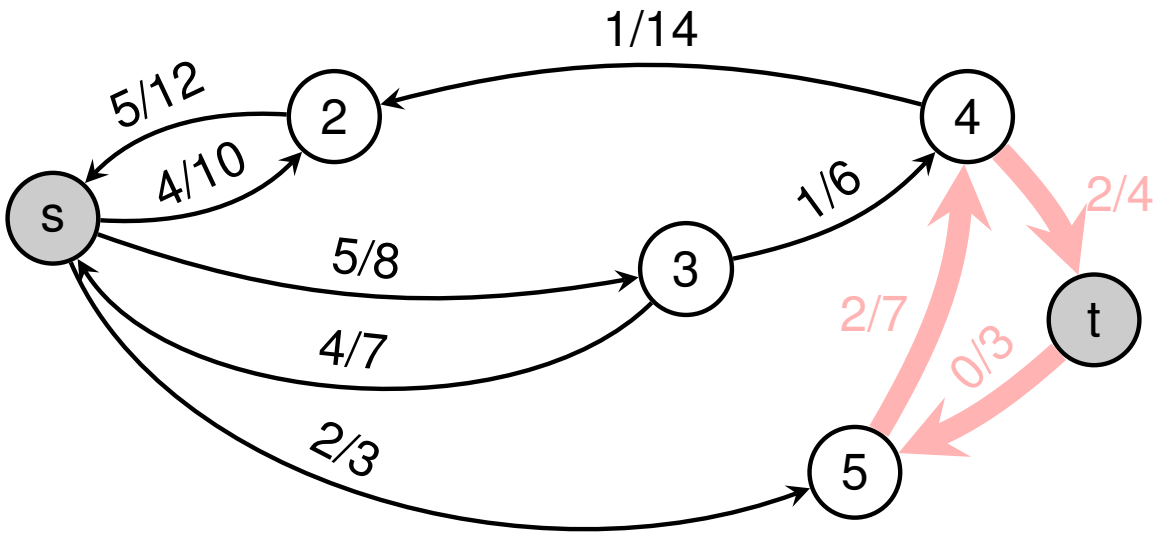

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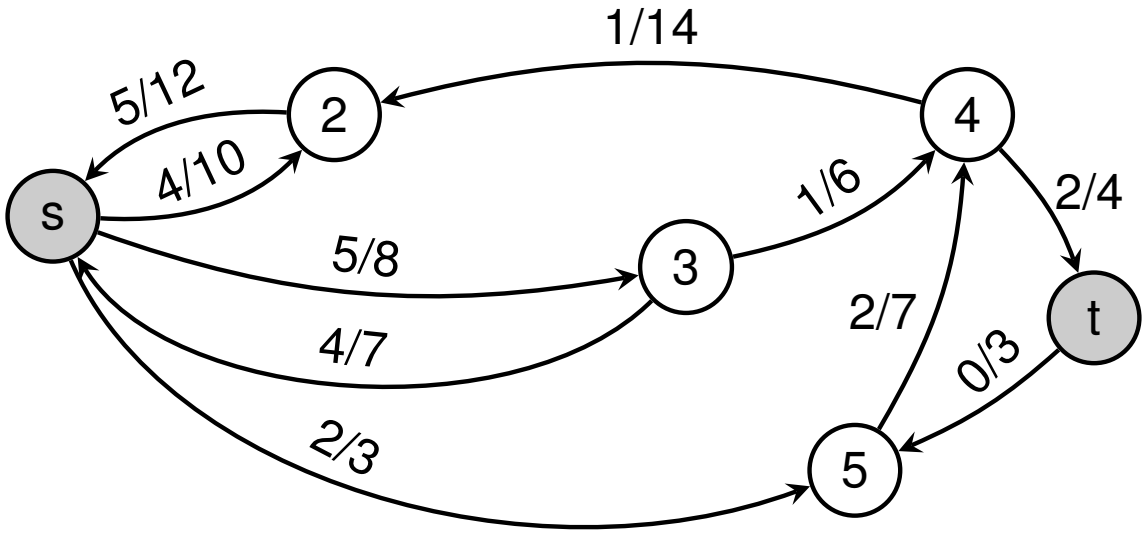

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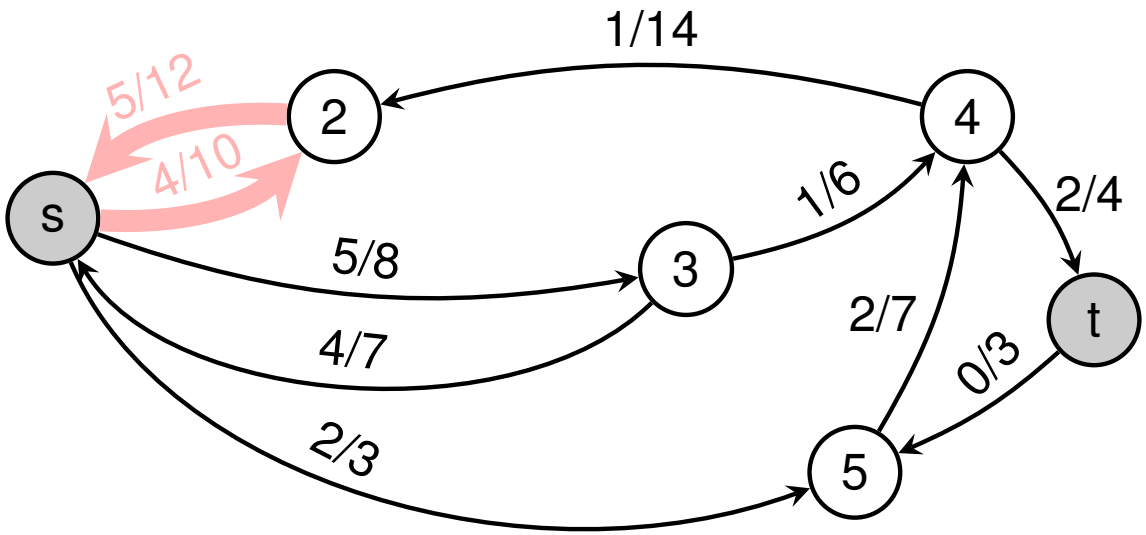

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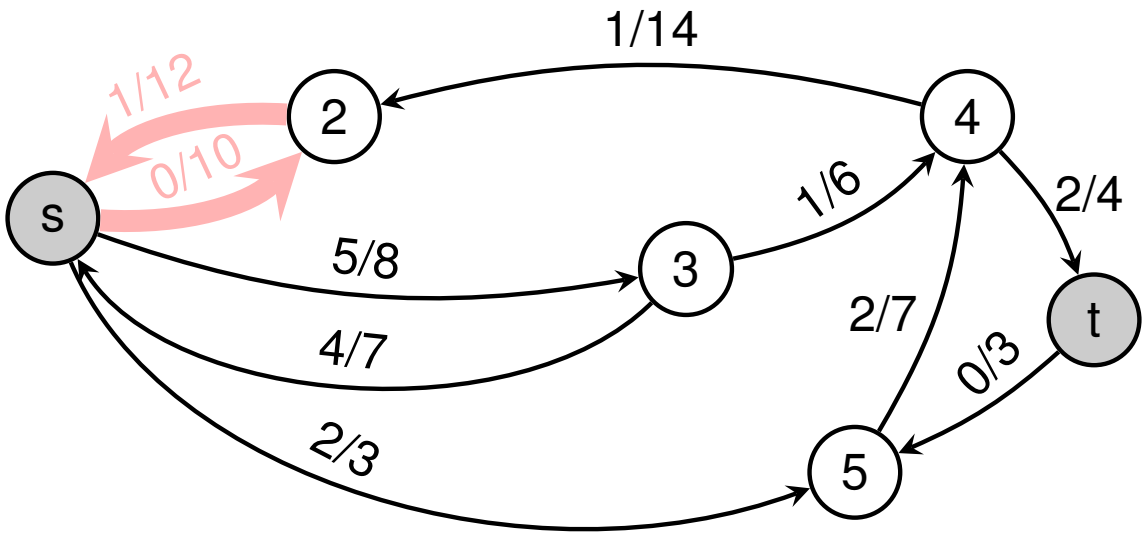

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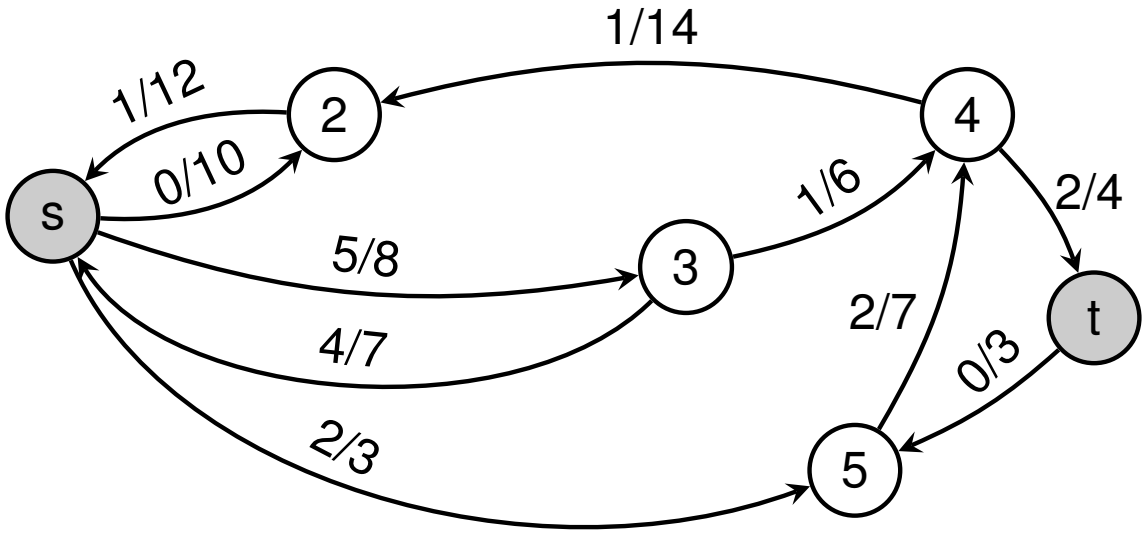

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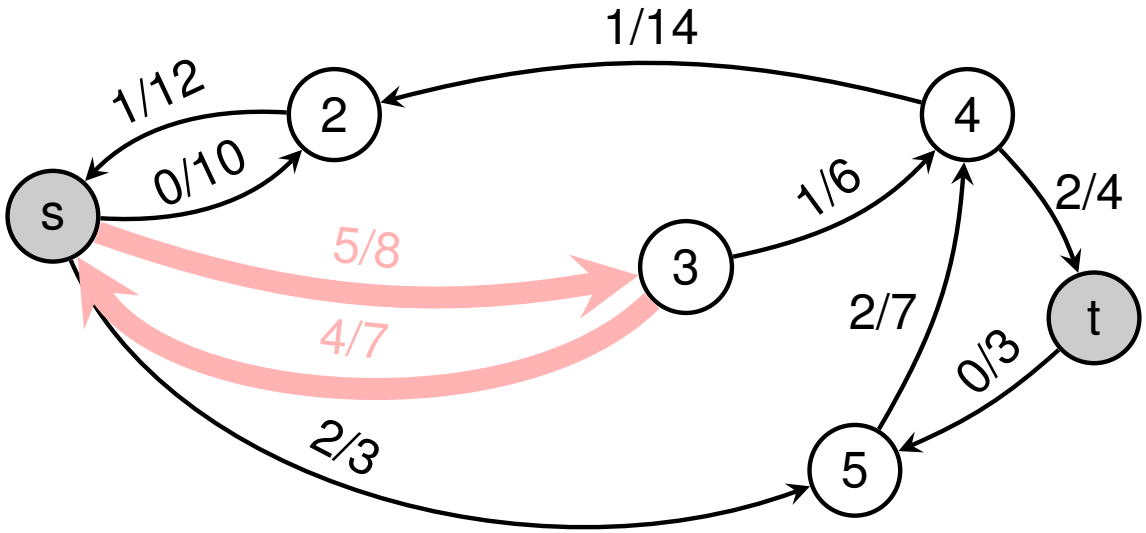

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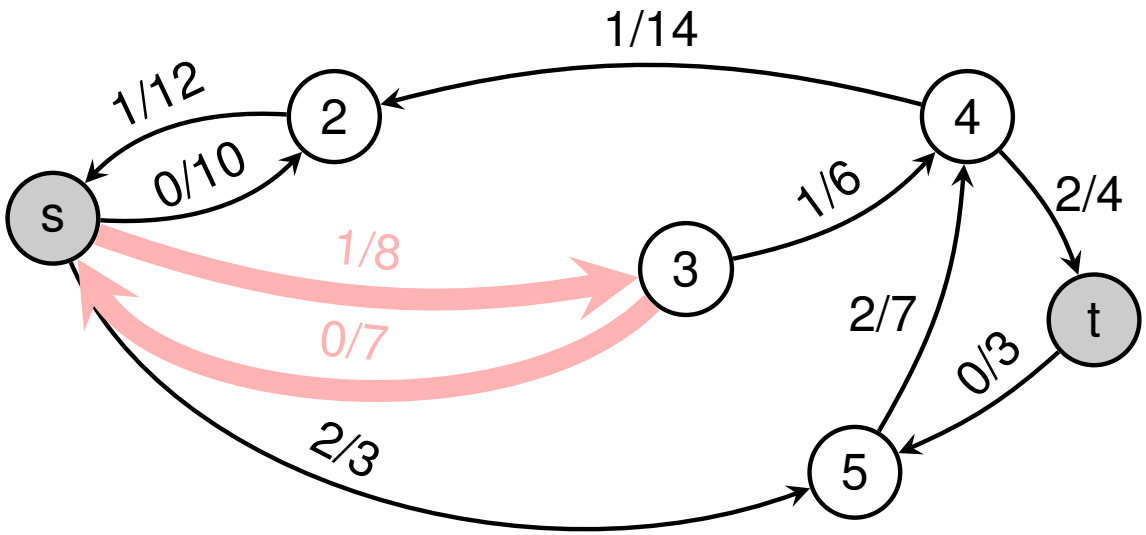

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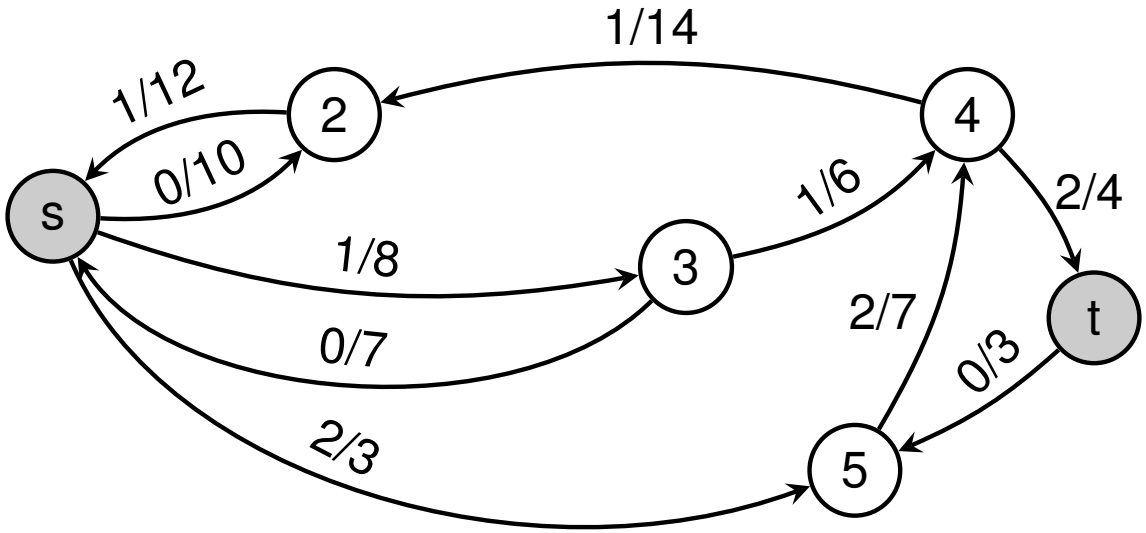

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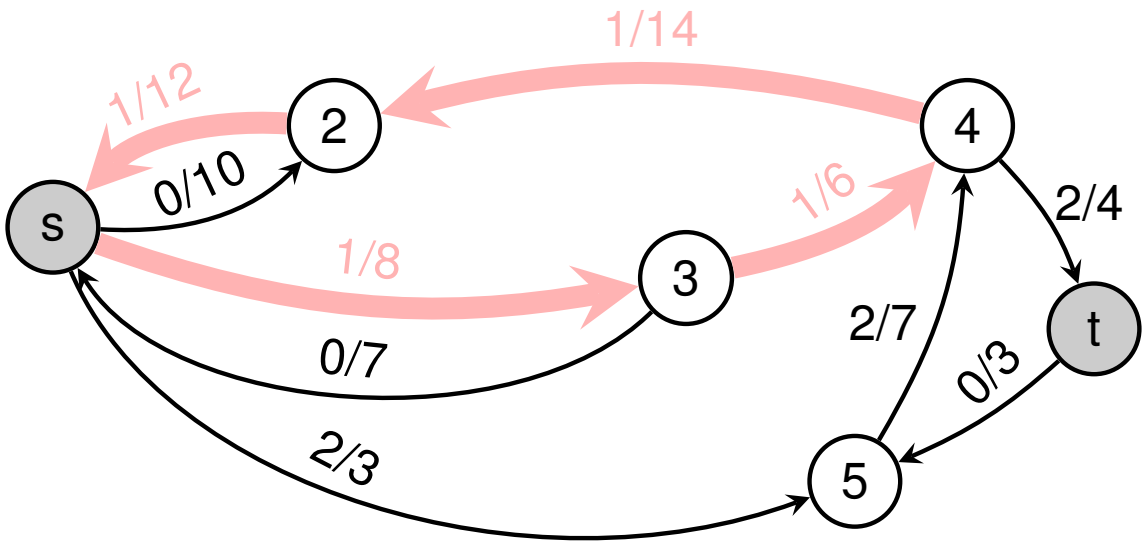

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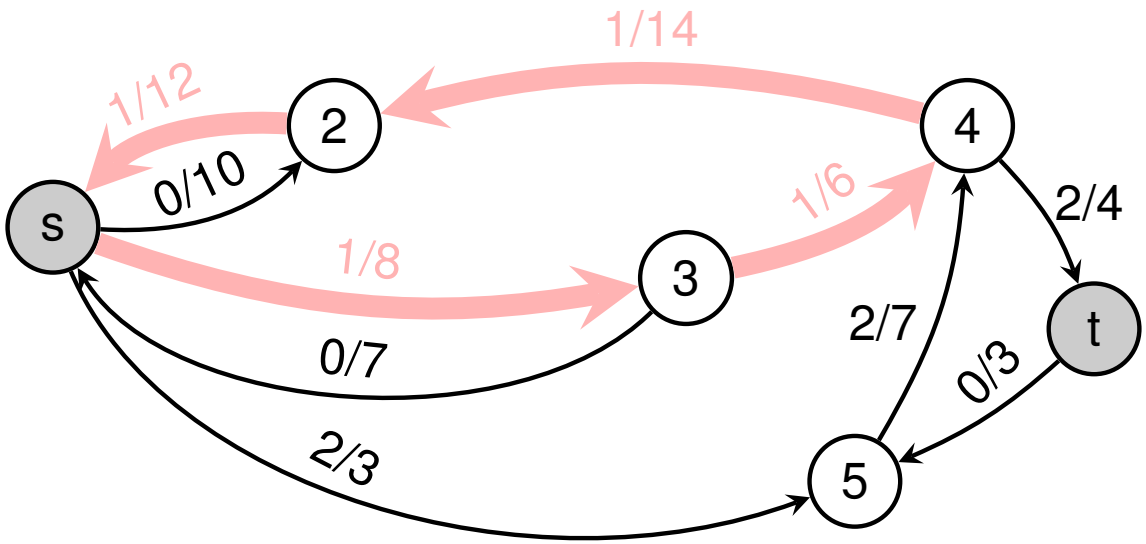

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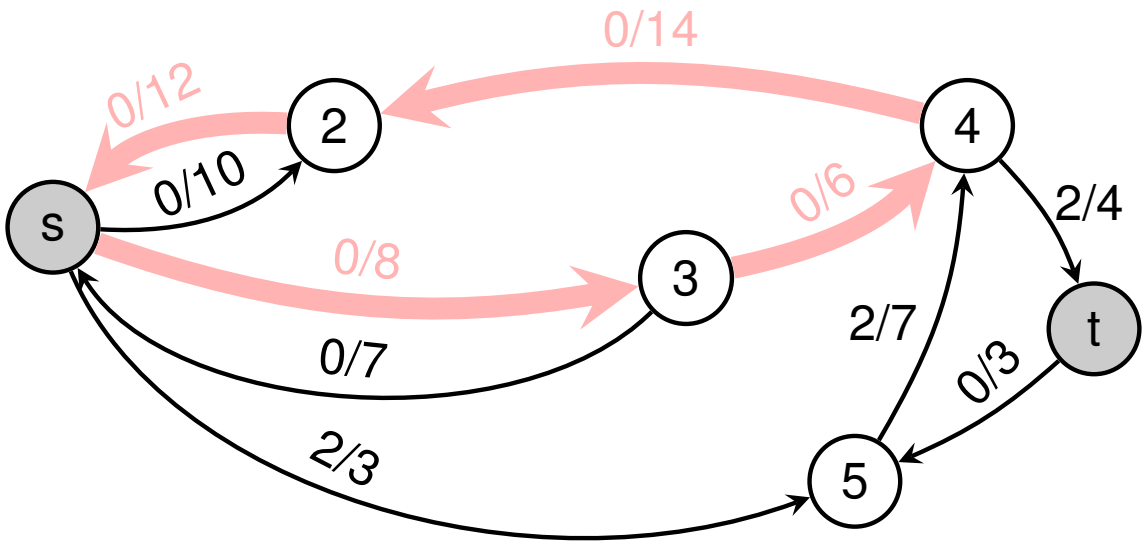

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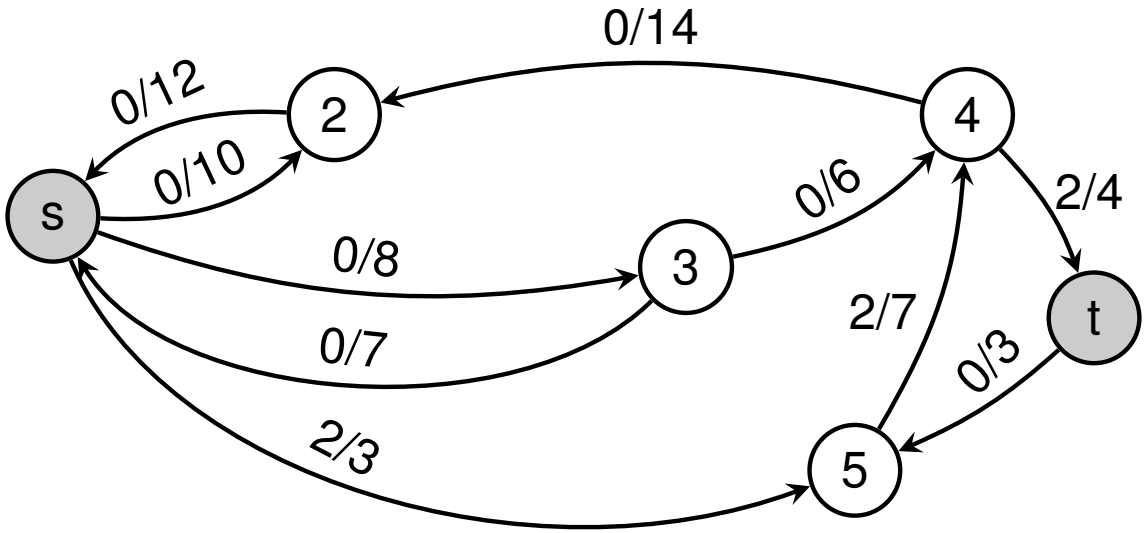

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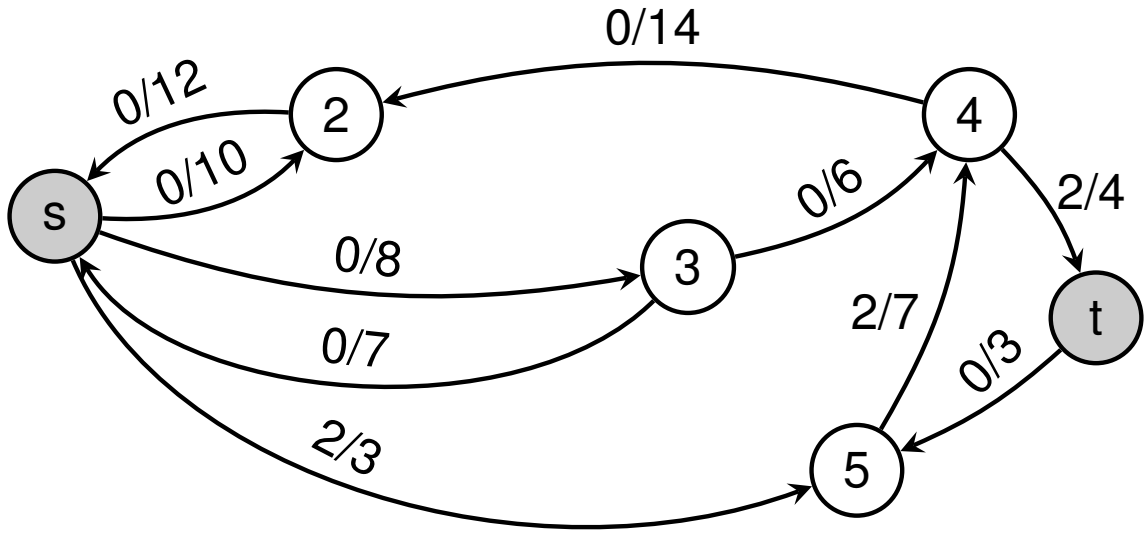

By successively eliminating cycles we can simplify and reduce the “transportation” cost of a flow.

The Ford-Fulkerson Method ("Enhanced Greedy")

```
0: def fordFulkerson(G)
1:     initialize flow to 0 on all edges
2:     while an augmenting path in G_f can be found:
3:         push as much extra flow as possible through it
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Augmenting path: Path from source to sink in G_f

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If f' is a flow in G_f and f a flow in G, then $f + f'$ is a flow in G

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Questions:

- How to find an augmenting path?
- Does this method terminate?
- If it terminates, how good is the solution?

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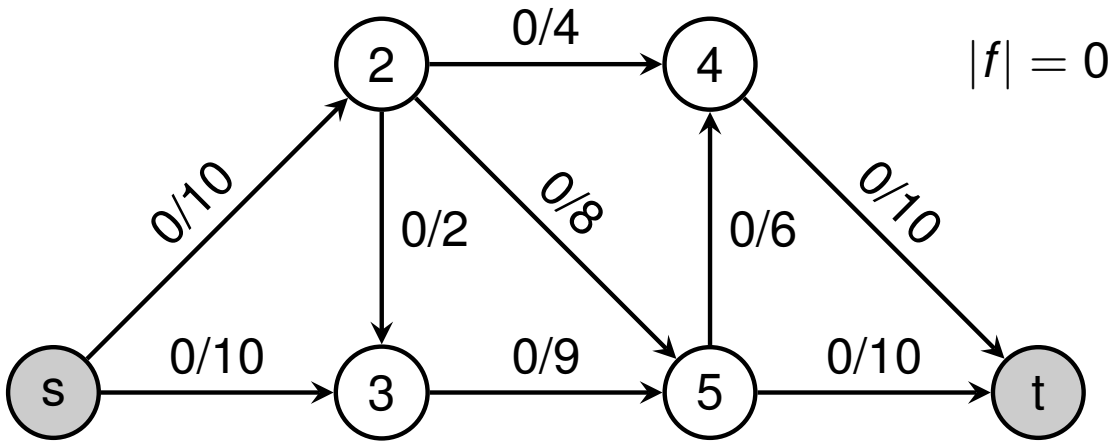
Using BFS or DFS, we can find an augmenting path in $O(V + E)$ time.

Questions:

- How to find an augmenting path?
- Does this method terminate?
- If it terminates, how good is the solution?

Illustration of the Ford-Fulkerson Method

Graph $G = (V, E, c)$:



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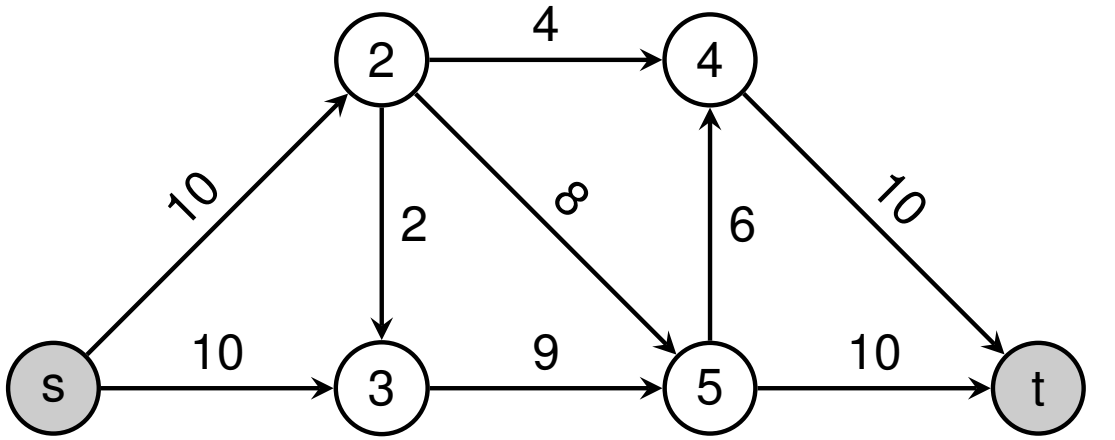

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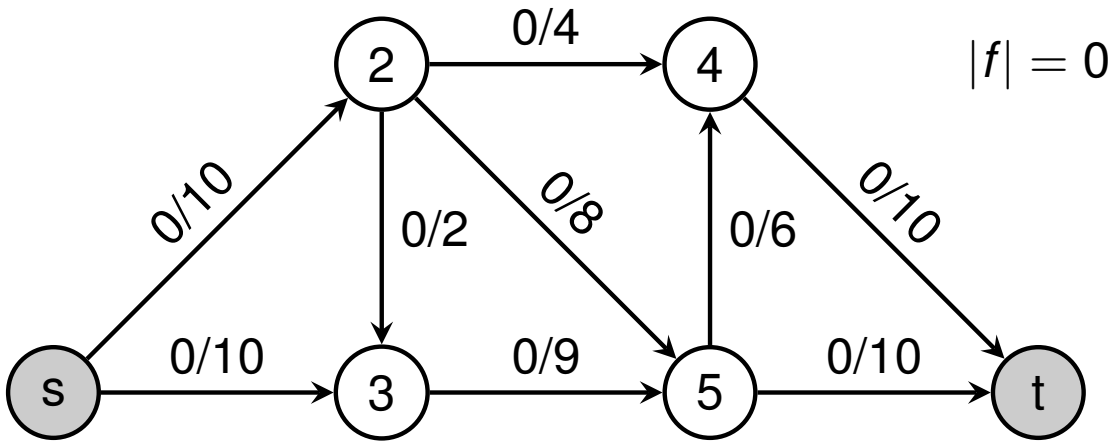

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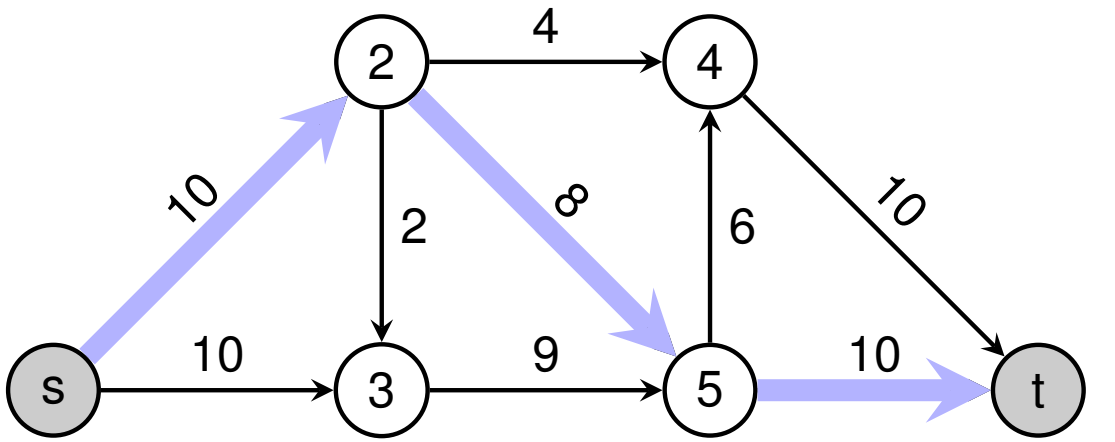

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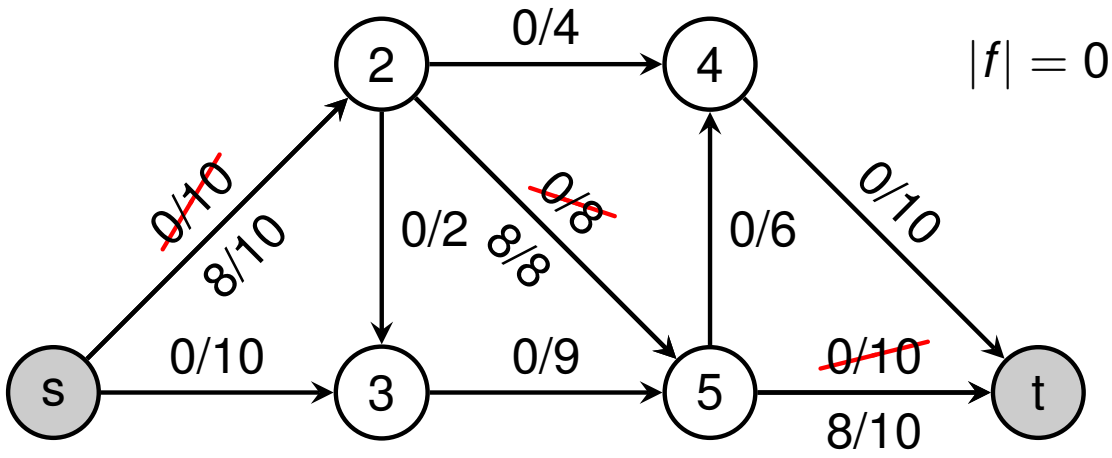

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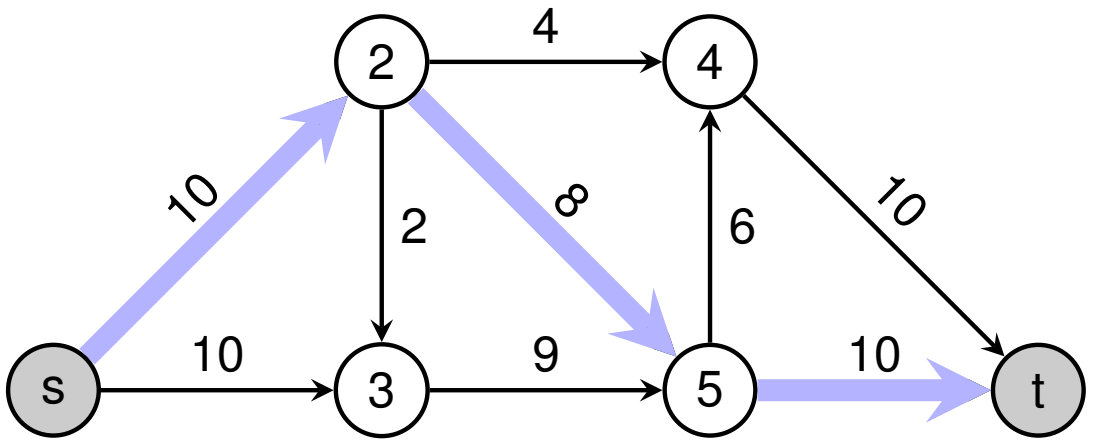

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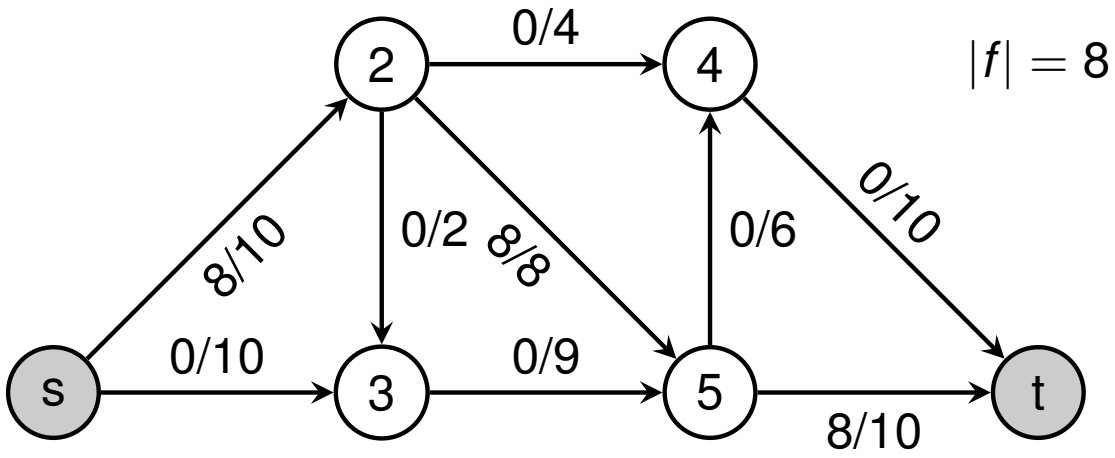

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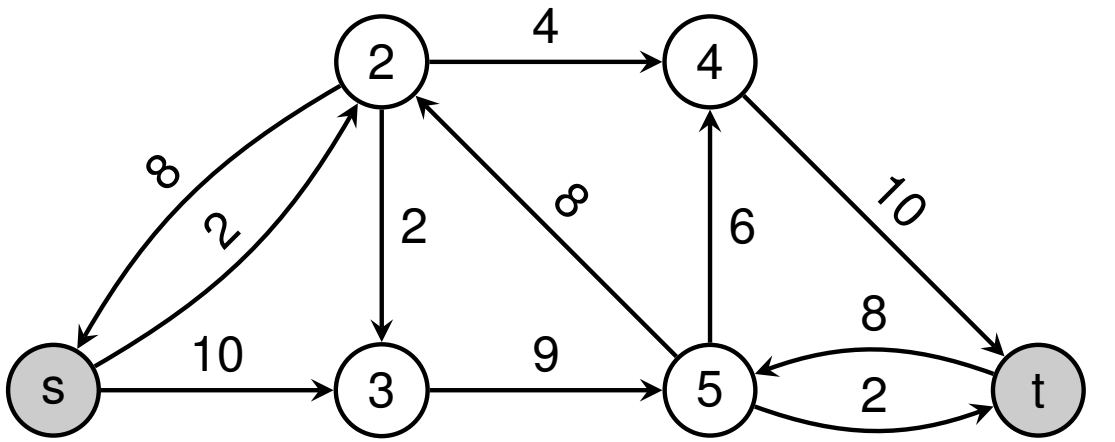

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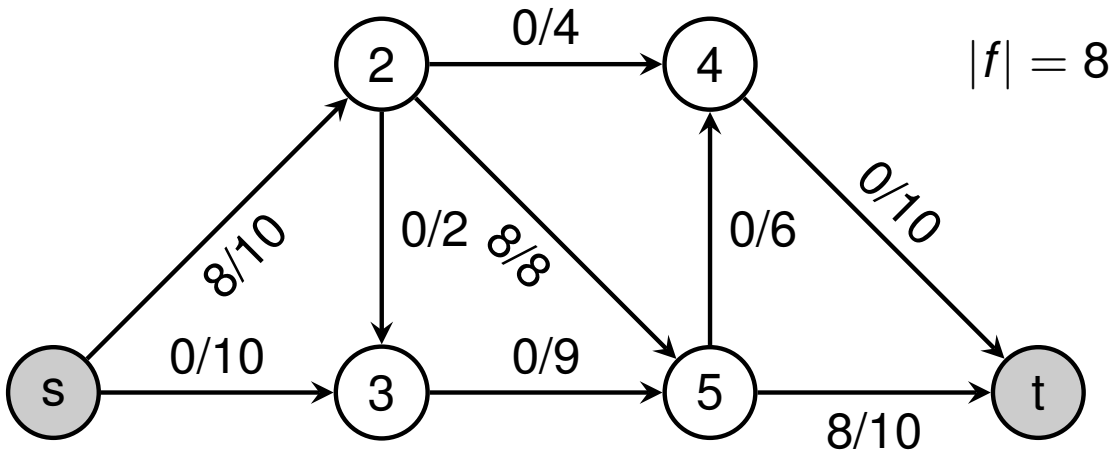

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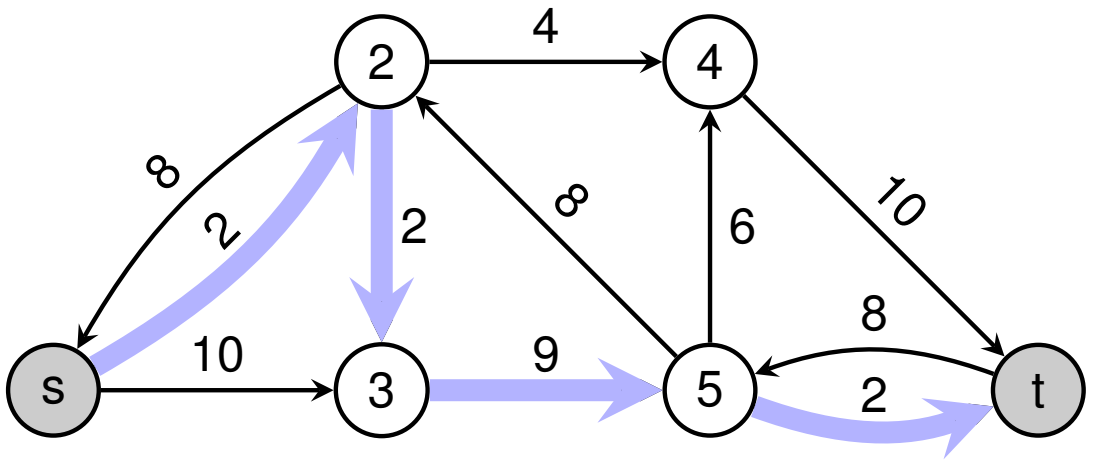

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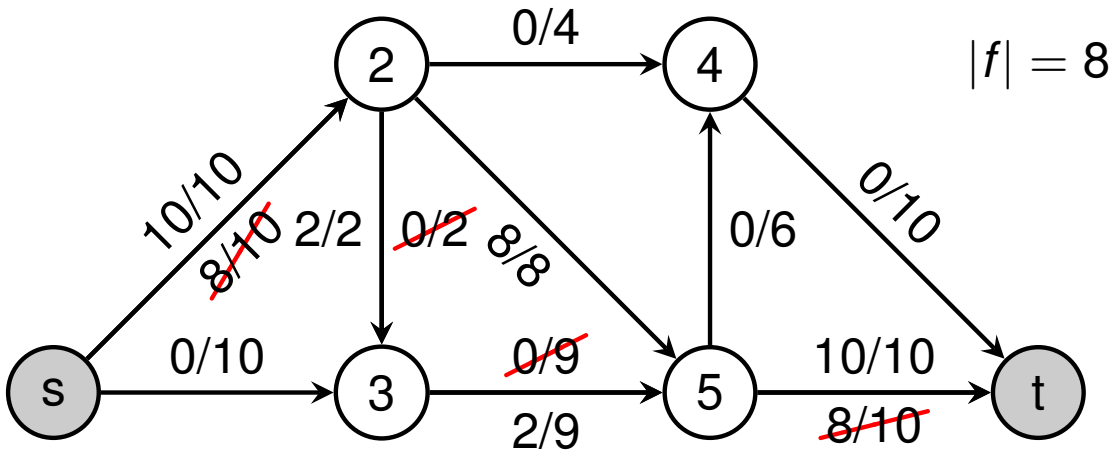

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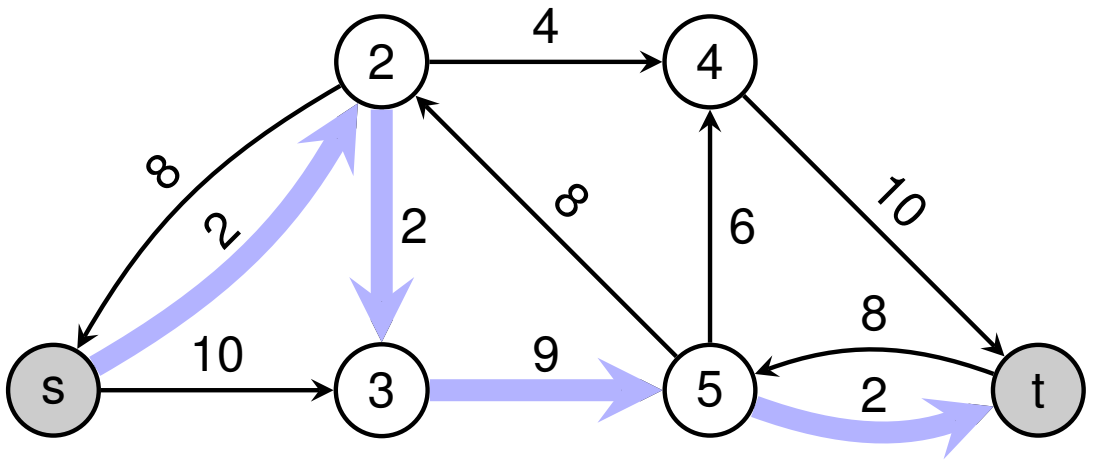

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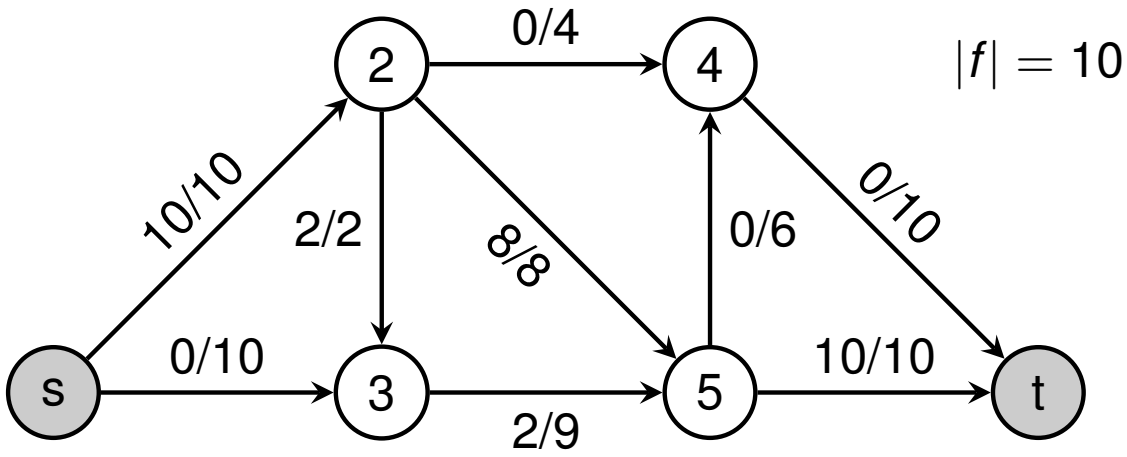

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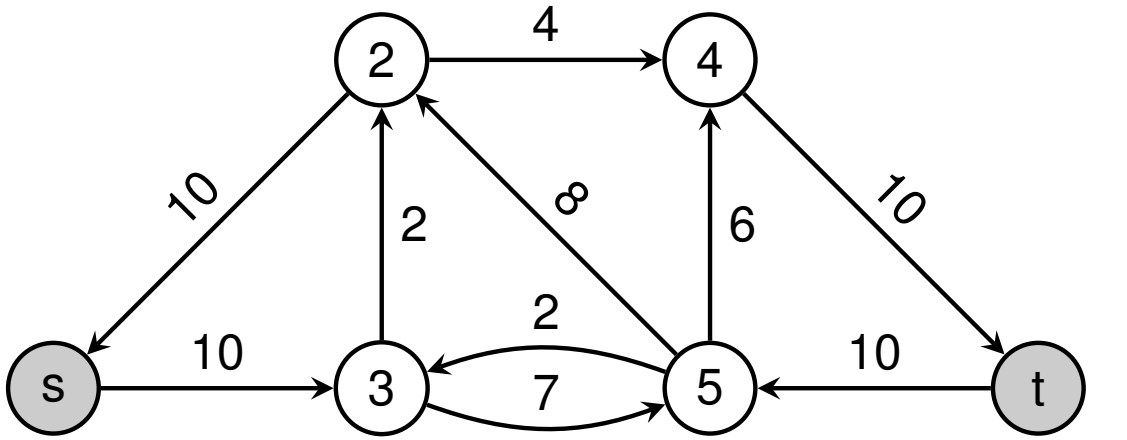

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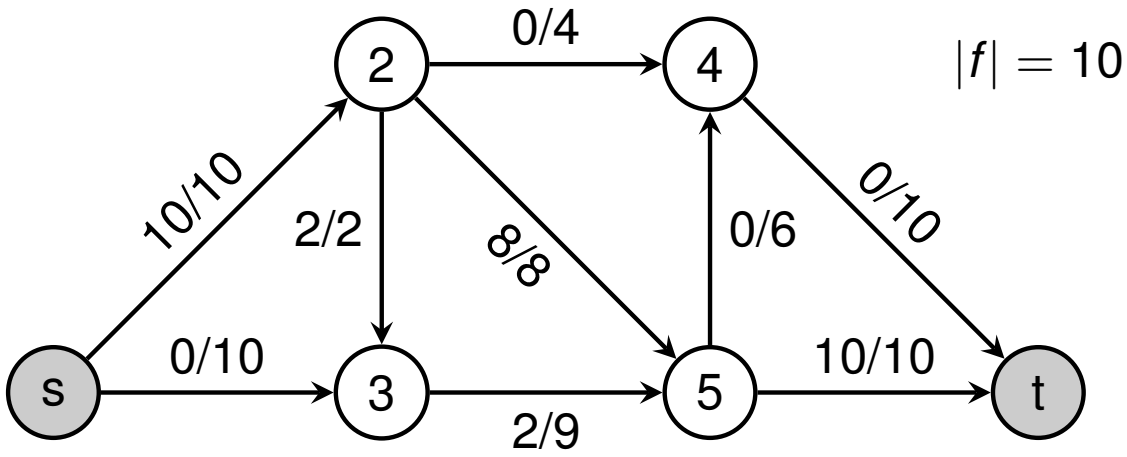

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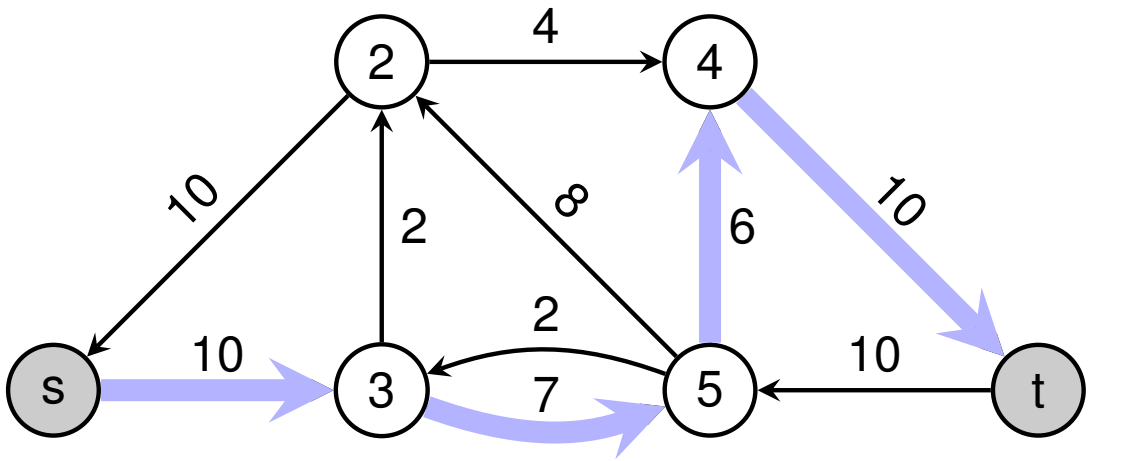

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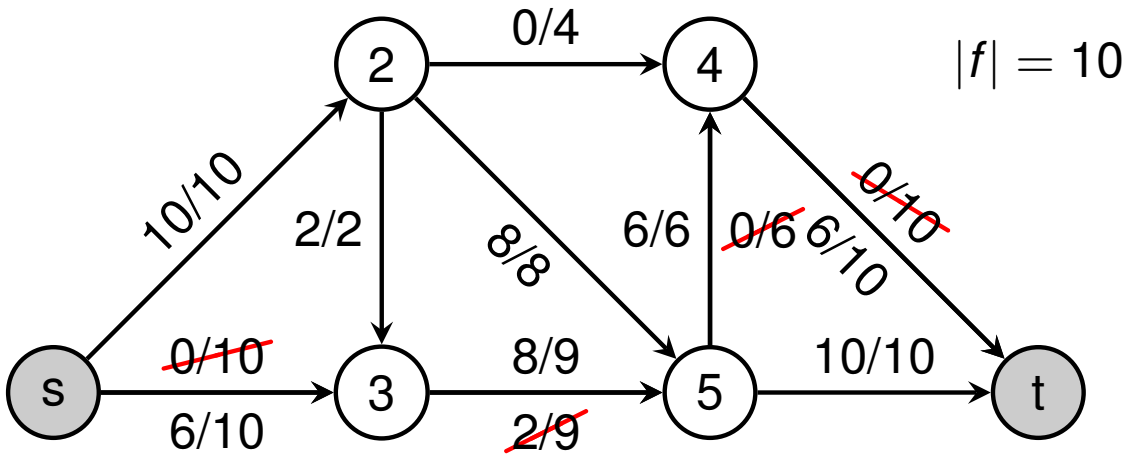

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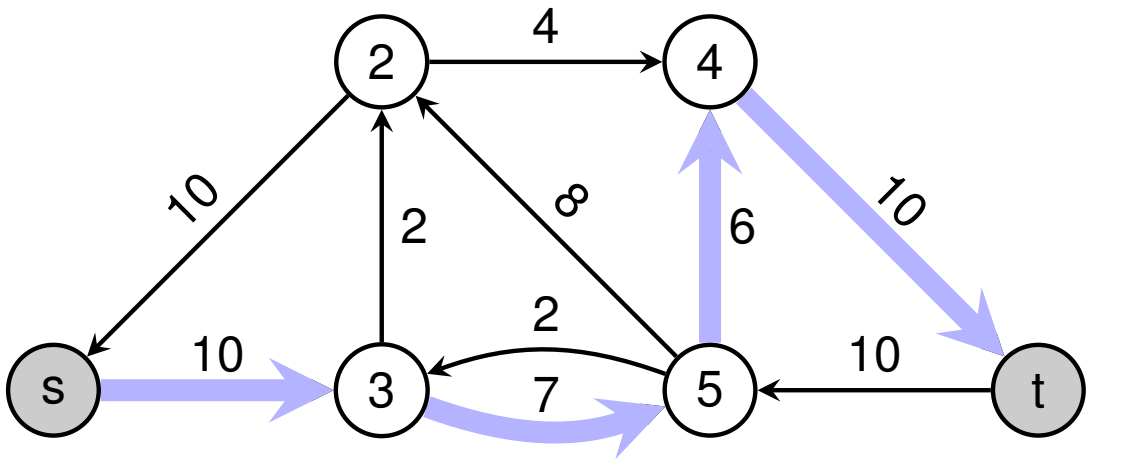

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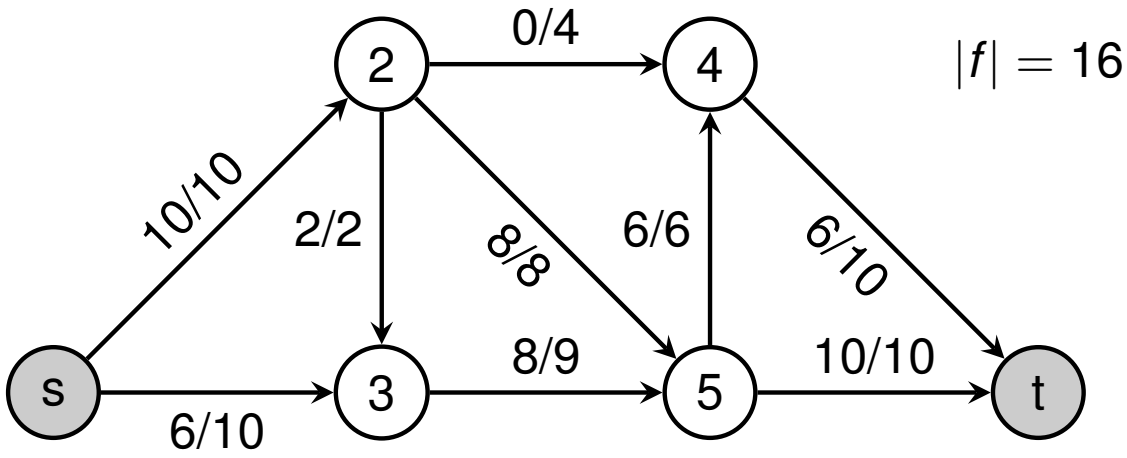

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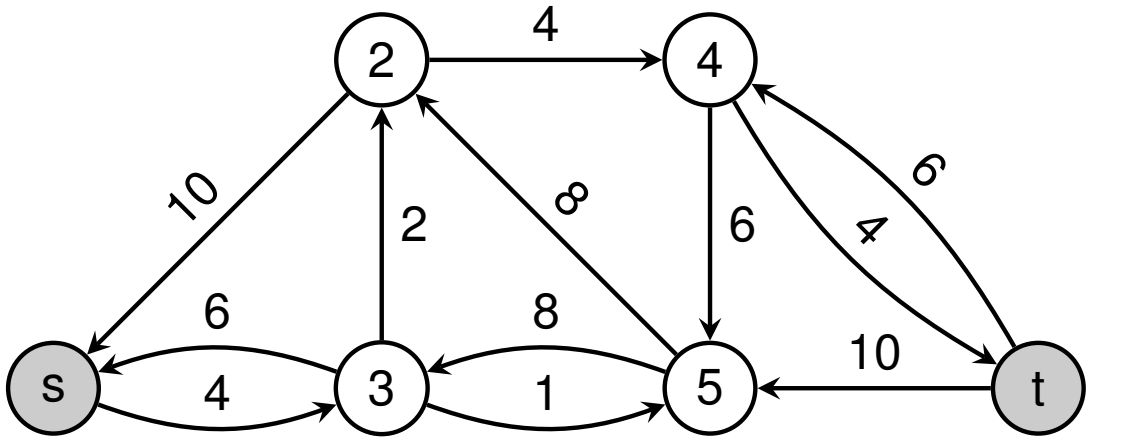

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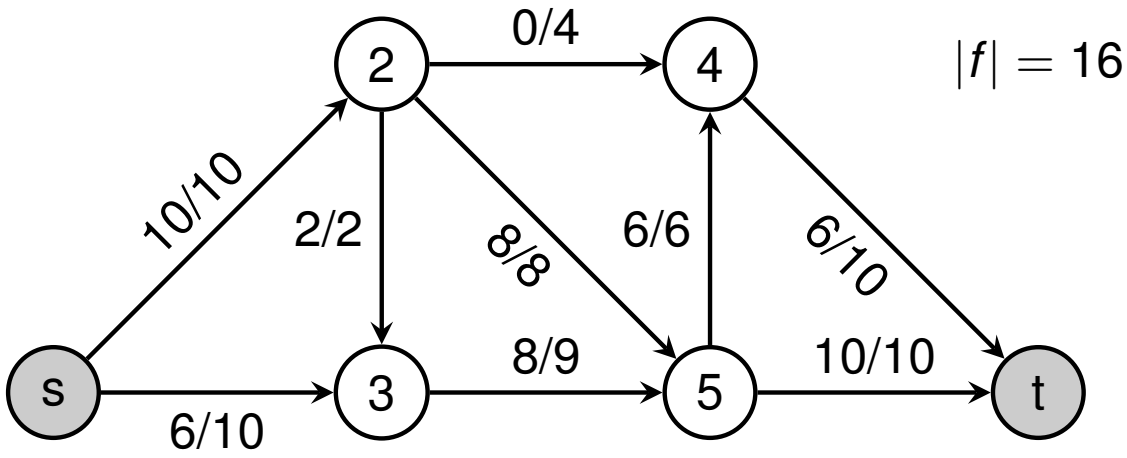

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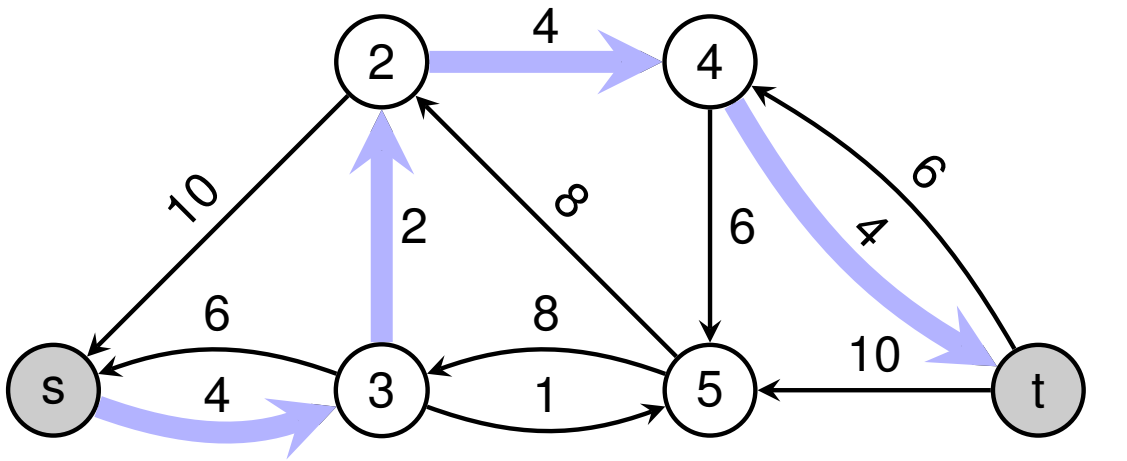

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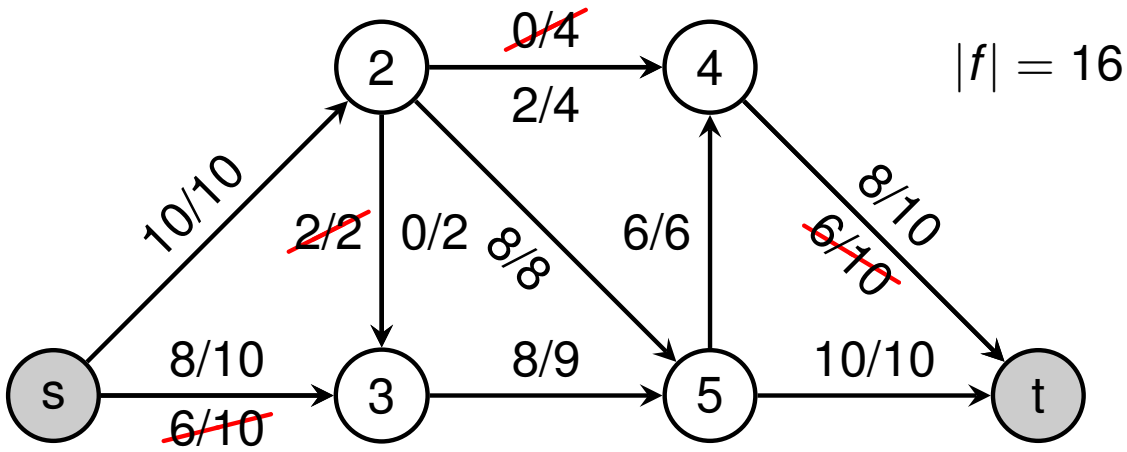

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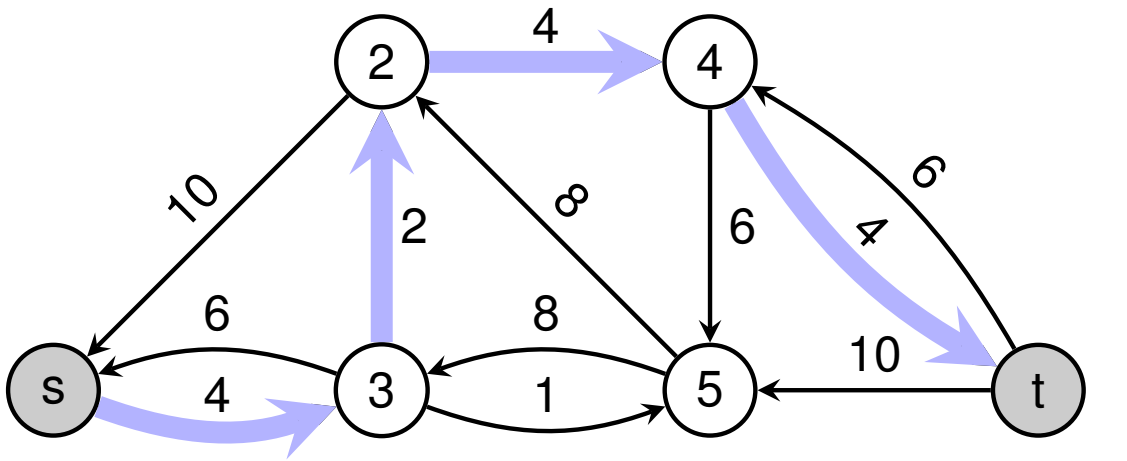

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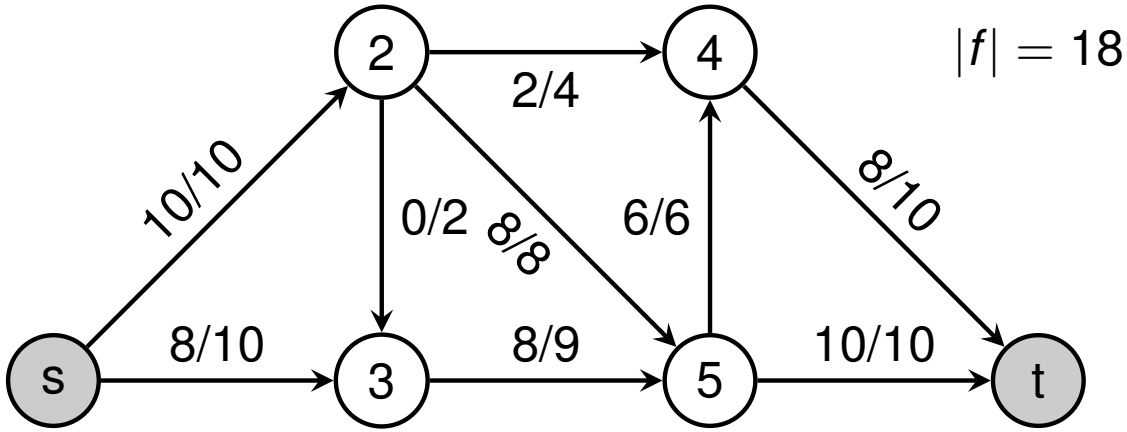

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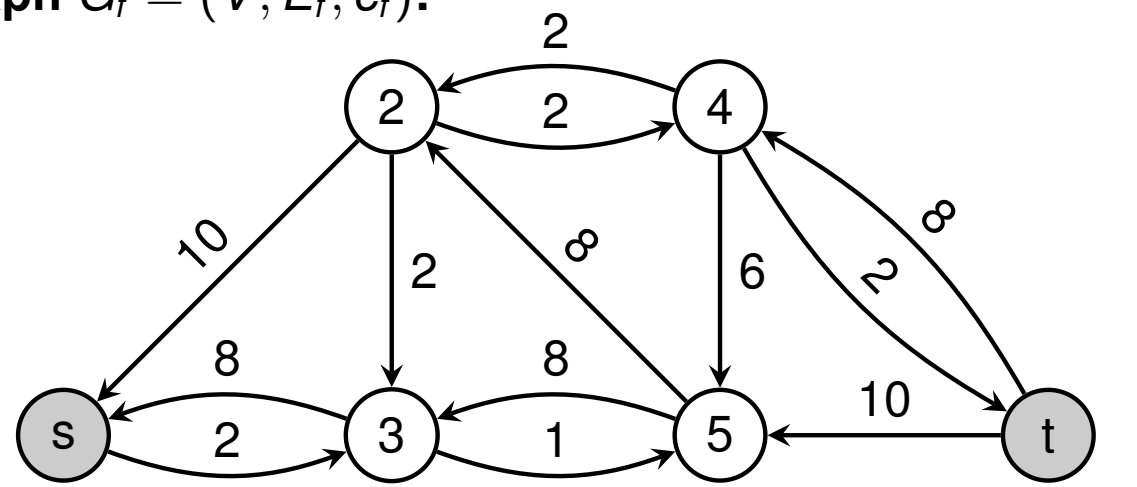

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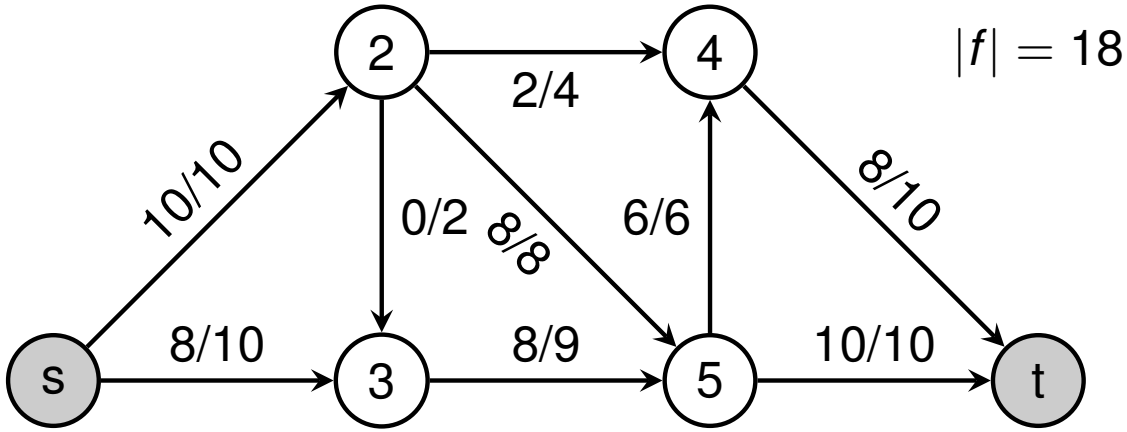

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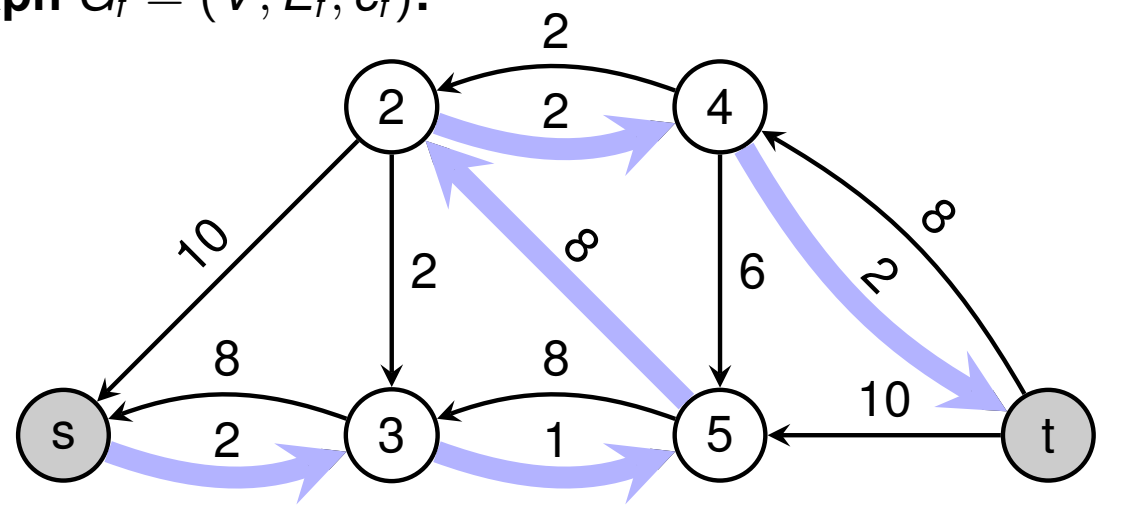

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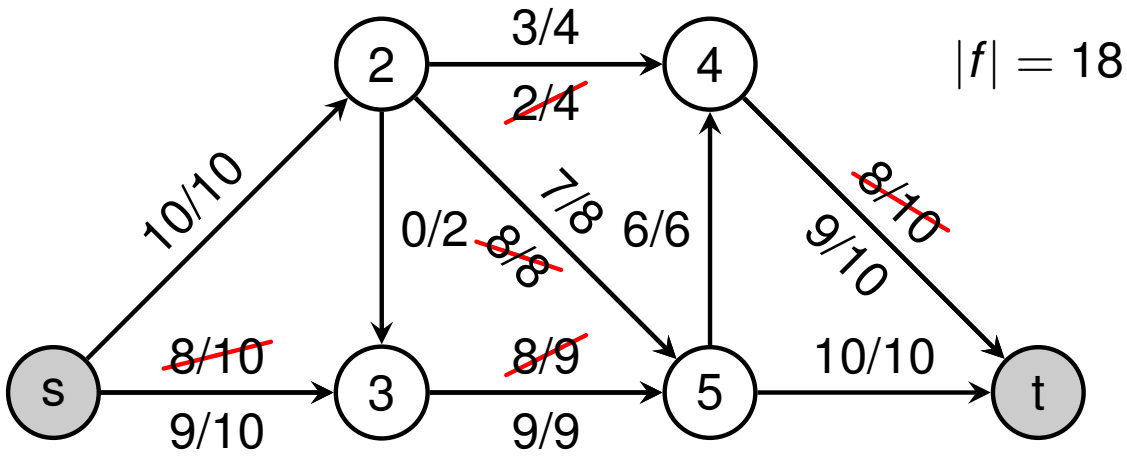

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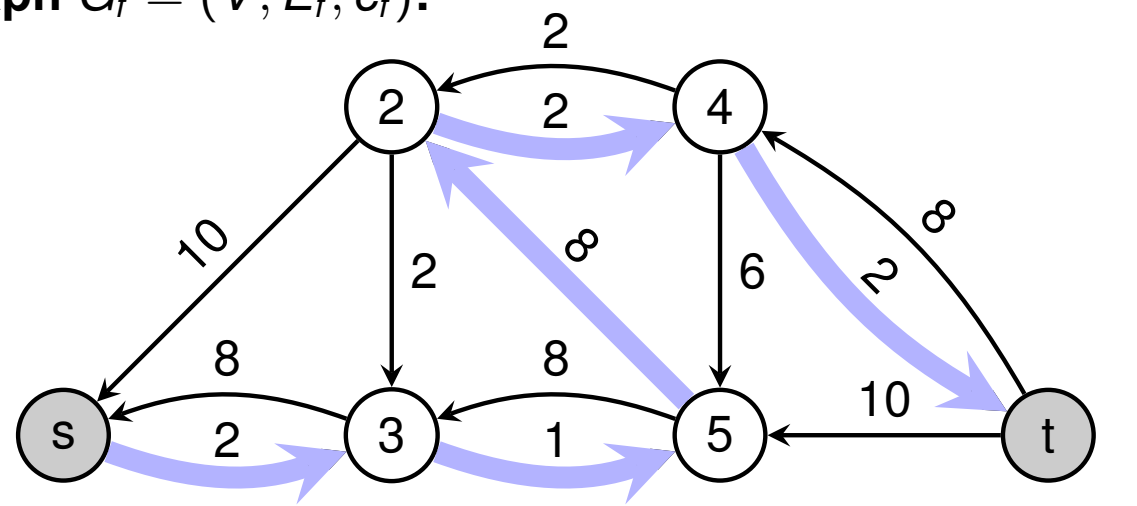

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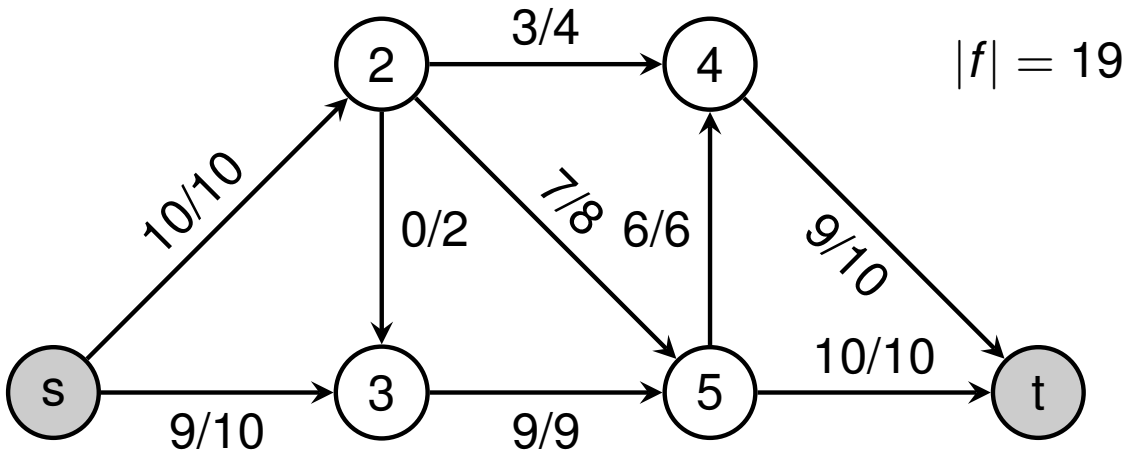

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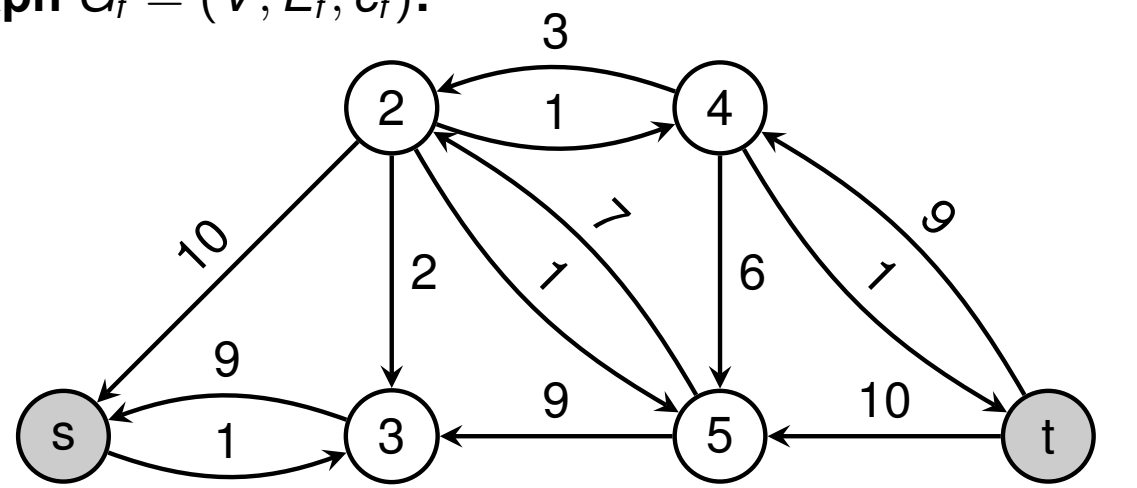

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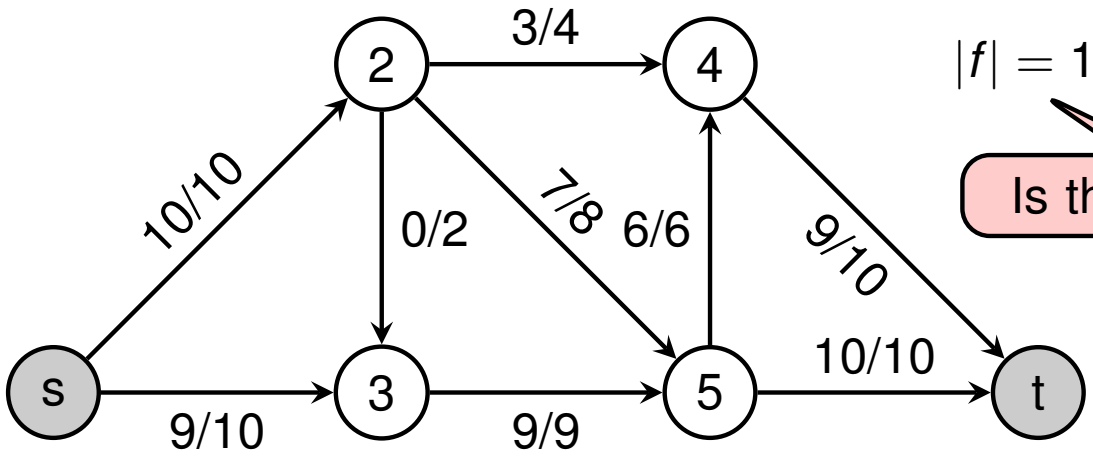

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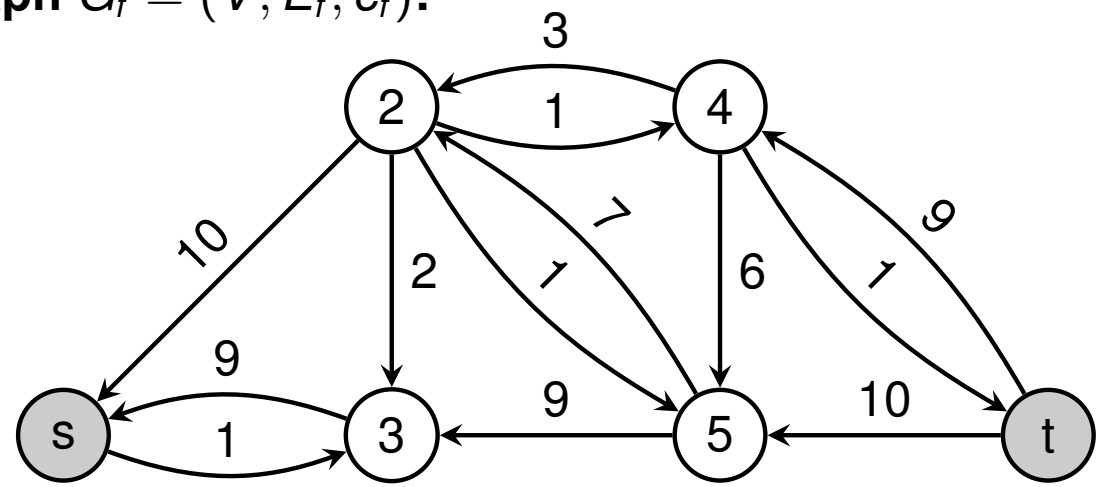

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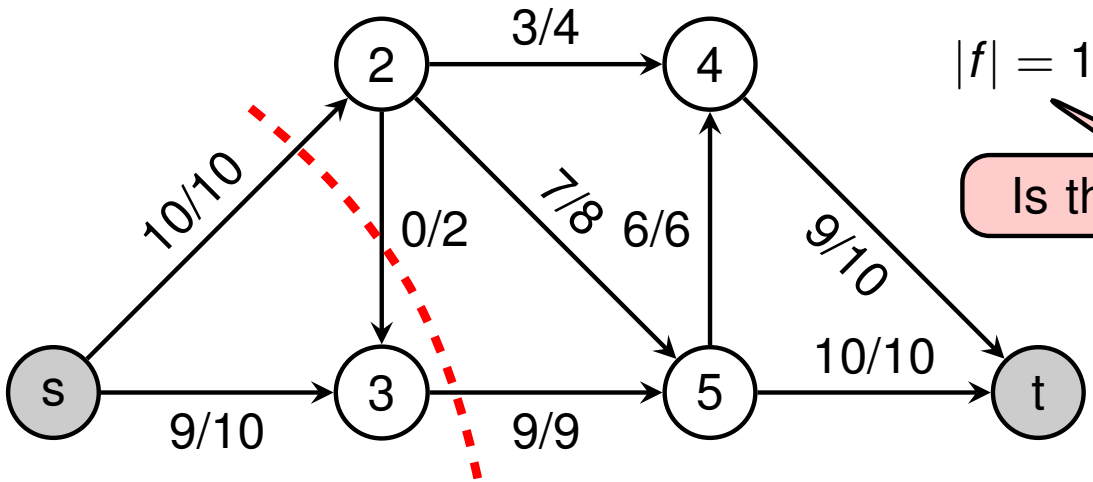

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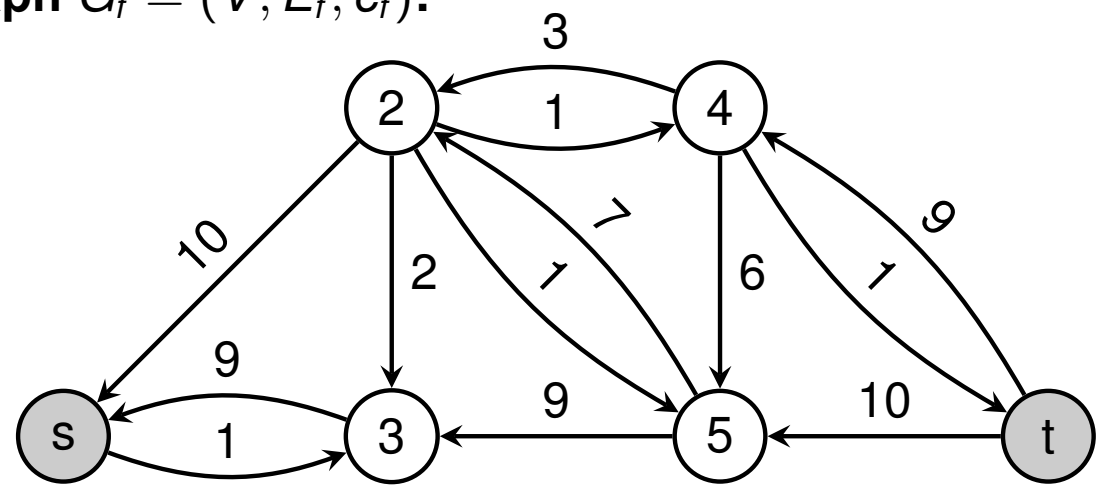

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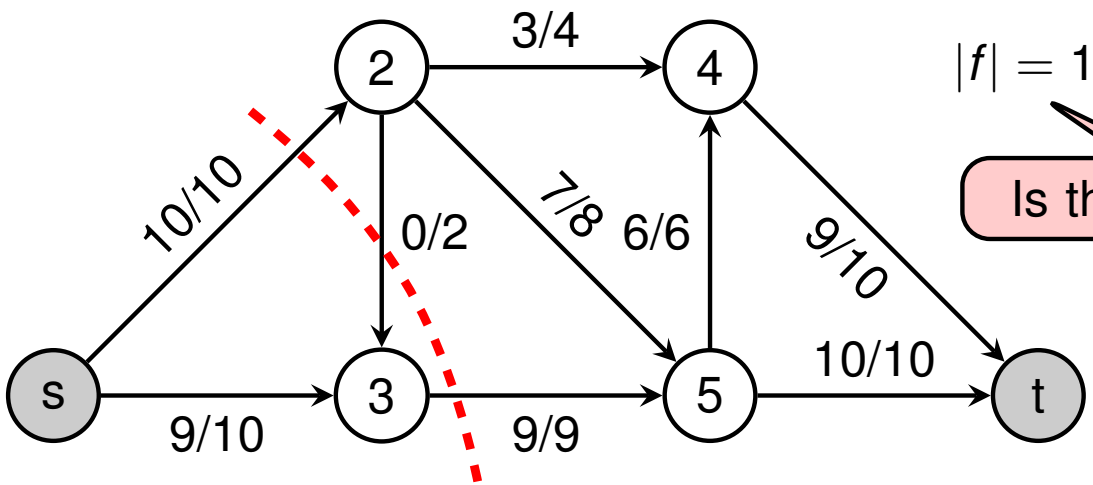

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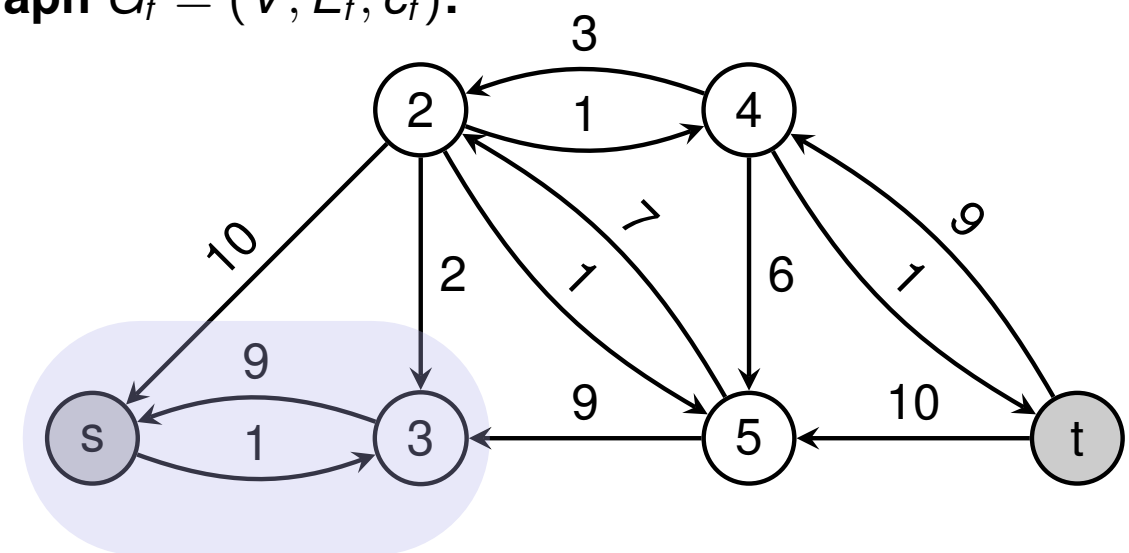

Outline

From Flows to Cuts

Cut

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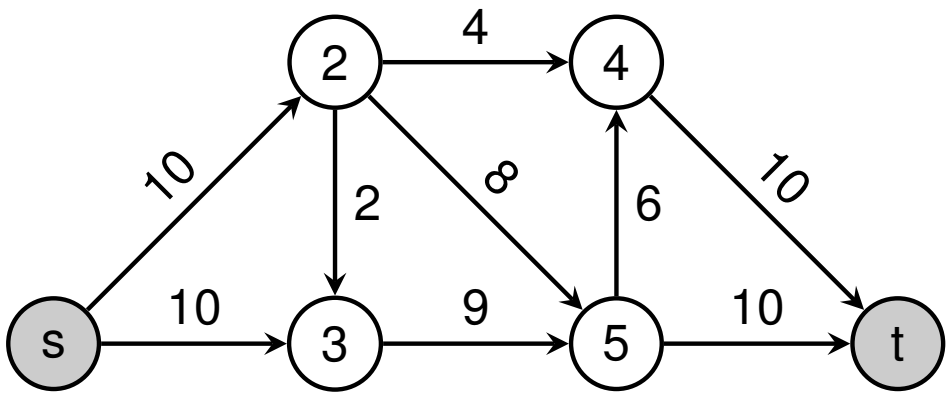

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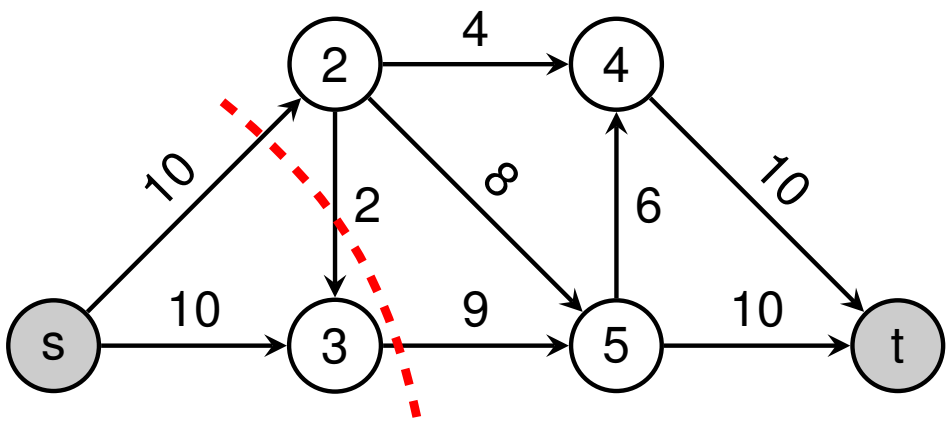

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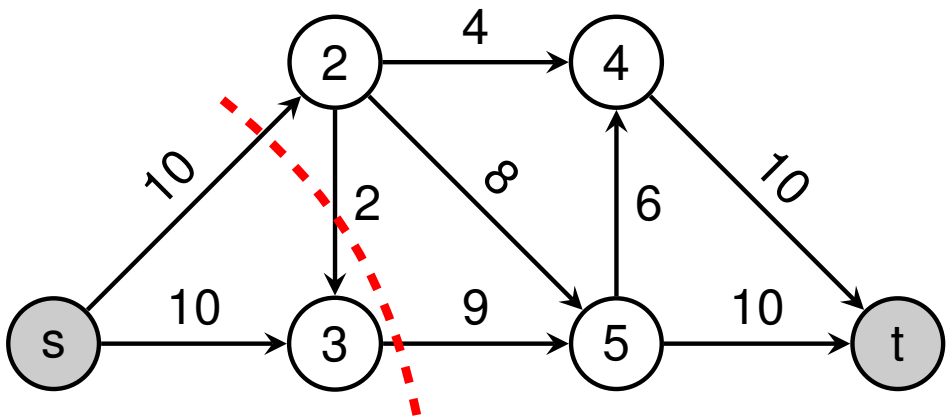

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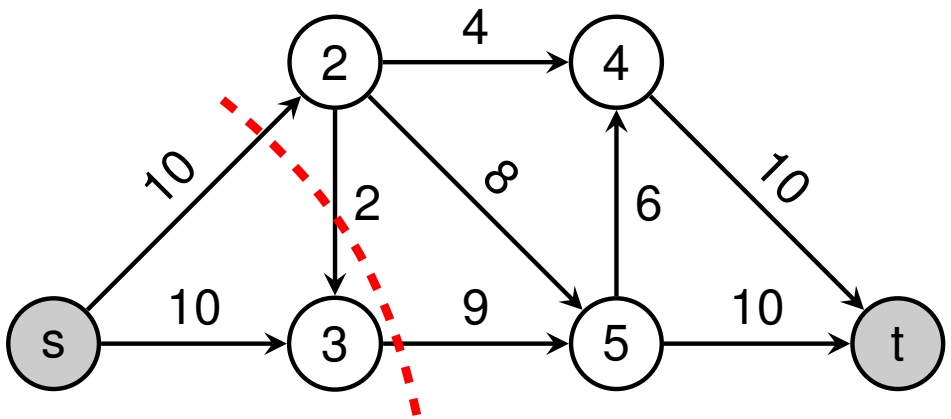

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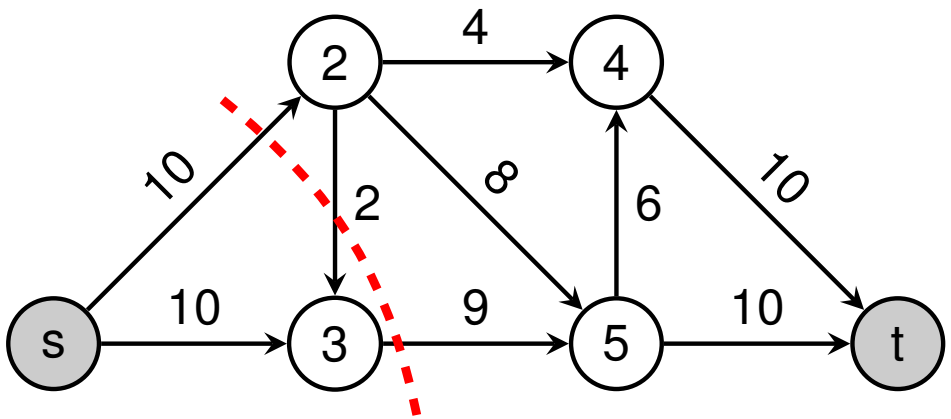

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The value of the max-flow is equal to the capacity of the min-cut, that is

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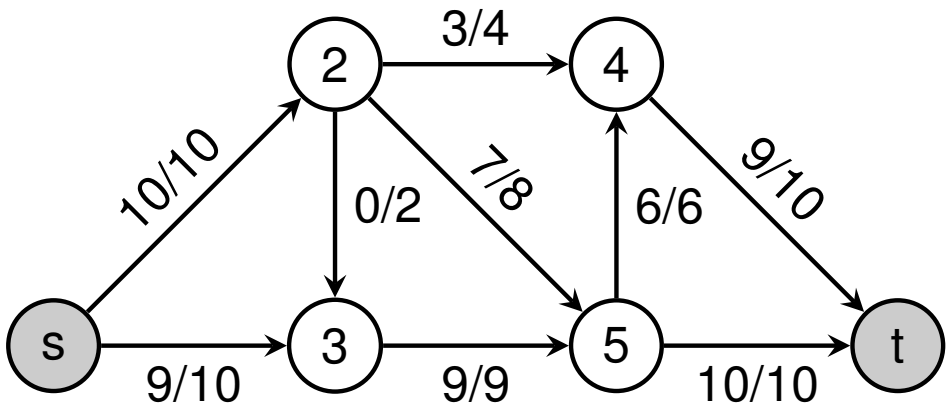

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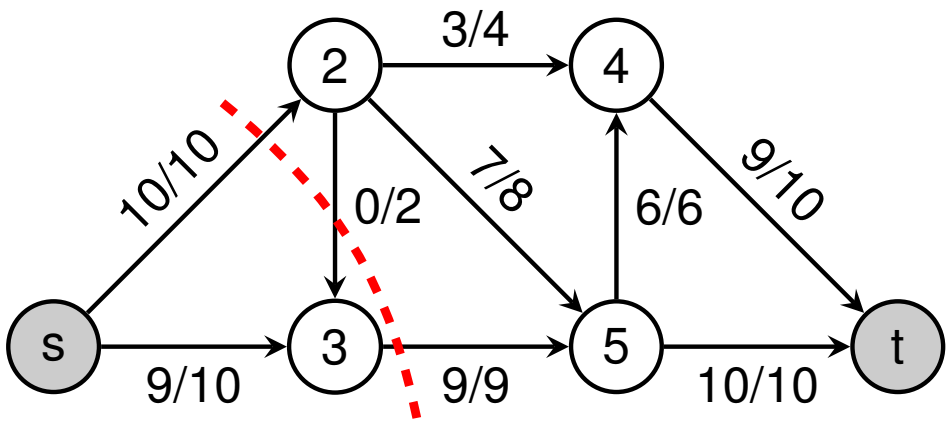

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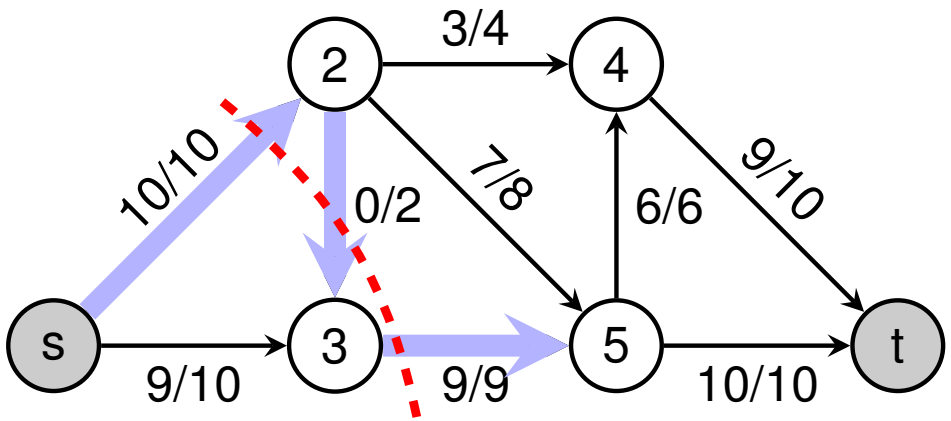

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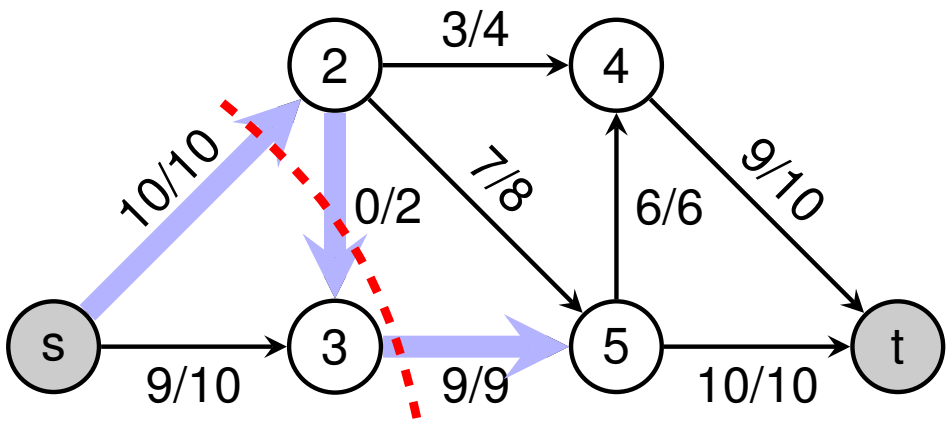

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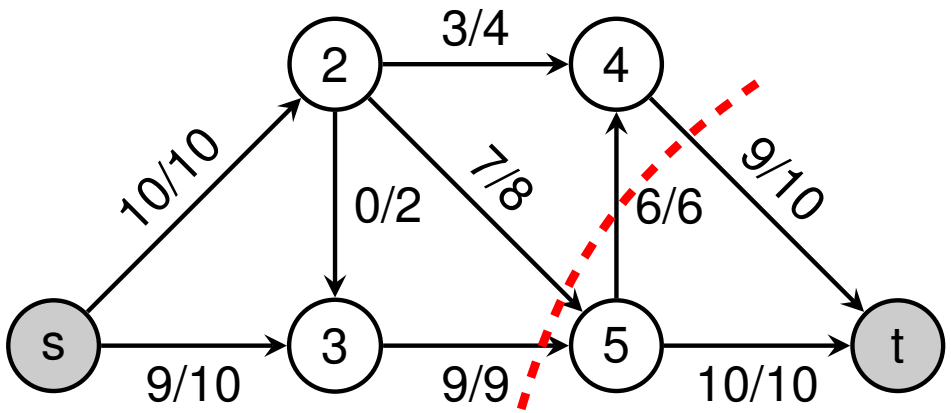

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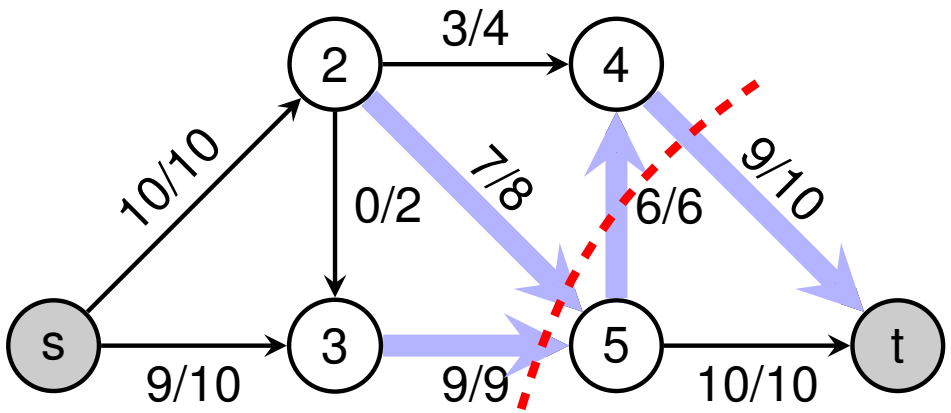

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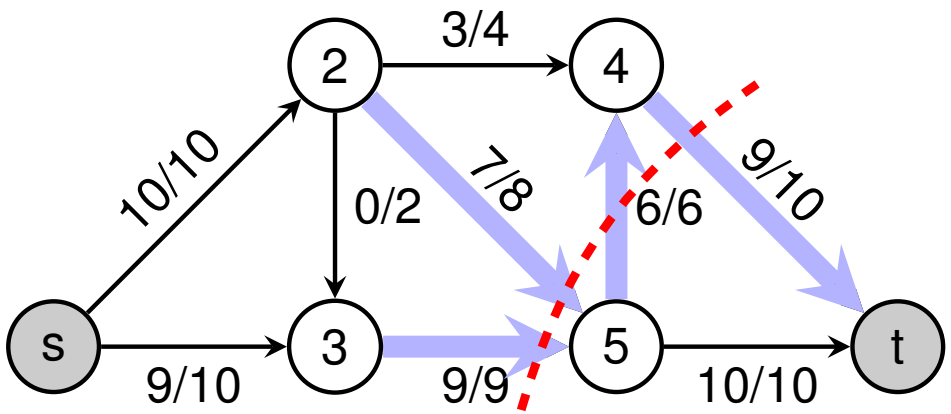

$9 + 7 - 6 + 9 = 19$

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0: def FordFulkerson(G)
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$\Rightarrow$ Flow after iteration integeral

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(proof omitted here, see CLRS3)