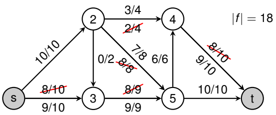
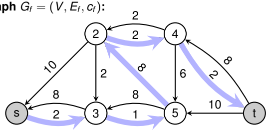
Graph $G = (V, E, c)$**:**



Residual Graph $G_f = (V, E_f, c_f)$**:**



6.6: Maximum flow

Frank Stajano

Thomas Sauerwald

Lent 2016

UNIVERSITY OF CAMBRIDGE

Outline

A Glimpse at the Max-Flow Min-Cut Theorem

Analysis of Ford-Fulkerson

Matchings in Bipartite Graphs

From Flows to Cuts

Cut

- A cut (S, T) is a partition of V into S and $T = V \setminus S$ such that $s \in S$ and $t \in T$.

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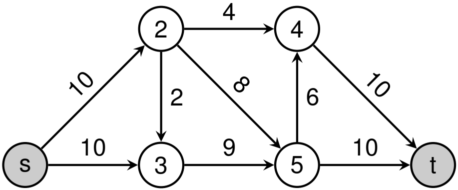

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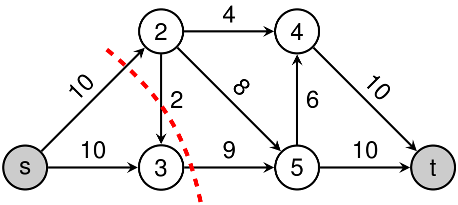

From Flows to Cuts

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- A cut (S, T) is a partition of V into S and $T = V \setminus S$ such that $s \in S$ and $t \in T$.
- The capacity of a cut (S, T) is the sum of capacities of the edges from S to T:

$$\mathrm{c}(S,T) = \sum_{u \in S, v \in T} c(u,v) = \sum_{(u,v) \in E(S,T)} c(u,v)$$

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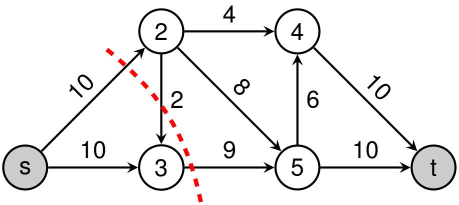

$\mathrm{c}(\{s, 3\}, \{2, 4, 5, t\}) =$

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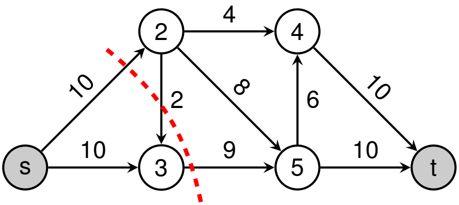

$c(\{s,3\},\{2,4,5,t\}) = 10 + 9 = 19$

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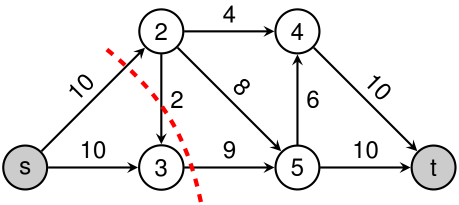
- A minimum cut of a network is a cut whose capacity is minimum over all cuts of the network.

Graph $G = (V, E, c)$:



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From Flows to Cuts

Theorem (Max-Flow Min-Cut Theorem)

The value of the max-flow is equal to the capacity of the min-cut, that is

$$\max_f |f| = \min_{S,T \subseteq V} c(S, T).$$

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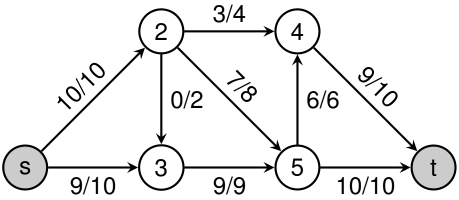

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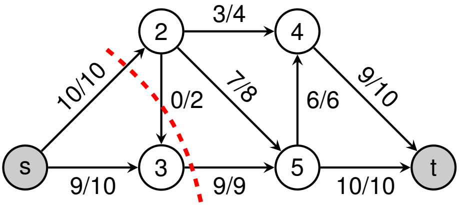

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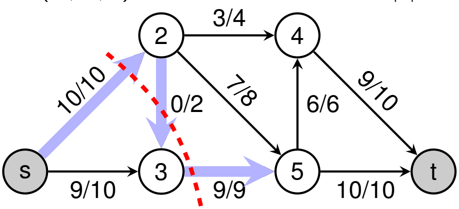

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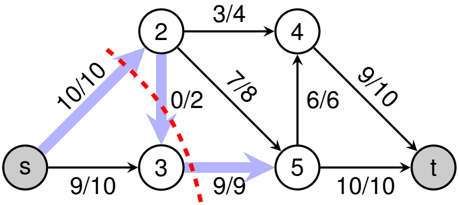

$10 - 0 + 9 = 19$

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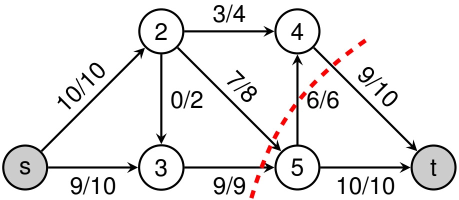

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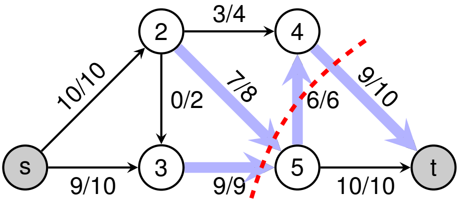

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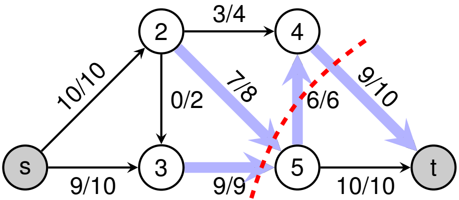

$9 + 7 - 6 + 9 = 19$

Extra: Proof of the Max-Flow Min-Cut Theorem (Easy Direction)

1. For every $u, v \in V$, $f(u, v) \leq c(u, v)$,
2. For every $u, v \in V$, $f(u, v) = -f(v, u)$,
3. For every $u \in V \setminus \{s, t\}$, $\sum_{v \in V} f(u, v) = 0$.

- Let f be any flow and (S, T) be any cut:

$$
\begin{aligned}
|f| &= \sum_{v \in V} f(s, v) \\
&\stackrel{(3)}{=} \sum_{u \in S} \sum_{v \in V} f(u, v) \\
&= \sum_{u \in S} \sum_{v \in S} f(u, v) + \sum_{u \in S} \sum_{v \in T} f(u, v) \\
&\stackrel{(2)}{=} \sum_{u \in S} \sum_{v \in T} f(u, v) \\
&\stackrel{(1)}{\leq} \sum_{u \in S} \sum_{v \in T} c(u, v) \\
&= c(S, T).
\end{aligned}
$$

Flow-Value-Lemma:
For any cut (S, T),

$$|f| = \sum_{u \in S} \sum_{v \in T} f(u, v).$$

- Since this holds for any pair of flow and cut, it follows that

$$\max_f |f| \leq \min_{(S,T)} c(S, T) \qquad \square$$

Outline

Analysis of Ford-Fulkerson

```
def FordFulkerson(G)
    initialize flow to 0 on all edges
    while an augmenting path in G_f can be found:
        push as much extra flow as possible through it
```

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If all capacities $c(u, v)$ are integral, then the flow at every iteration of Ford-Fulkerson is integral.

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Flow before iteration integral
& capacities in G_f are integral
$\Rightarrow$ Flow after iteration integral

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If all capacities $c(u, v)$ are integral, then the flow at every iteration of Ford-Fulkerson is integral.

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For integral capacities $c(u, v)$, Ford-Fulkerson terminates after $C := \max_{u,v} c(u, v)$ iterations and returns the maximum flow.

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(proof omitted here, see CLRS3)

Slow Convergence of Ford-Fulkerson (Figure 26.7)

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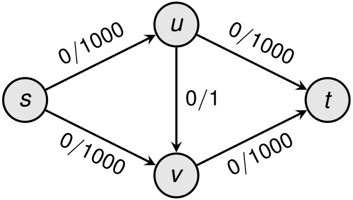

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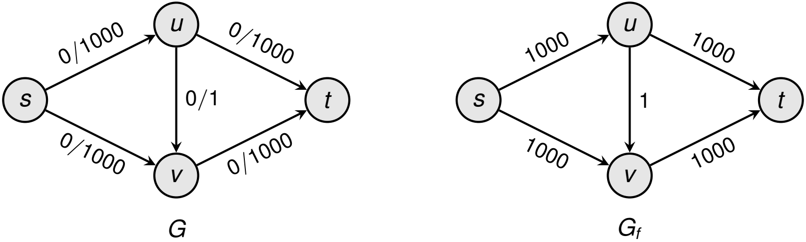

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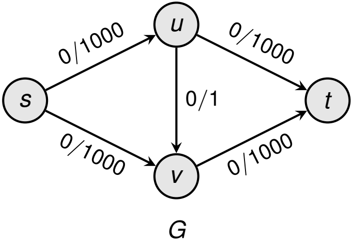

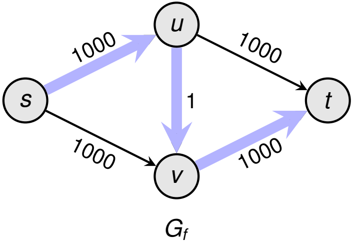

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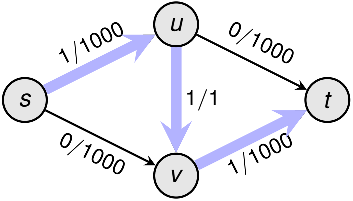

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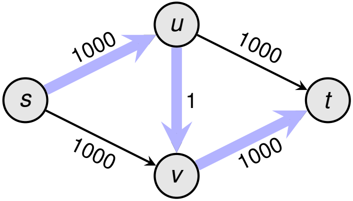

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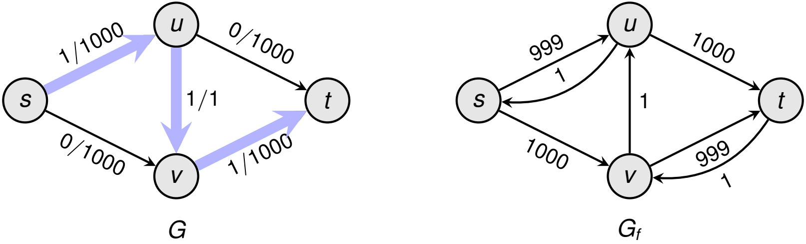

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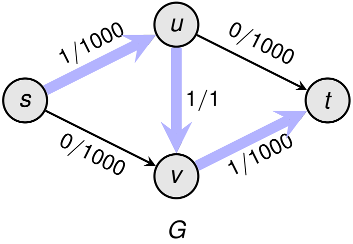

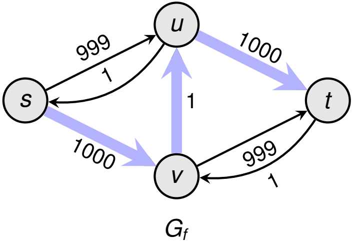

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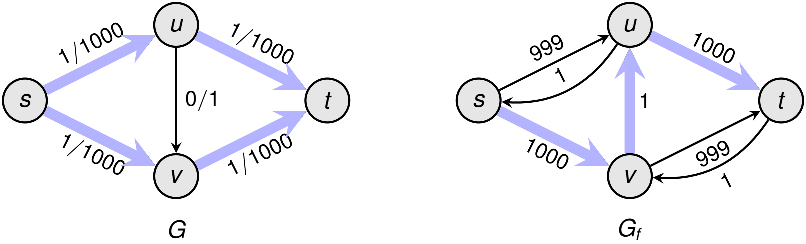

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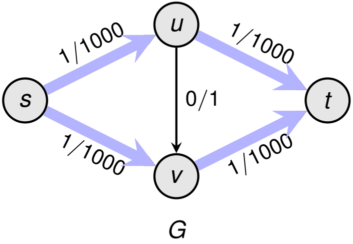

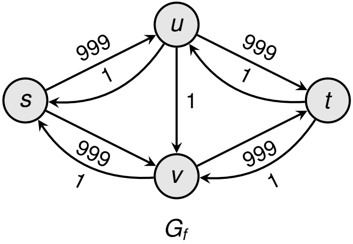

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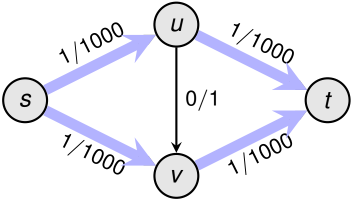

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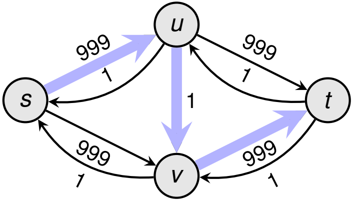

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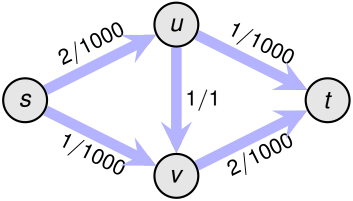

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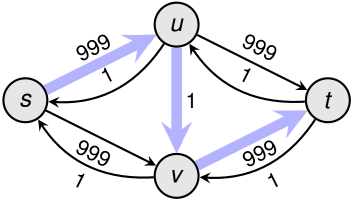

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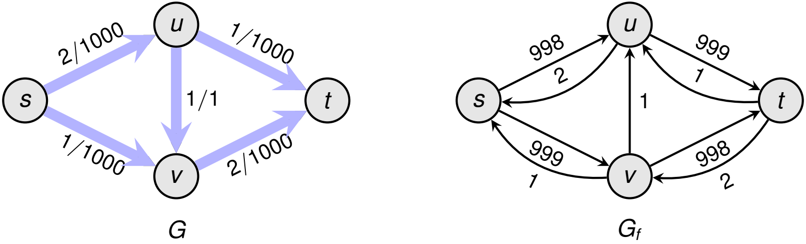

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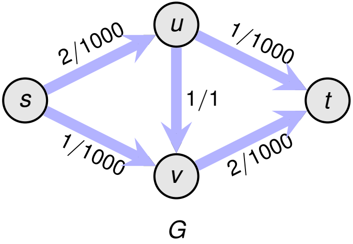

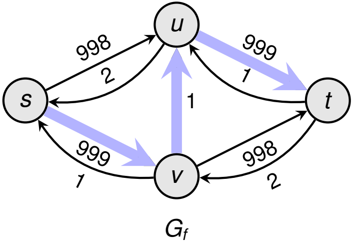

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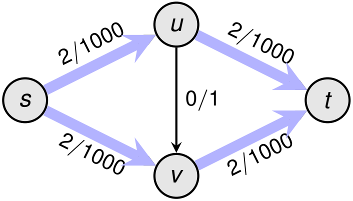

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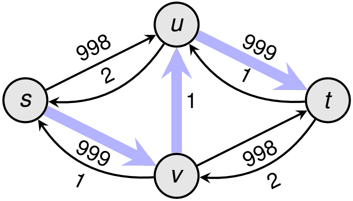

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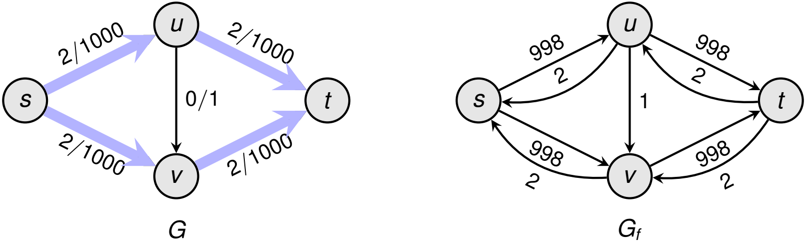

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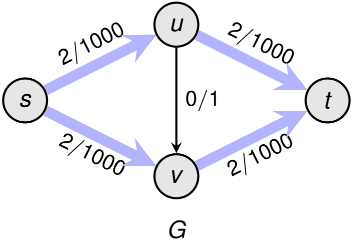

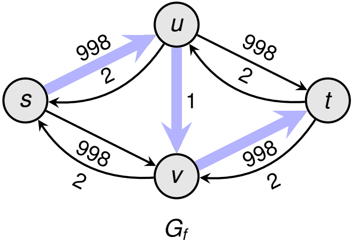

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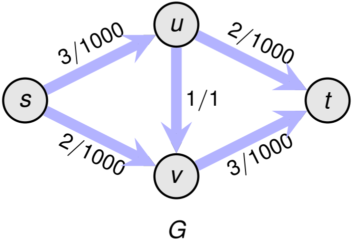

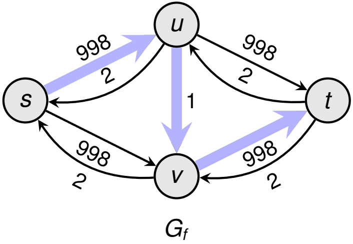

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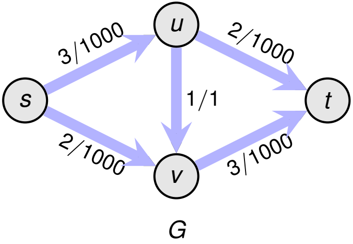

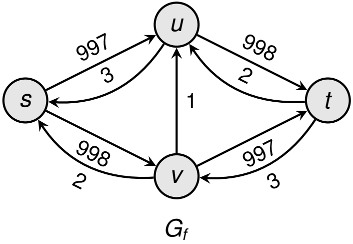

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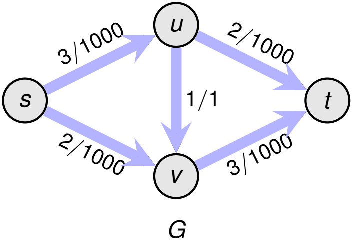

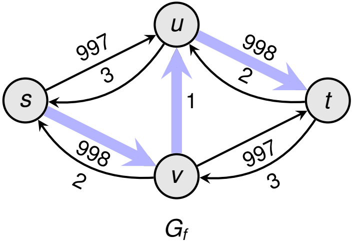

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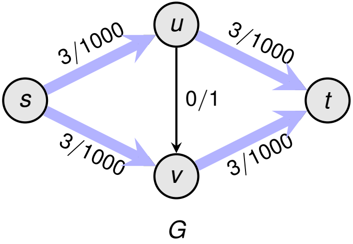

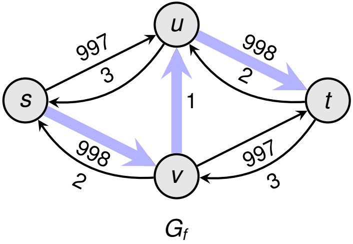

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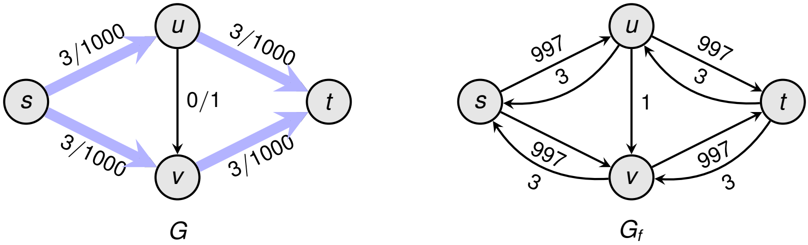

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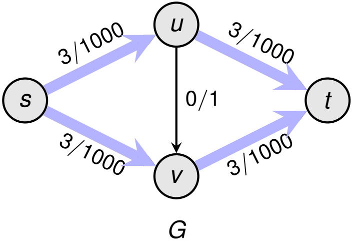

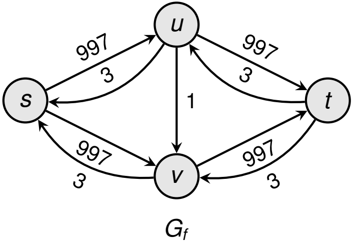

Number of iterations is $C := \max_{u,v} c(u,v)$!

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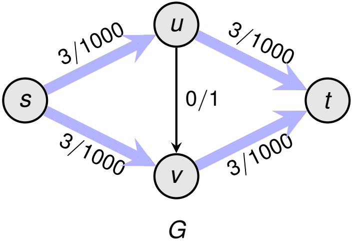
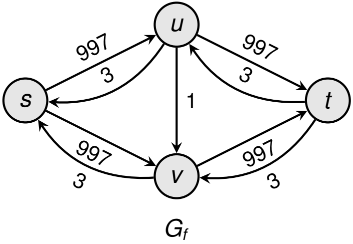
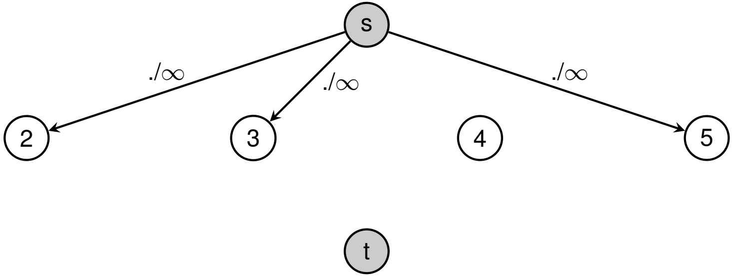




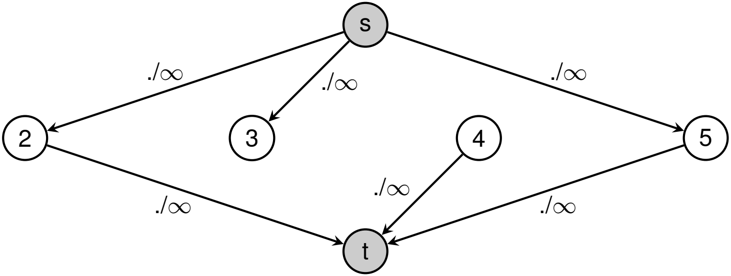
Number of iterations is $C := \max_{u,v} c(u,v)$!

For irrational capacities, Ford-Fulkerson may even fail to terminate!

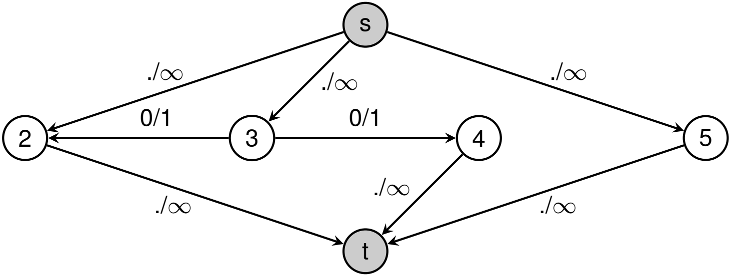
Non-Termination of Ford-Fulkerson for Irrational Capacities



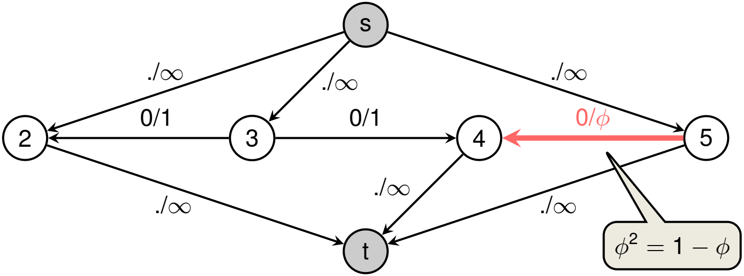
Non-Termination of Ford-Fulkerson for Irrational Capacities



Non-Termination of Ford-Fulkerson for Irrational Capacities



Non-Termination of Ford-Fulkerson for Irrational Capacities



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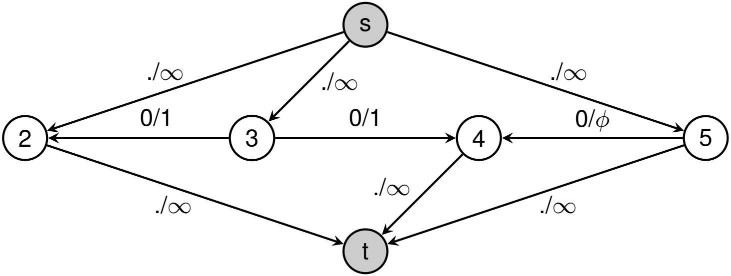
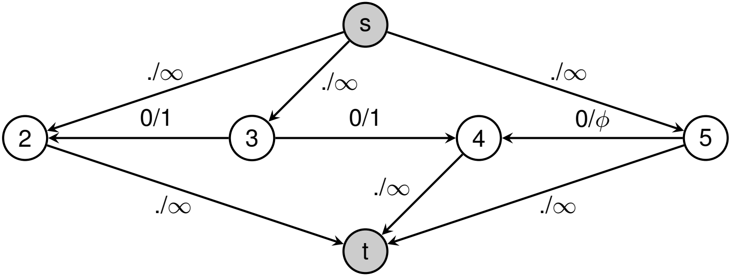
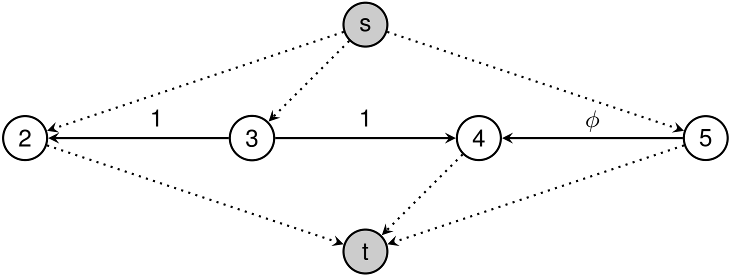
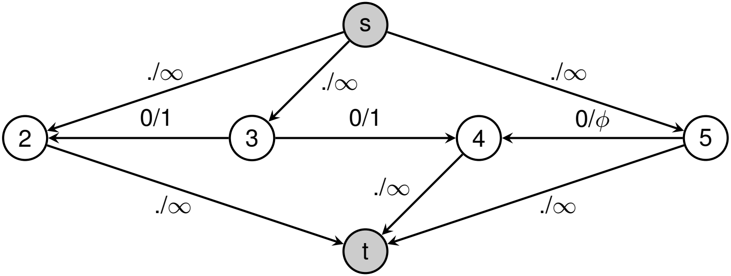


Non-Termination of Ford-Fulkerson for Irrational Capacities

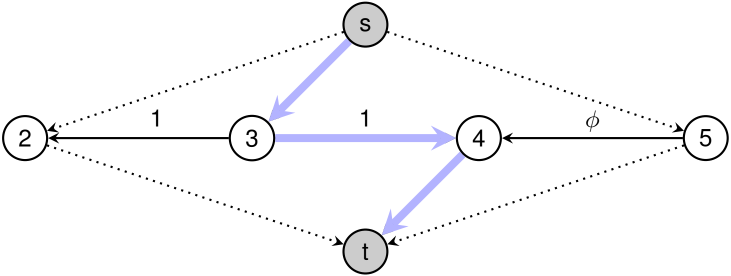




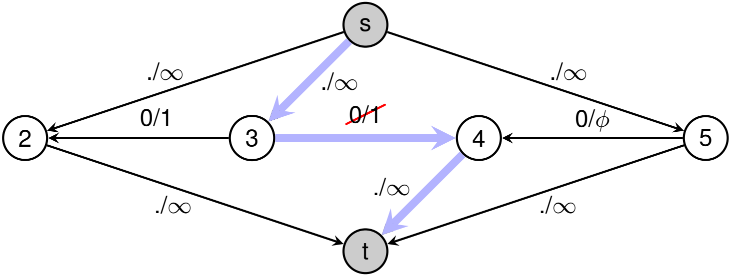
Non-Termination of Ford-Fulkerson for Irrational Capacities



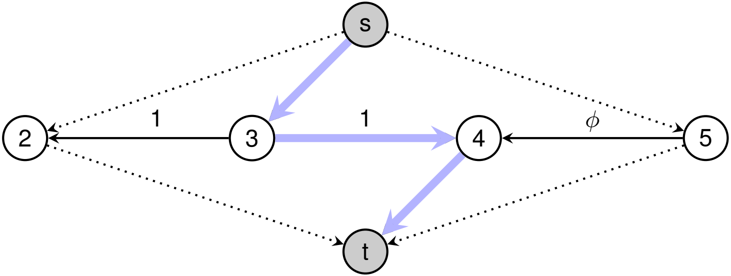
Iteration: 1, $|f| = 0$



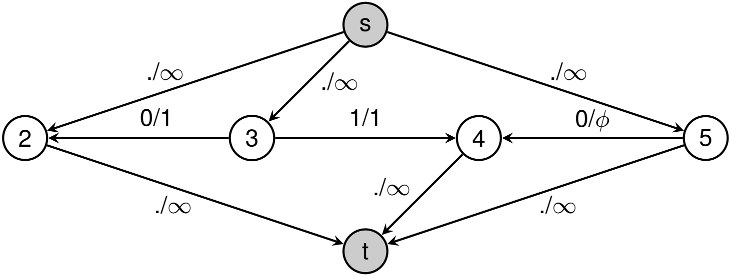
Non-Termination of Ford-Fulkerson for Irrational Capacities



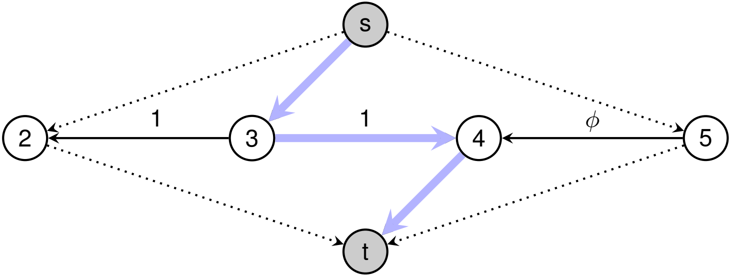
Iteration: 1, $|f| = 0$



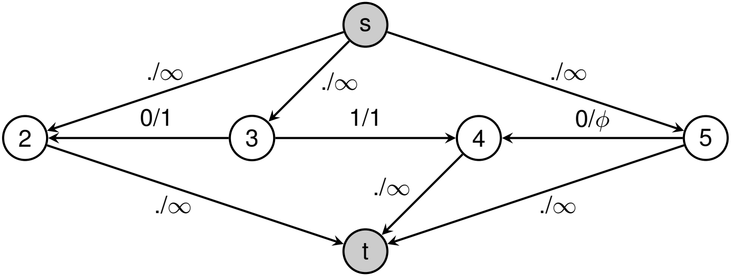
Non-Termination of Ford-Fulkerson for Irrational Capacities



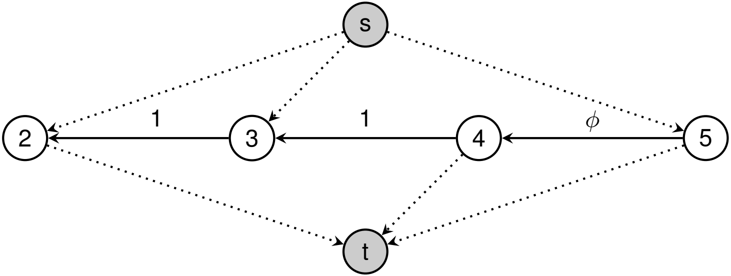
Iteration: 1, $|f| = 1$



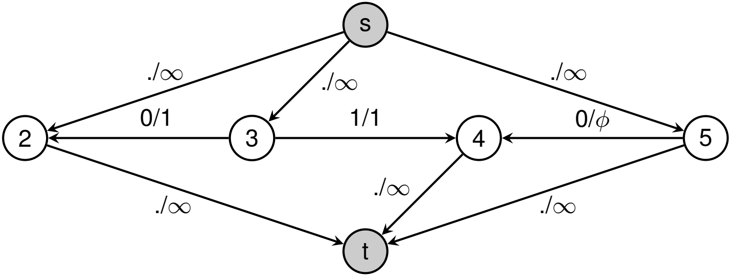
Non-Termination of Ford-Fulkerson for Irrational Capacities



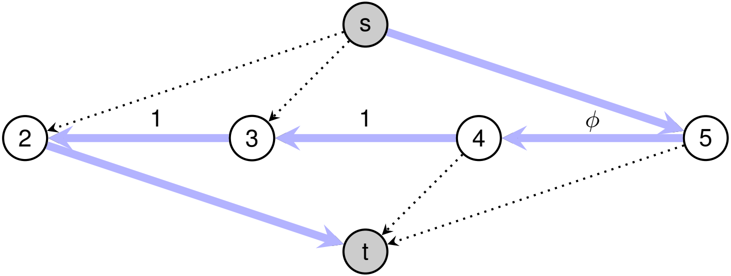
Iteration: 1, $|f| = 1$



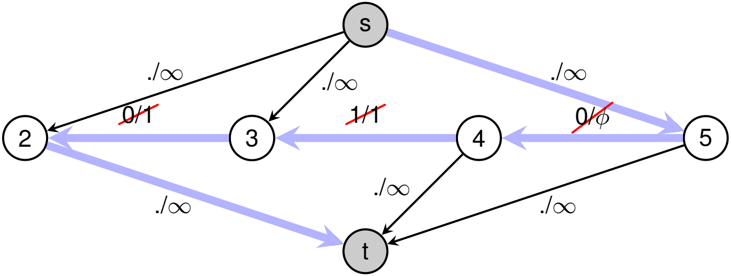
Non-Termination of Ford-Fulkerson for Irrational Capacities



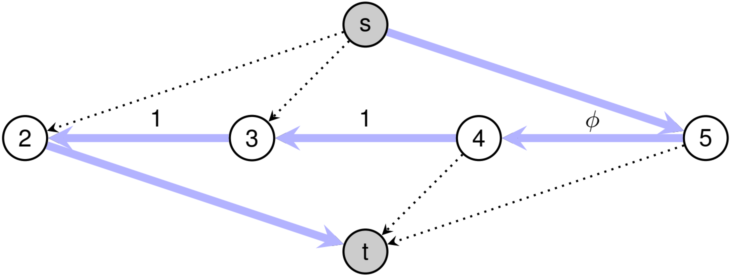
Iteration: 2, $|f| = 1$



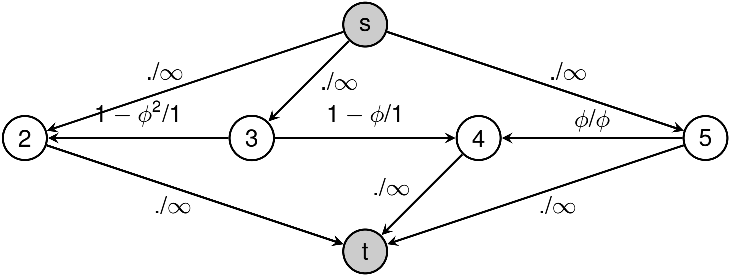
Non-Termination of Ford-Fulkerson for Irrational Capacities



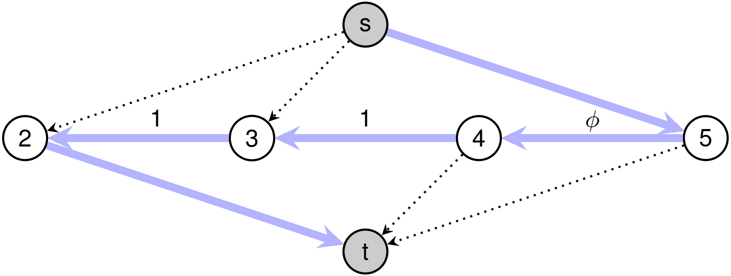
Iteration: 2, $|f| = 1$



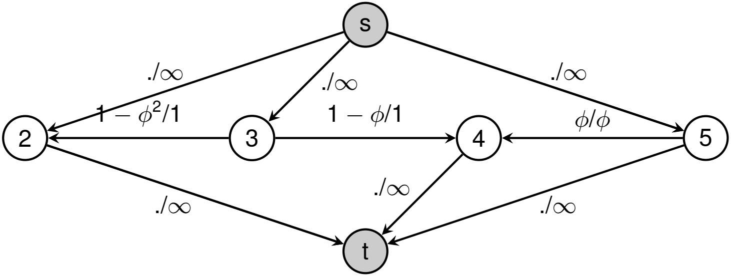
Non-Termination of Ford-Fulkerson for Irrational Capacities



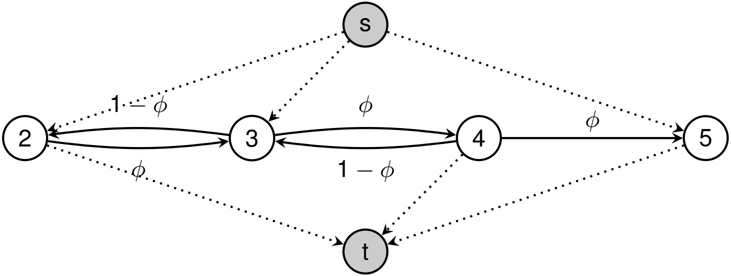
Iteration: 2, $|f| = 1 + \phi$



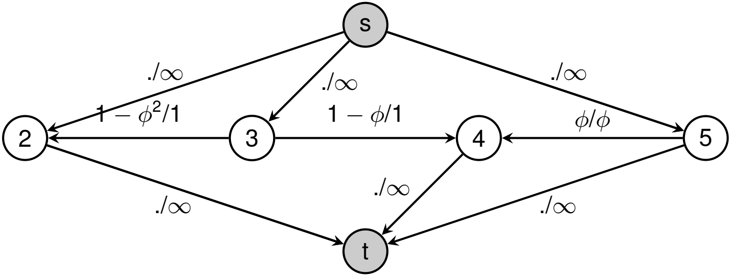
Non-Termination of Ford-Fulkerson for Irrational Capacities



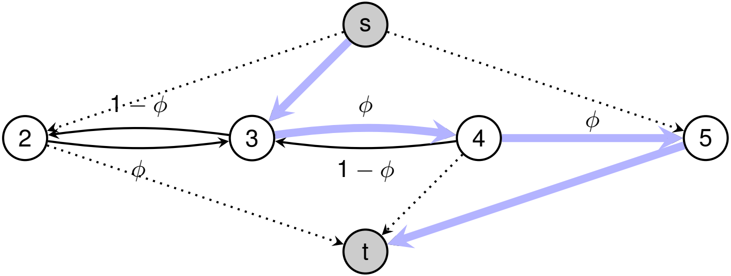
Iteration: 2, $|f| = 1 + \phi$



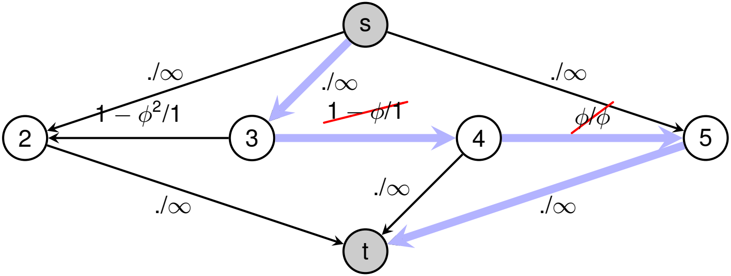
Non-Termination of Ford-Fulkerson for Irrational Capacities



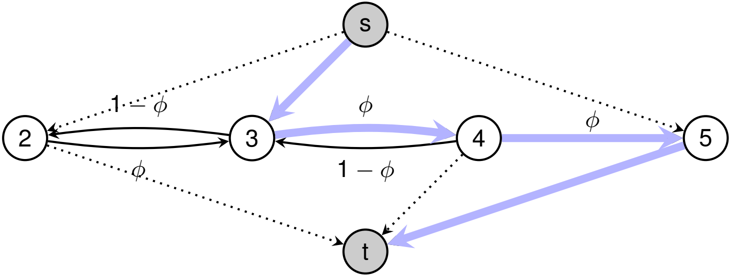
Iteration: 3, $|f| = 1 + \phi$



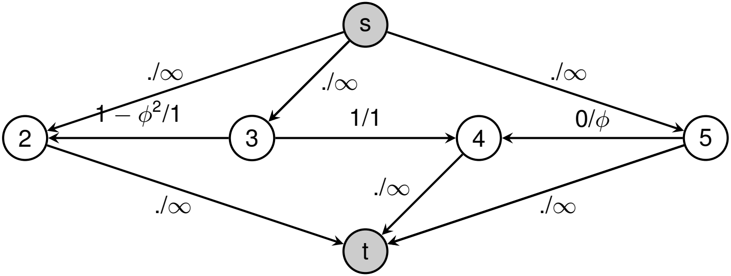
Non-Termination of Ford-Fulkerson for Irrational Capacities



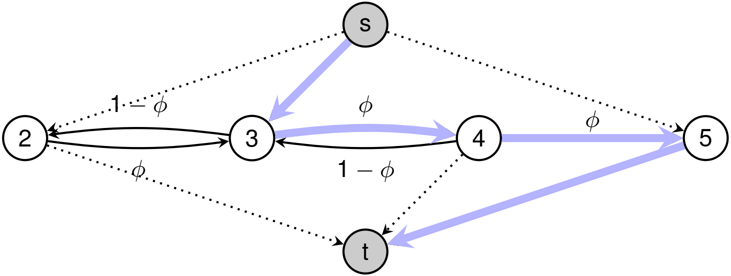
Iteration: 3, $|f| = 1 + \phi$



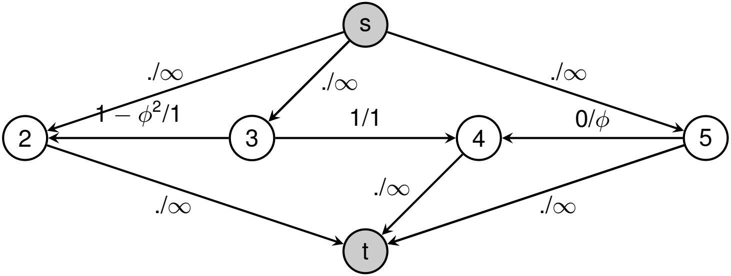
Non-Termination of Ford-Fulkerson for Irrational Capacities



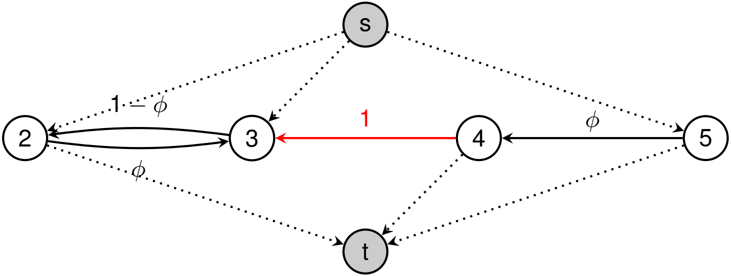
Iteration: 3, $|f| = 1 + 2 \cdot \phi$



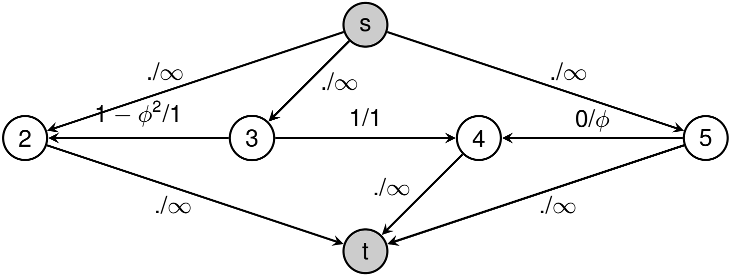
Non-Termination of Ford-Fulkerson for Irrational Capacities



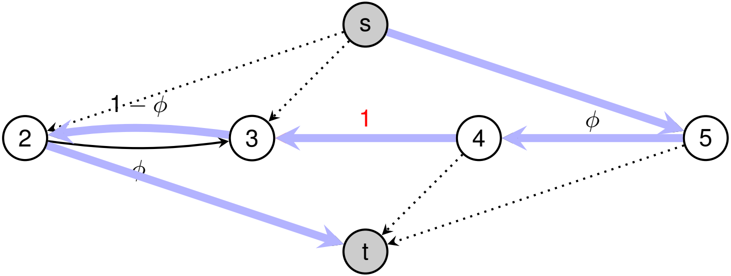
Iteration: 3, $|f| = 1 + 2 \cdot \phi$



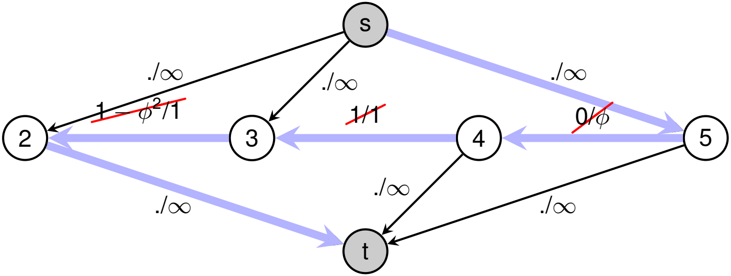
Non-Termination of Ford-Fulkerson for Irrational Capacities



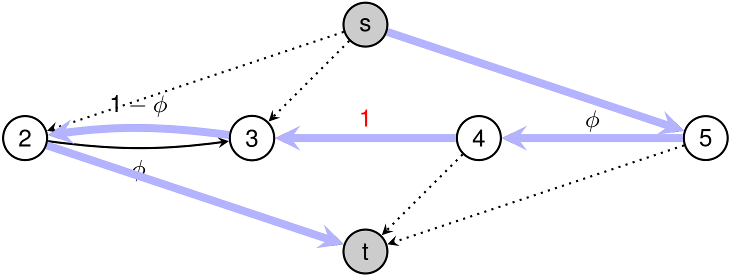
Iteration: 4, $|f| = 1 + 2 \cdot \phi$



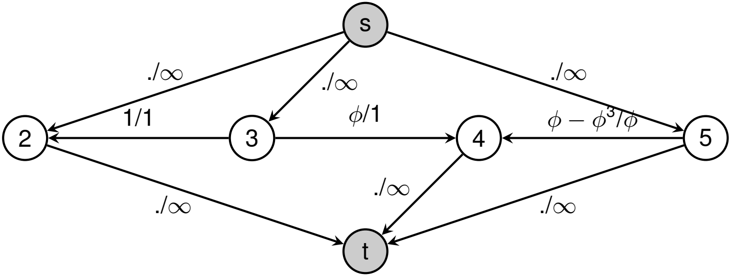
Non-Termination of Ford-Fulkerson for Irrational Capacities



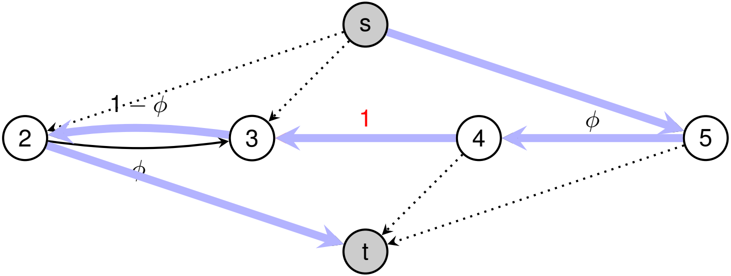
Iteration: 4, $|f| = 1 + 2 \cdot \phi$



Non-Termination of Ford-Fulkerson for Irrational Capacities



Iteration: 4, $|f| = 1 + 2 \cdot \phi + \phi^2$



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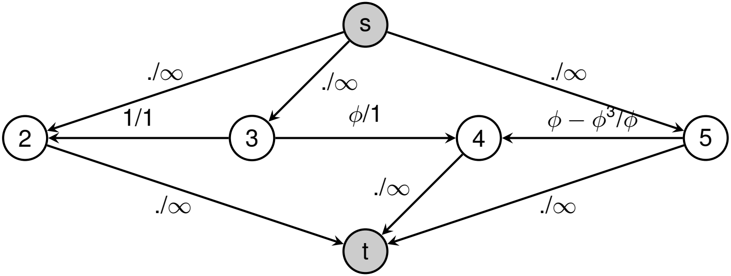
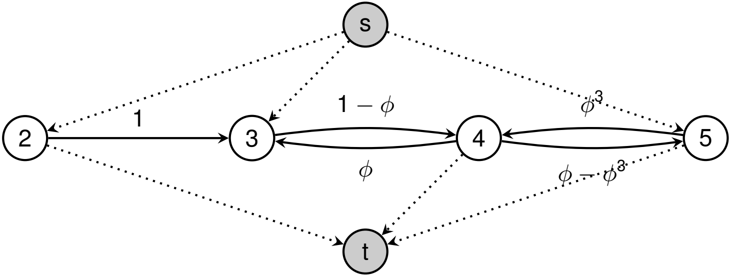
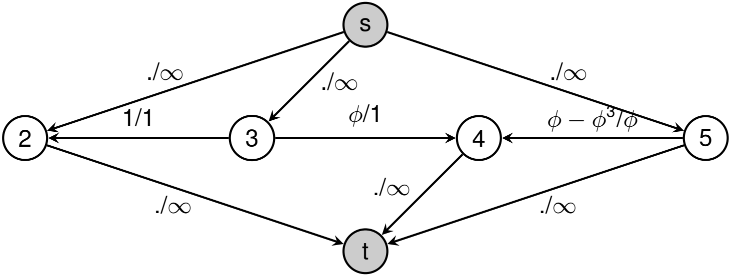


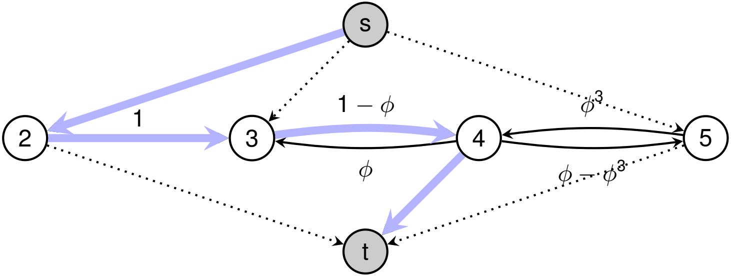
Iteration: 4, $|f| = 1 + 2 \cdot \phi + \phi^2$



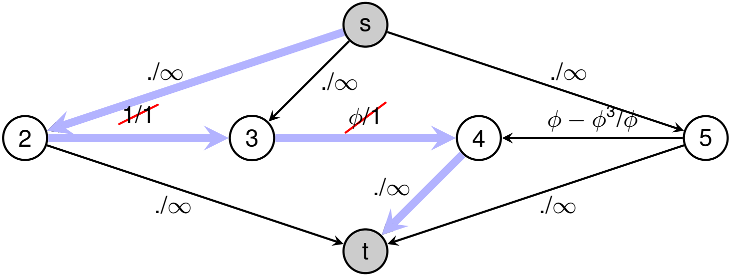
Non-Termination of Ford-Fulkerson for Irrational Capacities



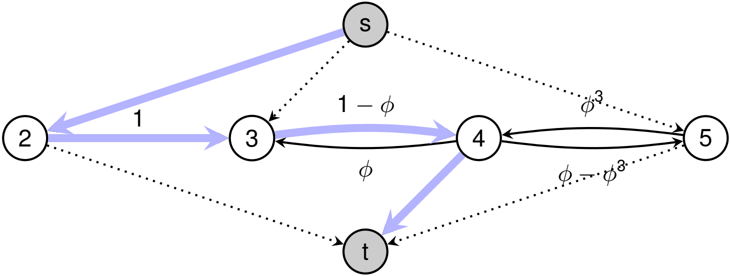
Iteration: 5, $|f| = 1 + 2 \cdot \phi + \phi^2$



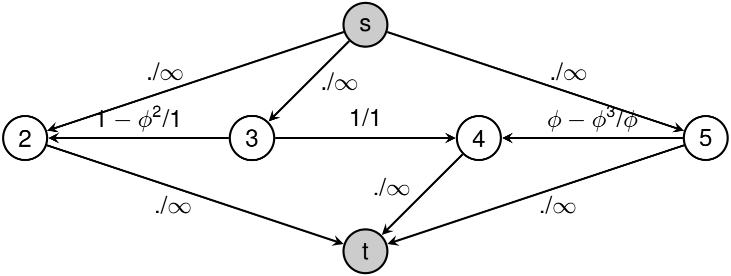
Non-Termination of Ford-Fulkerson for Irrational Capacities



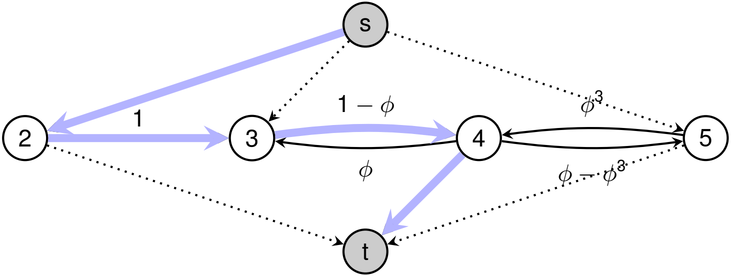
Iteration: 5, $|f| = 1 + 2 \cdot \phi + \phi^2$



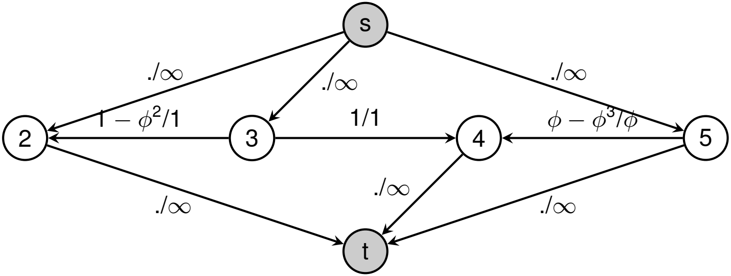
Non-Termination of Ford-Fulkerson for Irrational Capacities



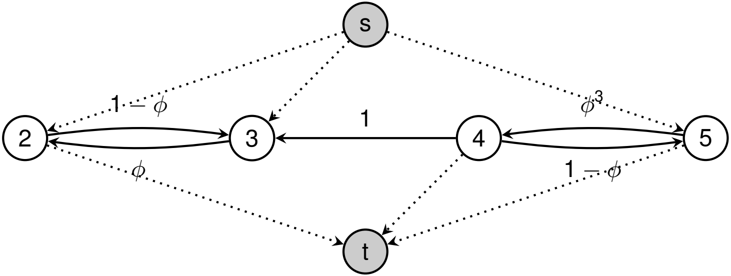
Iteration: 5, $|f| = 1 + 2 \cdot \phi + 2 \cdot \phi^2$



Non-Termination of Ford-Fulkerson for Irrational Capacities

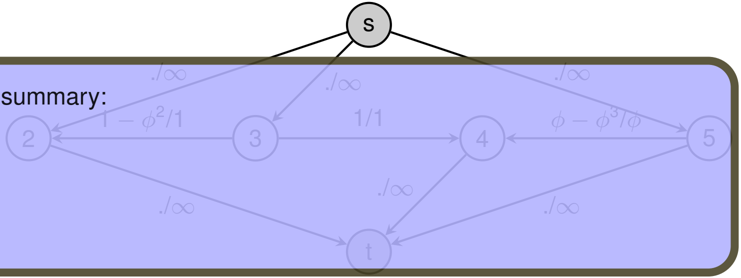


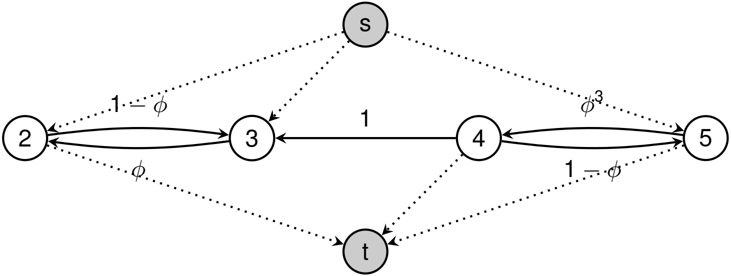
Iteration: 5, $|f| = 1 + 2 \cdot \phi + 2 \cdot \phi^2$



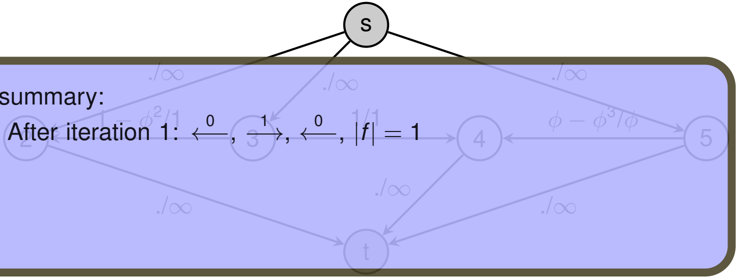
Non-Termination of Ford-Fulkerson for Irrational Capacities

In summary:



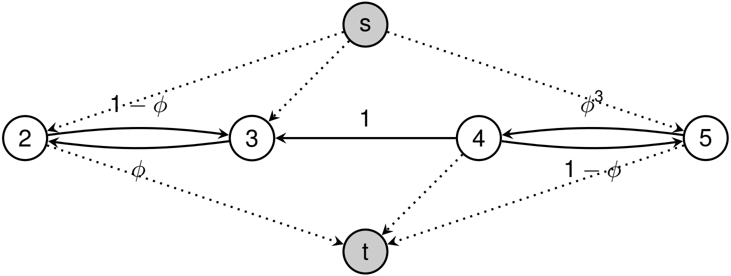


Non-Termination of Ford-Fulkerson for Irrational Capacities



In summary:

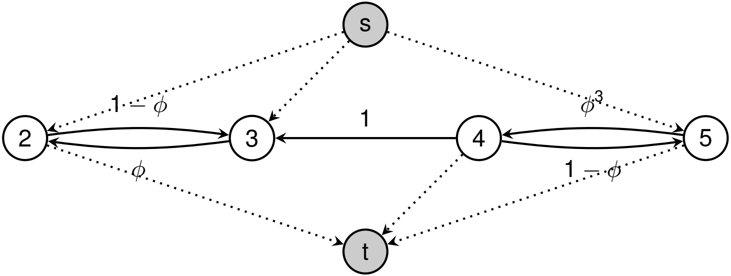
- After iteration 1: $\overset{0}{\longleftarrow}, \overset{1}{\longrightarrow}, \overset{0}{\longleftarrow}, |f| = 1$



Non-Termination of Ford-Fulkerson for Irrational Capacities

In summary:

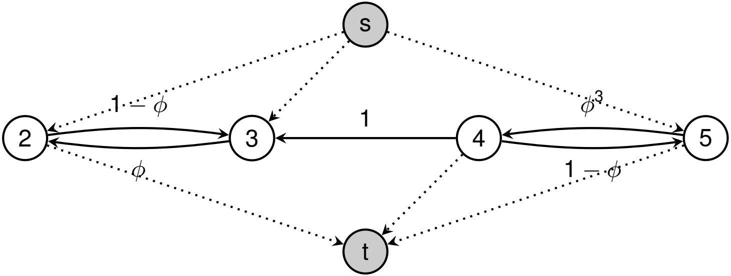
- After iteration 1: $\overset{0}{\longleftarrow}, \overset{1}{\longrightarrow}, \overset{0}{\longleftarrow}, |f| = 1$
- After iteration 5: $\overset{1-\phi^2}{\longleftarrow}, \overset{1}{\longrightarrow}, \overset{\phi-\phi^3}{\longleftarrow}, |f| = 1 + 2\phi + 2\phi^2$



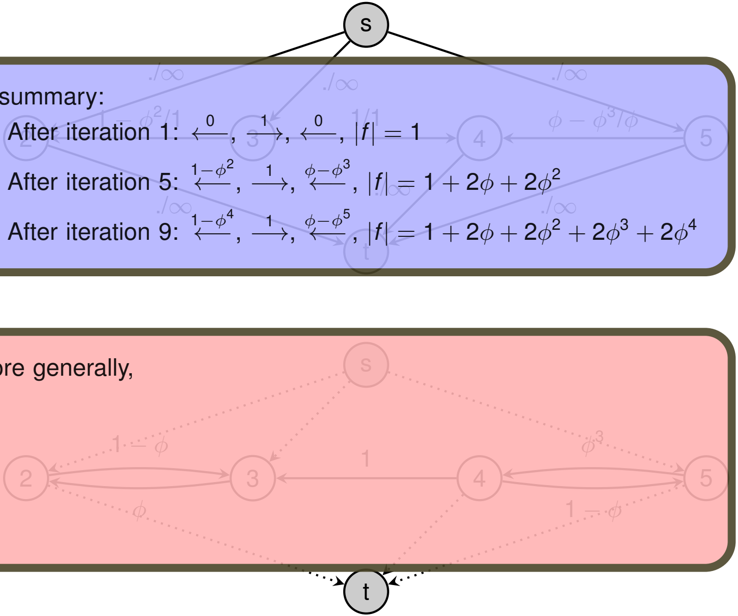
Non-Termination of Ford-Fulkerson for Irrational Capacities

In summary:

- After iteration 1: $\overset{0}{\longleftarrow}, \overset{1}{\longrightarrow}, \overset{0}{\longleftarrow}$, $|f| = 1$
- After iteration 5: $\overset{1-\phi^2}{\longleftarrow}, \overset{1}{\longrightarrow}, \overset{\phi-\phi^3}{\longleftarrow}$, $|f| = 1 + 2\phi + 2\phi^2$
- After iteration 9: $\overset{1-\phi^4}{\longleftarrow}, \overset{1}{\longrightarrow}, \overset{\phi-\phi^5}{\longleftarrow}$, $|f| = 1 + 2\phi + 2\phi^2 + 2\phi^3 + 2\phi^4$



Non-Termination of Ford-Fulkerson for Irrational Capacities

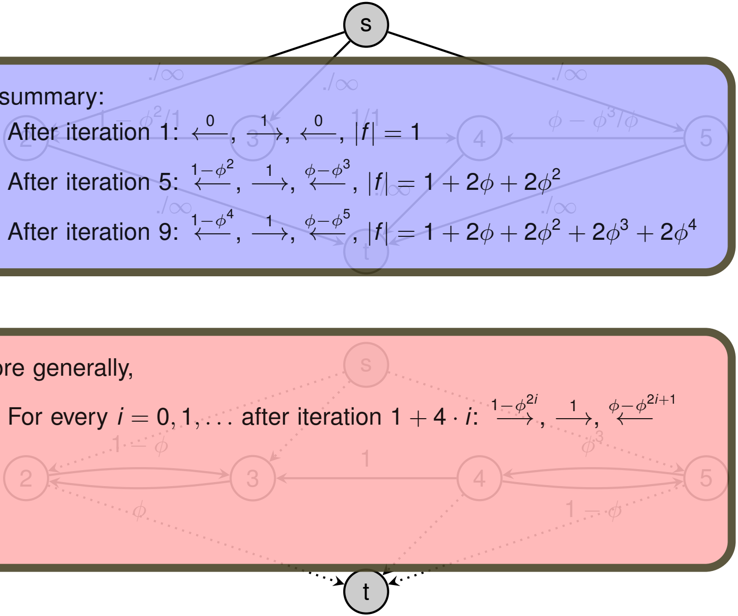


In summary:

- After iteration 1: $\xleftarrow{0}$, $\xrightarrow{1}$, $\xleftarrow{0}$, $|f| = 1$
- After iteration 5: $\xleftarrow{1-\phi^2}$, $\xrightarrow{1}$, $\xleftarrow{\phi-\phi^3}$, $|f| = 1 + 2\phi + 2\phi^2$
- After iteration 9: $\xleftarrow{1-\phi^4}$, $\xrightarrow{1}$, $\xleftarrow{\phi-\phi^5}$, $|f| = 1 + 2\phi + 2\phi^2 + 2\phi^3 + 2\phi^4$

More generally,

Non-Termination of Ford-Fulkerson for Irrational Capacities



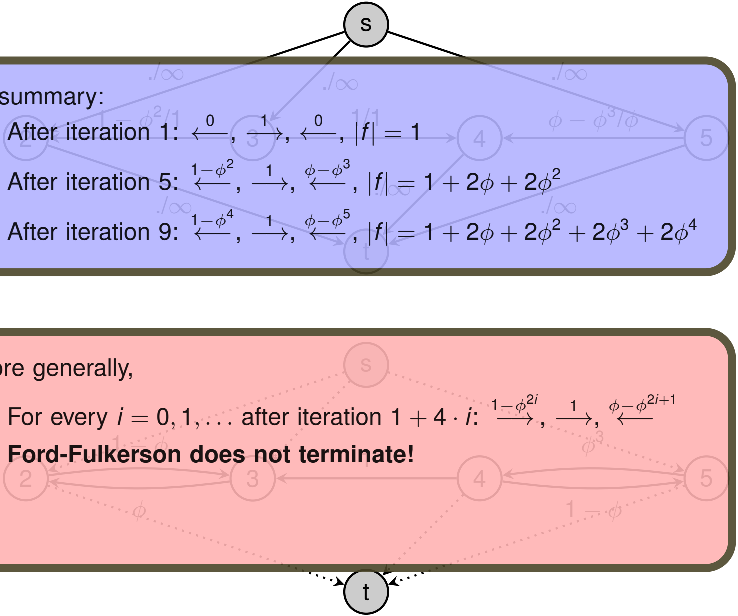
In summary:

- After iteration 1: $\overset{0}{\longleftarrow}, \overset{1}{\longrightarrow}, \overset{0}{\longleftarrow}, |f| = 1$
- After iteration 5: $\overset{1-\phi^2}{\longleftarrow}, \overset{1}{\longrightarrow}, \overset{\phi-\phi^3}{\longleftarrow}, |f| = 1 + 2\phi + 2\phi^2$
- After iteration 9: $\overset{1-\phi^4}{\longleftarrow}, \overset{1}{\longrightarrow}, \overset{\phi-\phi^5}{\longleftarrow}, |f| = 1 + 2\phi + 2\phi^2 + 2\phi^3 + 2\phi^4$

More generally,

- For every $i = 0, 1, \ldots$ after iteration $1 + 4 \cdot i$: $\overset{1-\phi^{2i}}{\longrightarrow}, \overset{1}{\longrightarrow}, \overset{\phi-\phi^{2i+1}}{\longleftarrow}$

Non-Termination of Ford-Fulkerson for Irrational Capacities



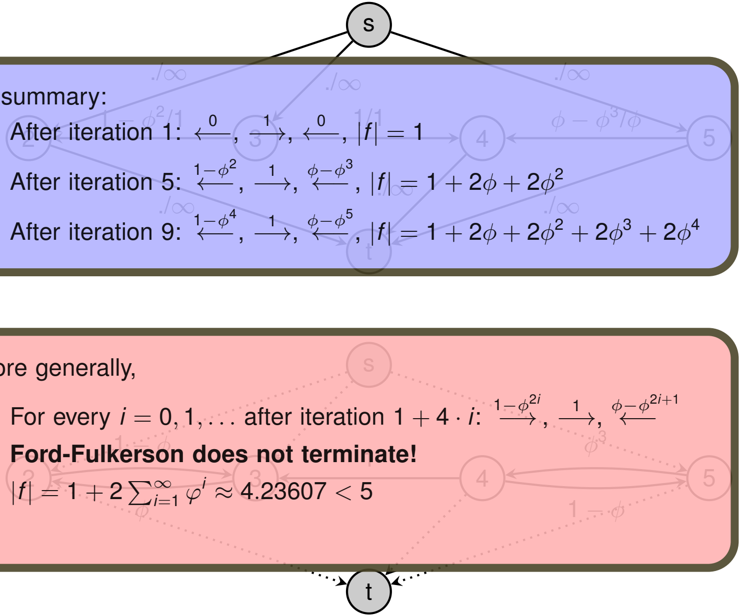
In summary:

- After iteration 1: $\overset{0}{\longleftarrow}, \overset{1}{\longrightarrow}, \overset{0}{\longleftarrow}, |f| = 1$
- After iteration 5: $\overset{1-\phi^2}{\longleftarrow}, \overset{1}{\longrightarrow}, \overset{\phi-\phi^3}{\longleftarrow}, |f| = 1 + 2\phi + 2\phi^2$
- After iteration 9: $\overset{1-\phi^4}{\longleftarrow}, \overset{1}{\longrightarrow}, \overset{\phi-\phi^5}{\longleftarrow}, |f| = 1 + 2\phi + 2\phi^2 + 2\phi^3 + 2\phi^4$

More generally,

- For every $i = 0, 1, \ldots$ after iteration $1 + 4 \cdot i$: $\overset{1-\phi^{2i}}{\longrightarrow}, \overset{1}{\longrightarrow}, \overset{\phi-\phi^{2i+1}}{\longleftarrow}$
- **Ford-Fulkerson does not terminate!**

Non-Termination of Ford-Fulkerson for Irrational Capacities



In summary:

- After iteration 1: $\overset{0}{\longleftarrow}$, $\overset{1}{\longrightarrow}$, $\overset{0}{\longleftarrow}$, $|f| = 1$
- After iteration 5: $\overset{1-\phi^2}{\longleftarrow}$, $\overset{1}{\longrightarrow}$, $\overset{\phi-\phi^3}{\longleftarrow}$, $|f| = 1 + 2\phi + 2\phi^2$
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More generally,

- For every $i = 0, 1, \ldots$ after iteration $1 + 4 \cdot i$: $\overset{1-\phi^{2i}}{\longrightarrow}$, $\overset{1}{\longrightarrow}$, $\overset{\phi-\phi^{2i+1}}{\longleftarrow}$
- **Ford-Fulkerson does not terminate!**
- $|f| = 1 + 2\sum_{i=1}^{\infty} \varphi^i \approx 4.23607 < 5$

Non-Termination of Ford-Fulkerson for Irrational Capacities

In summary:

- After iteration 1: $\xleftarrow{0}$, $\xrightarrow{1}$, $\xleftarrow{0}$, $|f| = 1$
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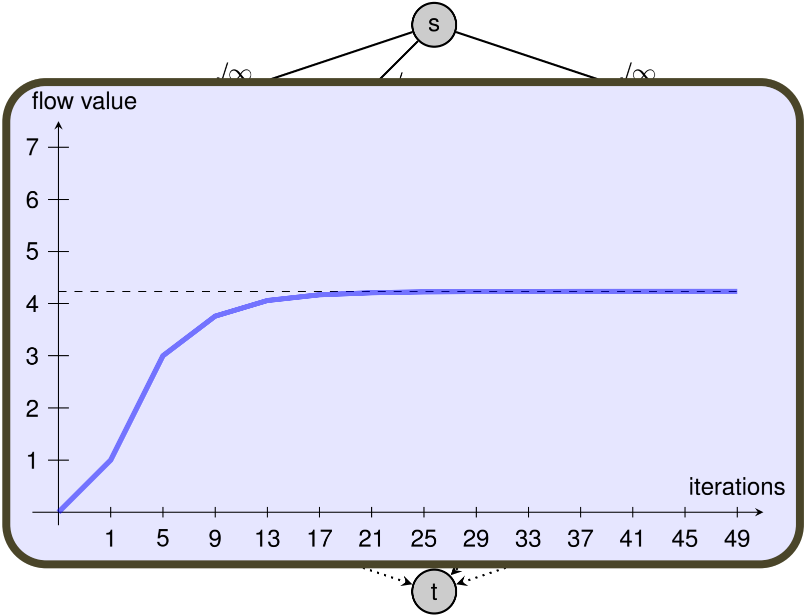
- $|f| = 1 + 2\sum_{i=1}^{\infty} \varphi^i \approx 4.23607 < 5$
- **It does not even converge to a maximum flow!**

Non-Termination of Ford-Fulkerson for Irrational Capacities



Summary and Outlook

Ford-Fulkerson Method

- works only for integral (rational) capacities
- Runtime: $O(E \cdot |f^*|) = O(E \cdot V \cdot C)$

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- Idea: Find an augmenting path with high capacity
- Consider subgraph of G_f consisting of edges (u, v) with $c_f(u, v) > \Delta$
- scaling parameter Δ, which is initially $2^{\lceil \log_2 C \rceil}$ and 1 after termination
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Edmonds-Karp Algorithm

- Idea: Find the shortest augmenting path in G_f
- Runtime: $O(E^2 \cdot V)$

Outline

A Glimpse at the Max-Flow Min-Cut Theorem

Analysis of Ford-Fulkerson

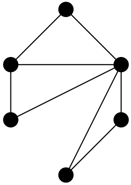
Matchings in Bipartite Graphs

Application: Maximum-Bipartite-Matching Problem

Matching

A matching is a subset $M \subseteq E$ such that for all $v \in V$, at most one edge of M is incident to v.

Application: Maximum-Bipartite-Matching Problem



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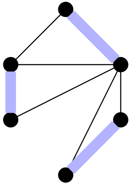

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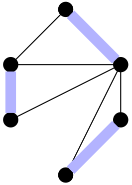

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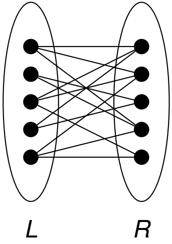
A graph G is bipartite if V can be partitioned into L and R so that all edges go between L and R.



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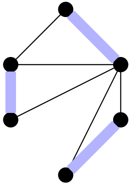

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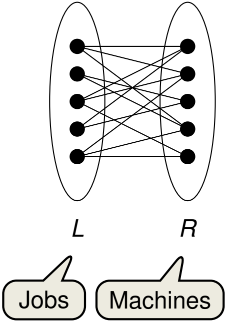

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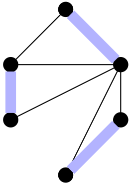

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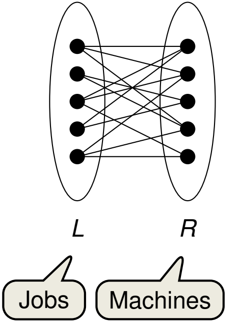

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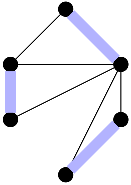

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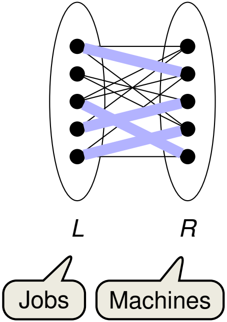

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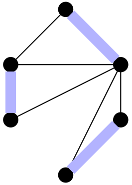

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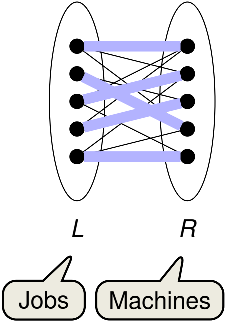

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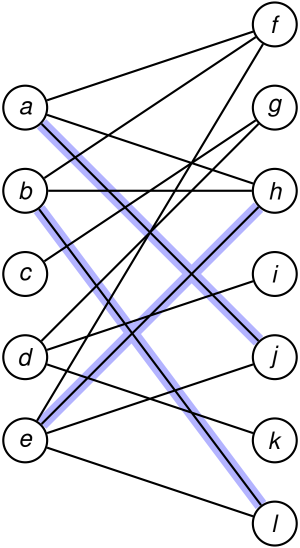
Matchings in Bipartite Graphs via Maximum Flows



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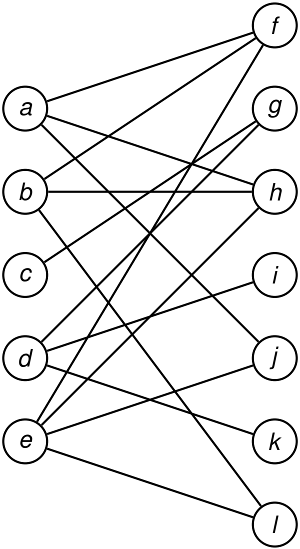

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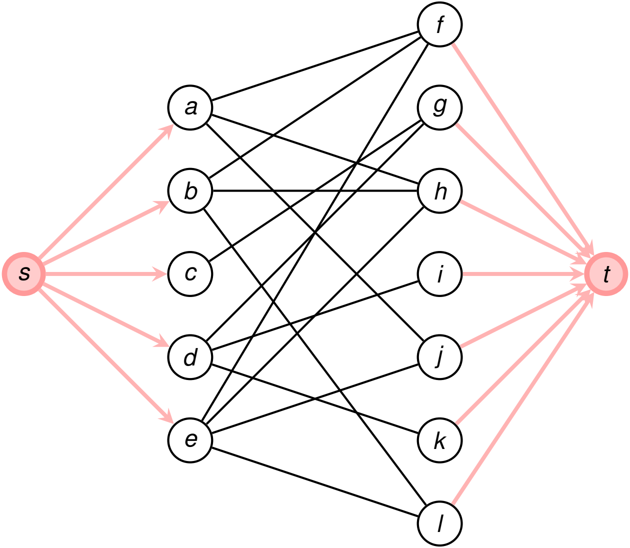

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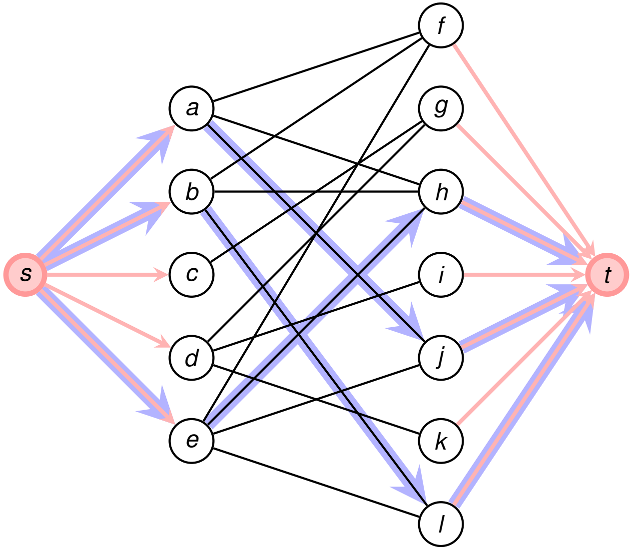
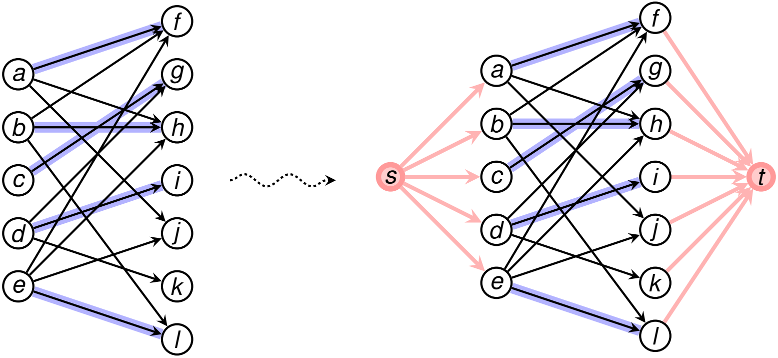


Correspondence between Maximum Matchings and Max Flow

Theorem (Corollary 26.11)

The cardinality of a maximum matching M in a bipartite graph G equals the value of a maximum flow f in the corresponding flow network $\widetilde{G}$.

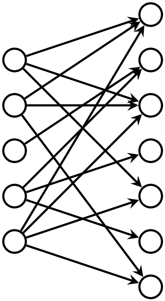


Graph G

Graph $\widetilde{G}$

From Matching to Flow

- Given a maximum matching of cardinality k



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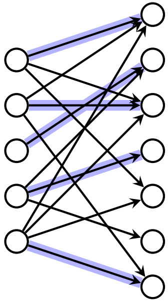

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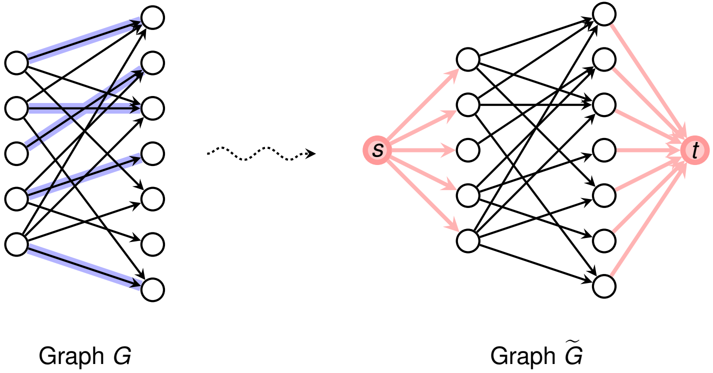
From Matching to Flow

- Given a maximum matching of cardinality k
- Consider flow f that sends one unit along each each of k paths



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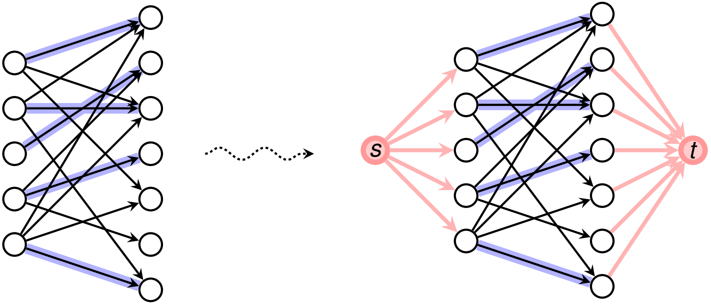

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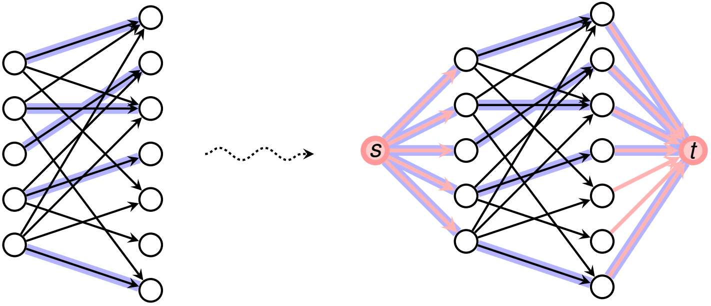

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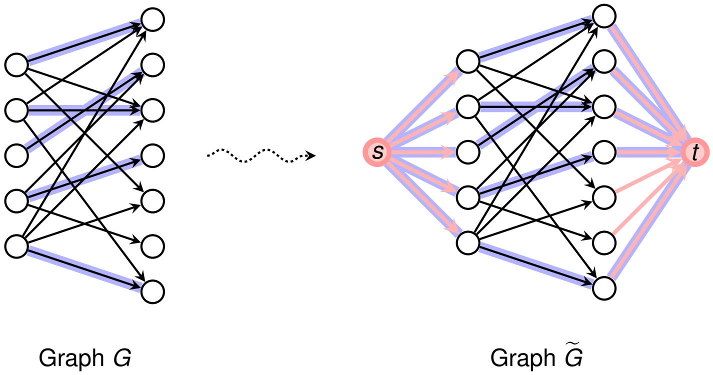
$\Rightarrow$ f is a flow and has value k



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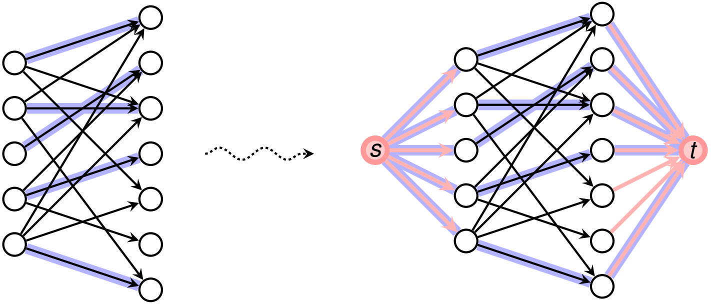
max cardinality matching $\leq$ value of maxflow



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- Let f be a maximum flow in $\widetilde{G}$ of value k

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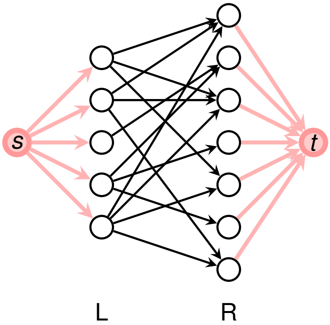

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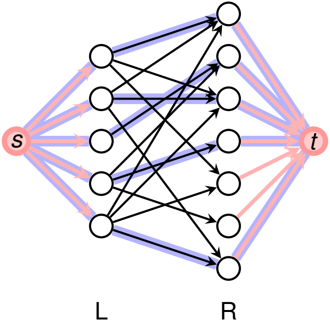

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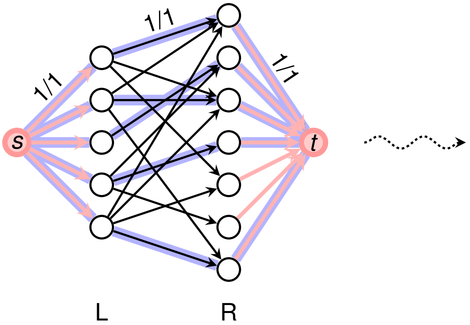
- Let f be a maximum flow in $\widetilde{G}$ of value k
- Integrality Theorem $\Rightarrow f(u, v) \in \{0, 1\}$ and k integral



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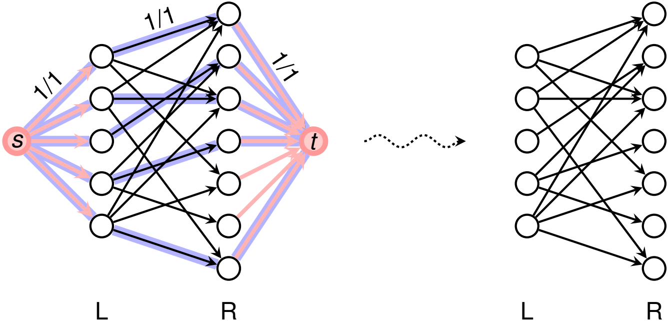

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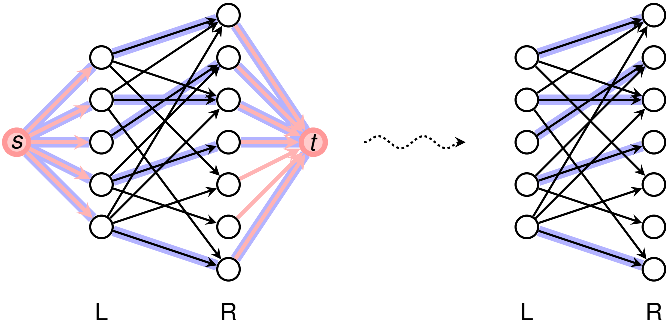

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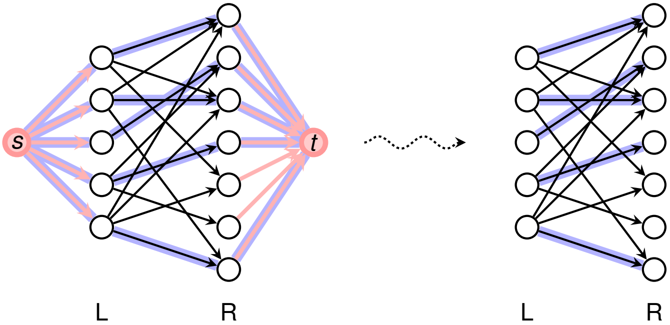

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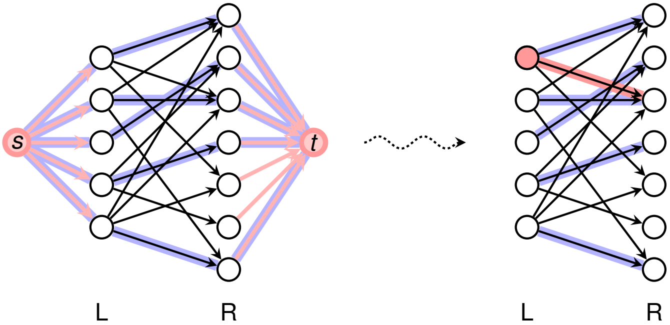

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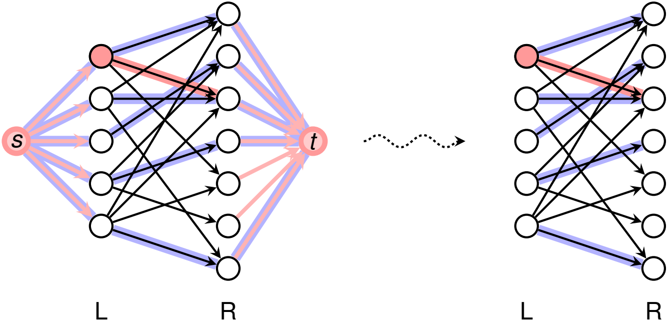

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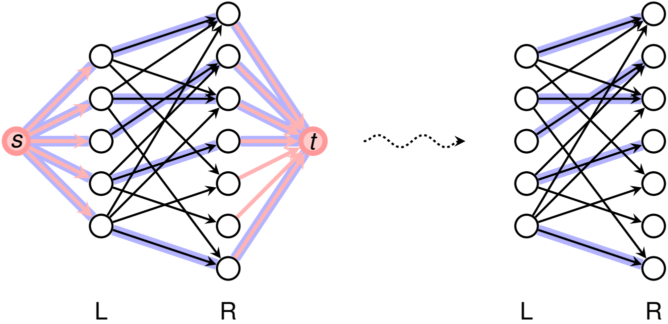

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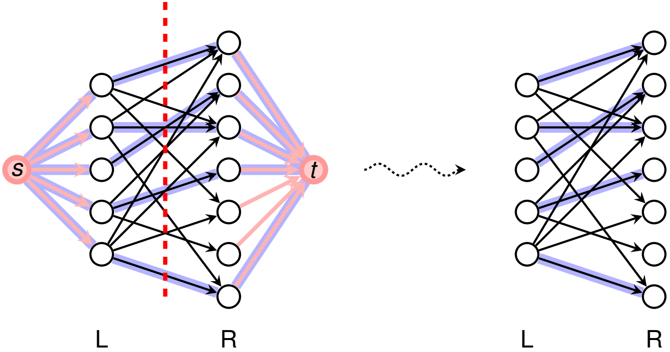
c) Cut $(L \cup \{s\}, R \cup \{t\})$



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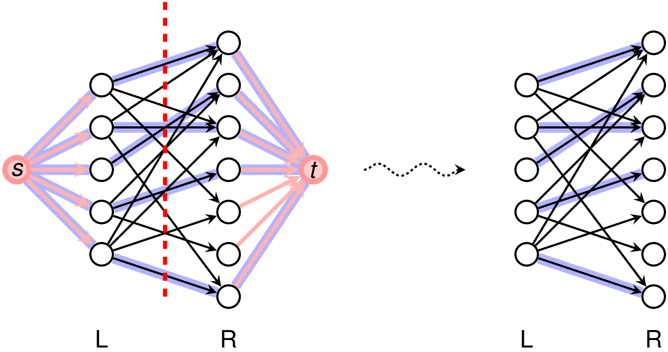

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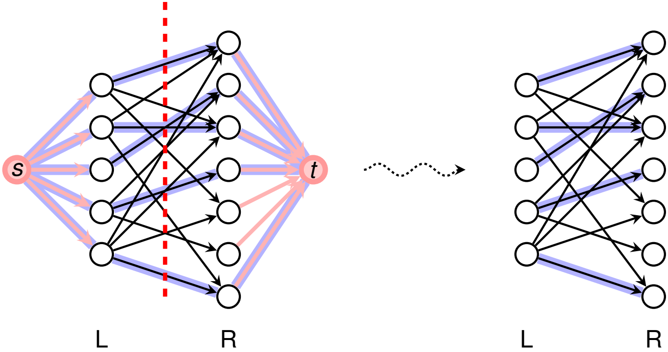

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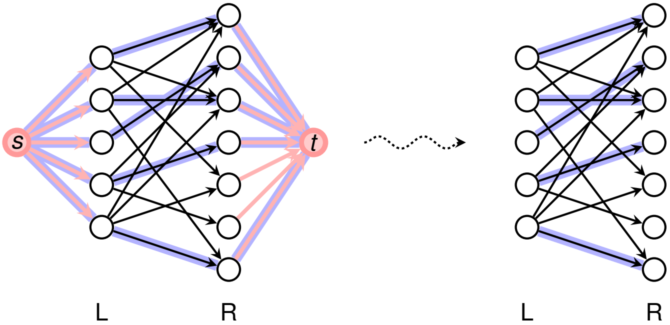

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value of maxflow $\leq$ max cardinality matching