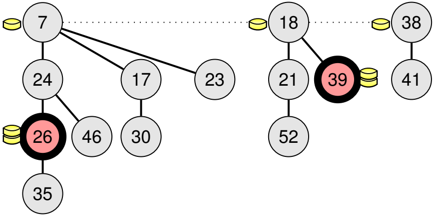


5.2 Fibonacci Heaps (Analysis)

Frank Stajano

Thomas Sauerwald

Lent 2016

UNIVERSITY OF CAMBRIDGE

Outline

Recap of INSERT, EXTRACT-MIN and DECREASE-KEY

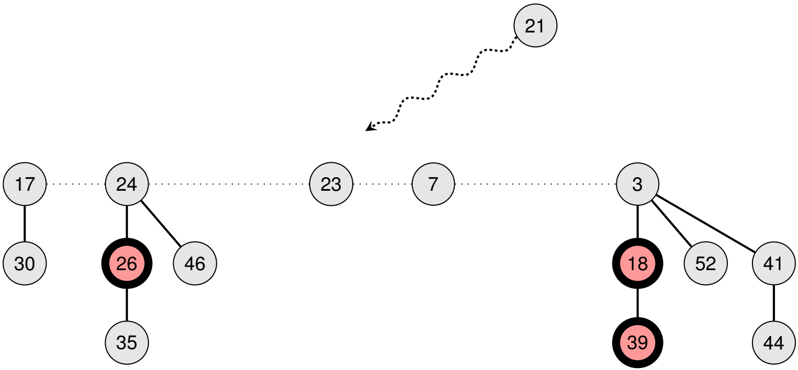
Glimpse at the Analysis

Amortized Analysis

Bounding the Maximum Degree

Fibonacci Heap: INSERT

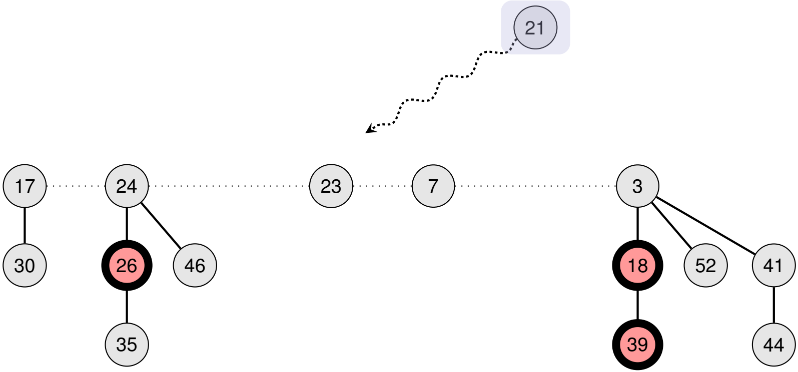
INSERT



Fibonacci Heap: INSERT

INSERT

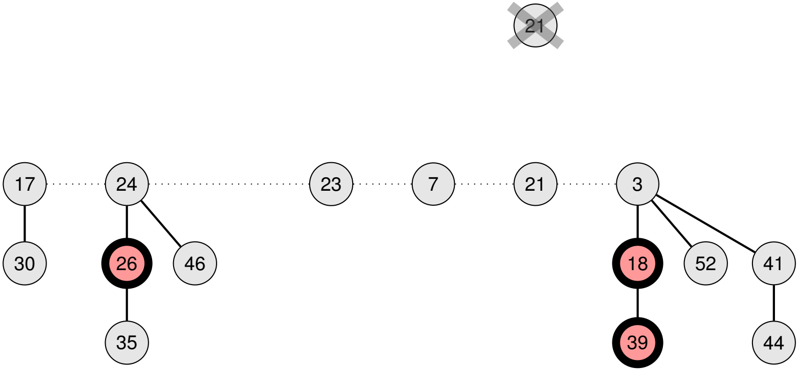
- Create a singleton tree



Fibonacci Heap: INSERT

INSERT

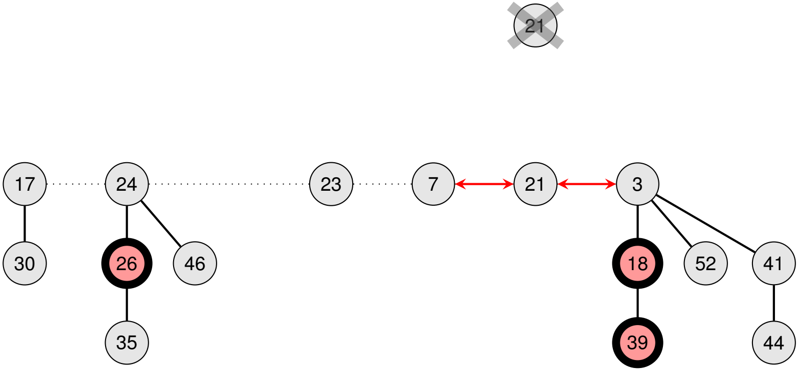
- Create a singleton tree
- Add to root list



Fibonacci Heap: INSERT

INSERT

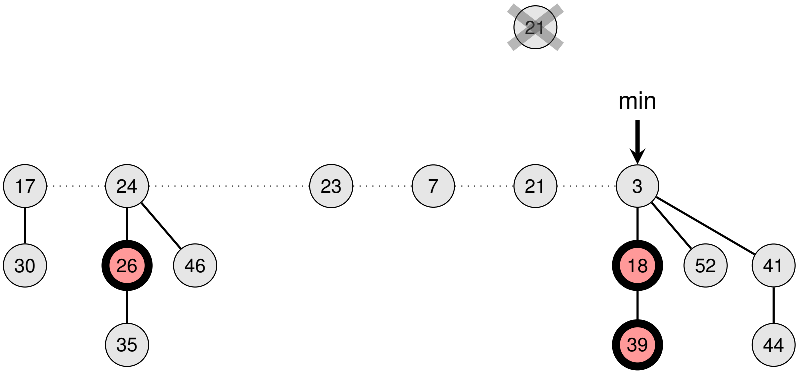
- Create a singleton tree
- Add to root list



Fibonacci Heap: INSERT

INSERT

- Create a singleton tree
- Add to root list and update min-pointer (if necessary)



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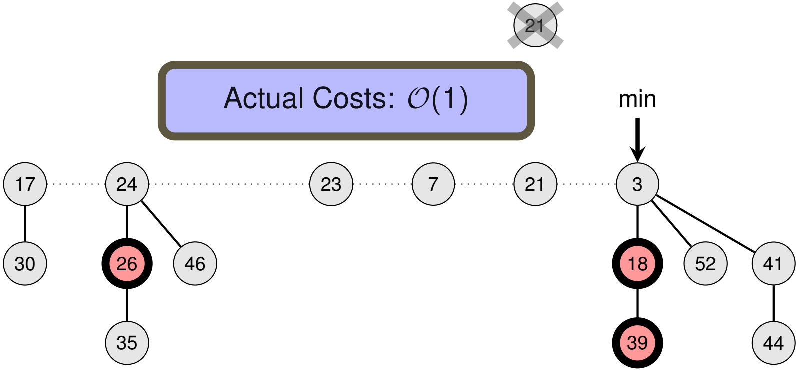
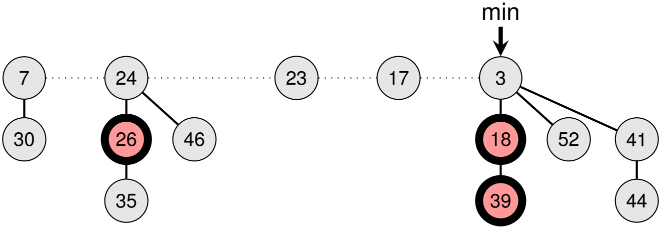


Fibonacci Heap: EXTRACT-MIN

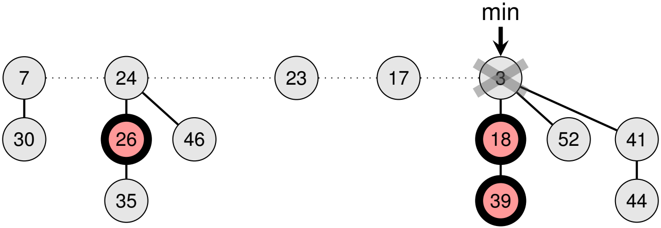
EXTRACT-MIN



Fibonacci Heap: EXTRACT-MIN

EXTRACT-MIN

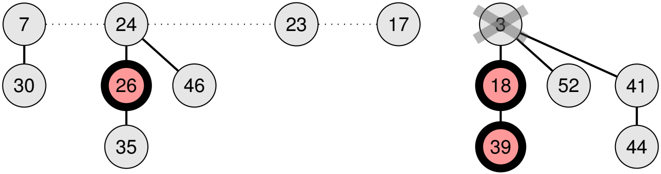
- Delete min



Fibonacci Heap: EXTRACT-MIN

EXTRACT-MIN

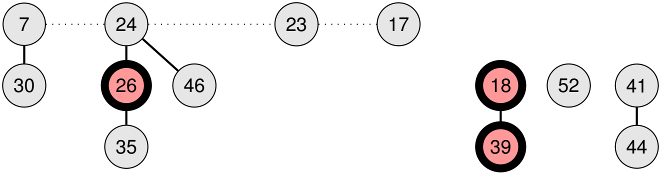
- Delete min ✓



Fibonacci Heap: EXTRACT-MIN

EXTRACT-MIN

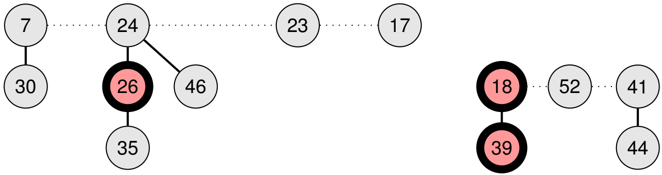
- Delete min ✓
- Meld childen into root list and unmark them



Fibonacci Heap: Extract-Min

Extract-Min

- Delete min ✓
- Meld childen into root list and unmark them



Fibonacci Heap: EXTRACT-MIN

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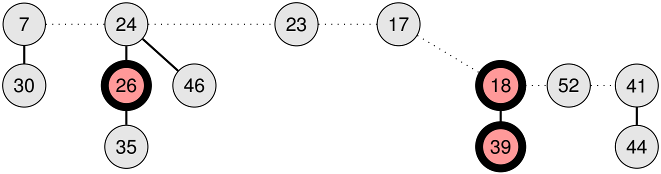

Fibonacci Heap: Extract-Min

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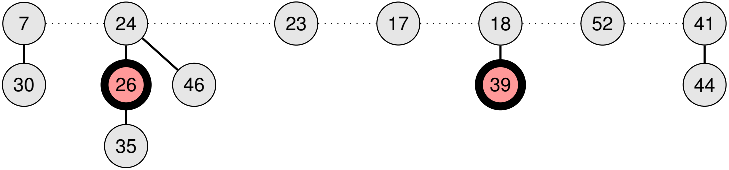
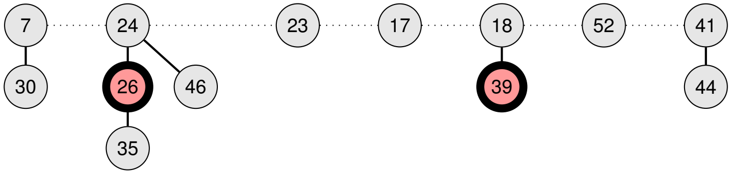


Fibonacci Heap: EXTRACT-MIN

EXTRACT-MIN

- Delete min ✓
- Meld childen into root list and unmark them ✓
- Consolidate so that no roots have the same degree



Fibonacci Heap: EXTRACT-MIN

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- Delete min ✓
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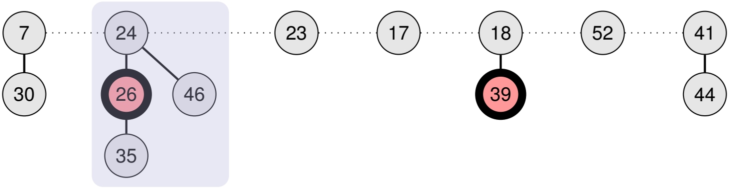

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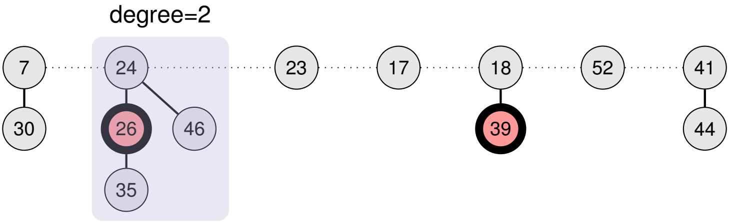

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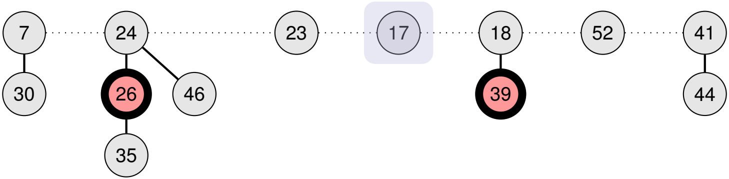

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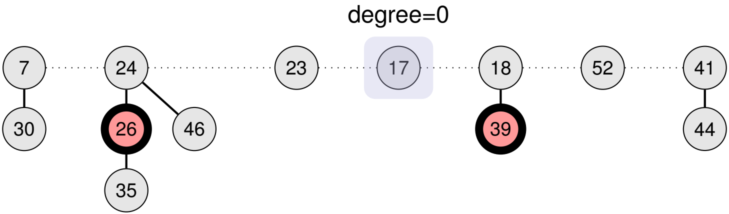

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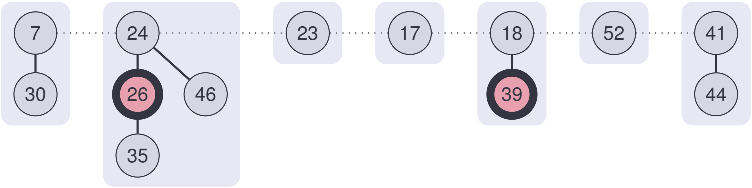

Fibonacci Heap: Extract-Min

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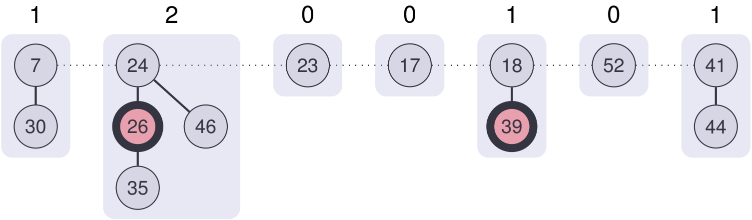


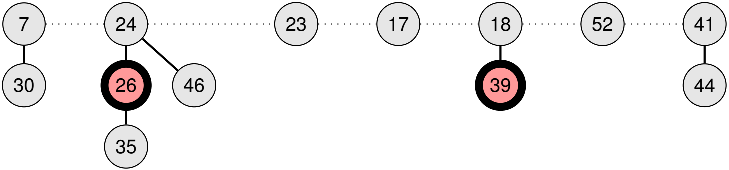
Fibonacci Heap: Extract-Min

Extract-Min

- Delete min ✓
- Meld childen into root list and unmark them ✓
- Consolidate so that no roots have the same degree (# children)

degree

0	1	2	3



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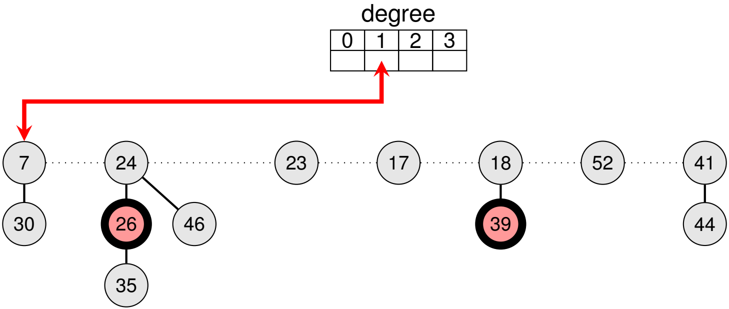

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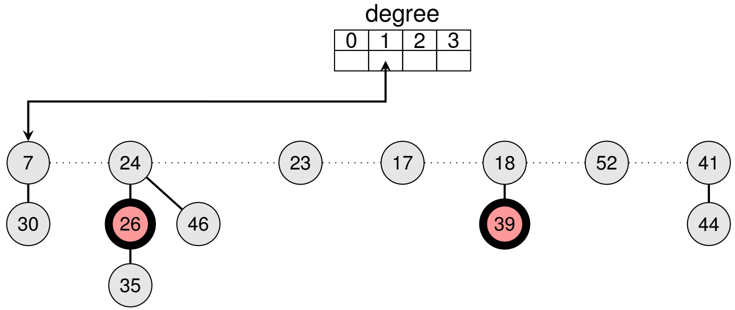

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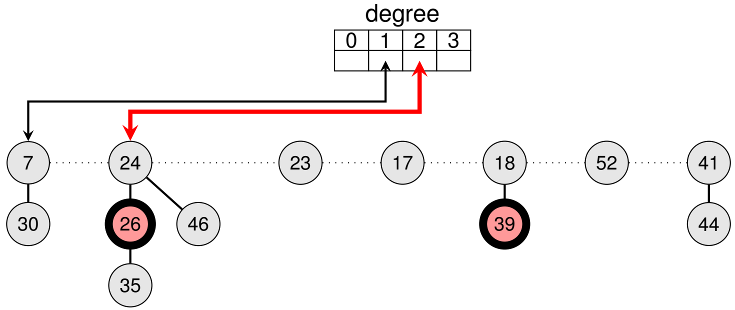

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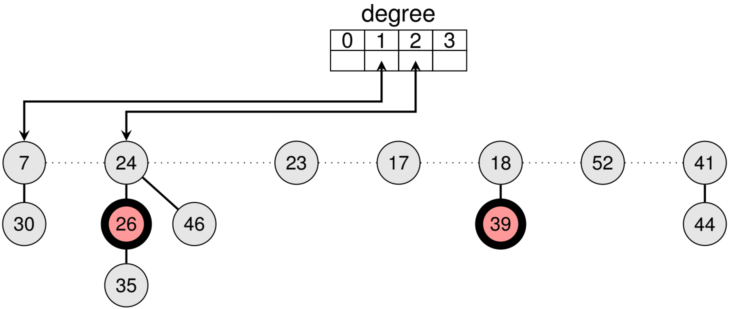

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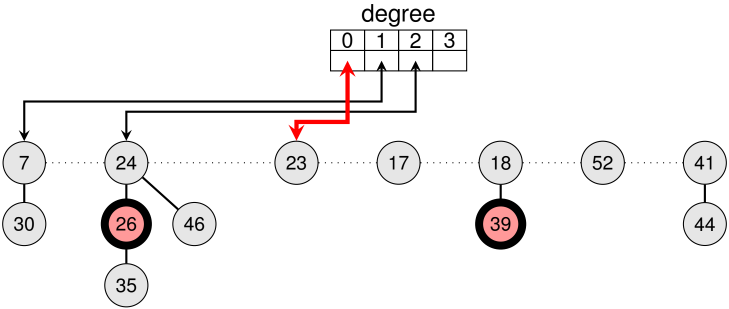

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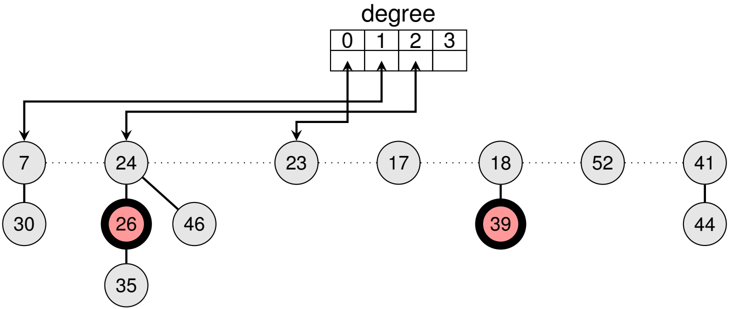

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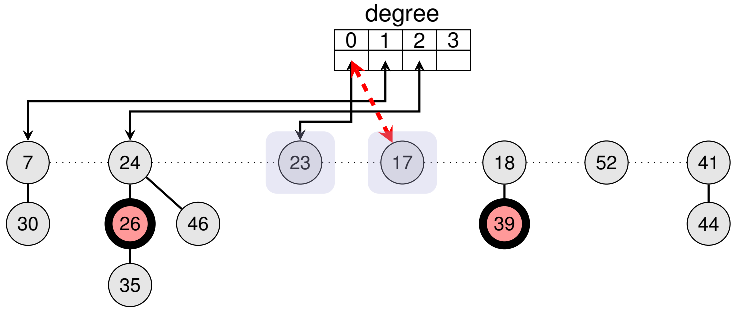

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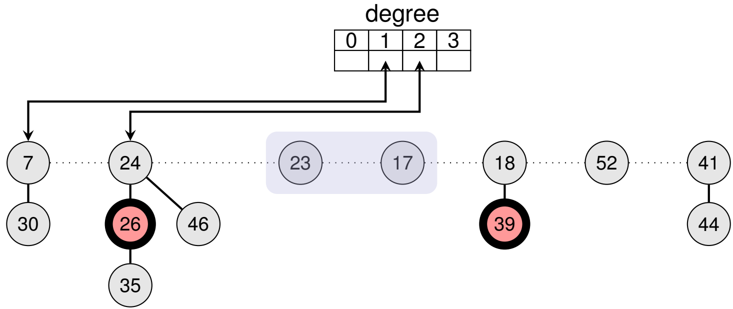

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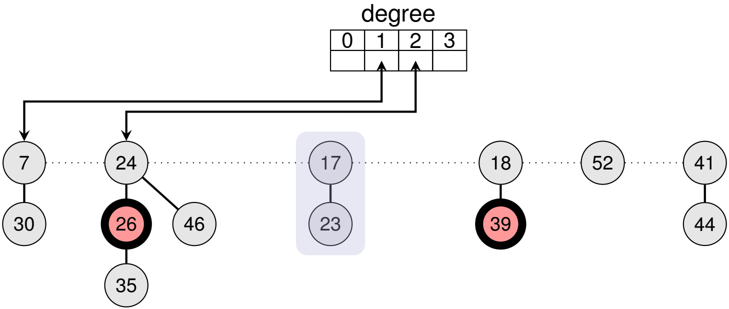

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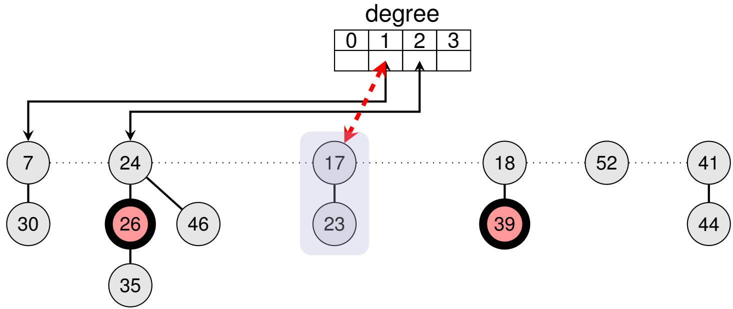

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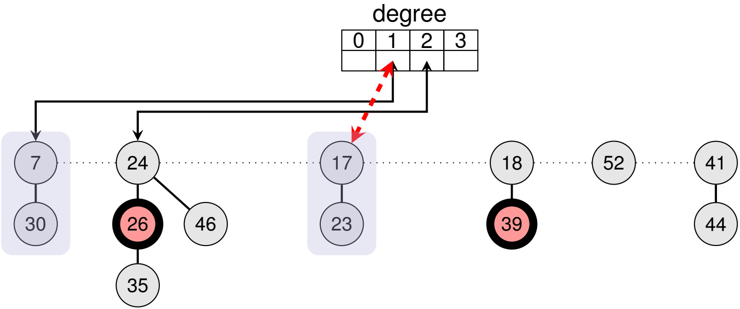

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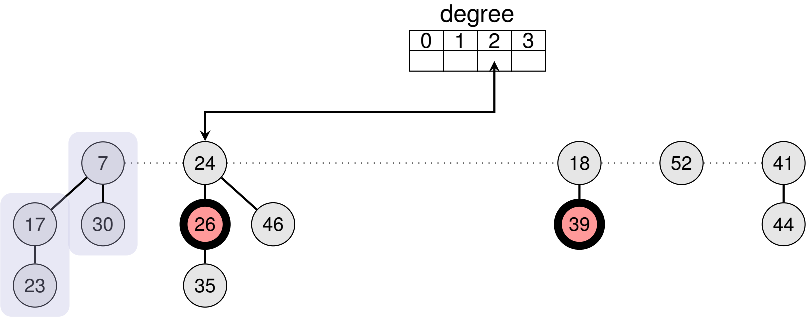

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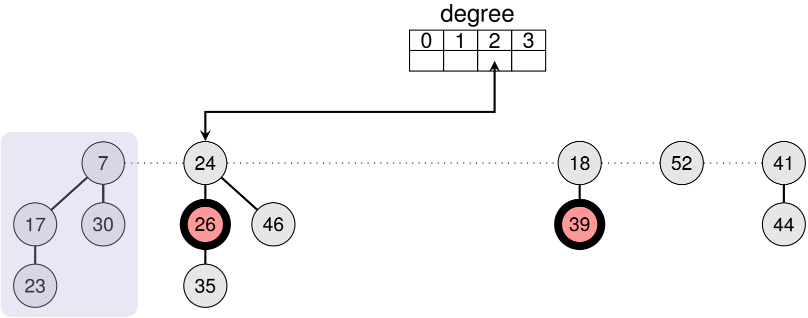

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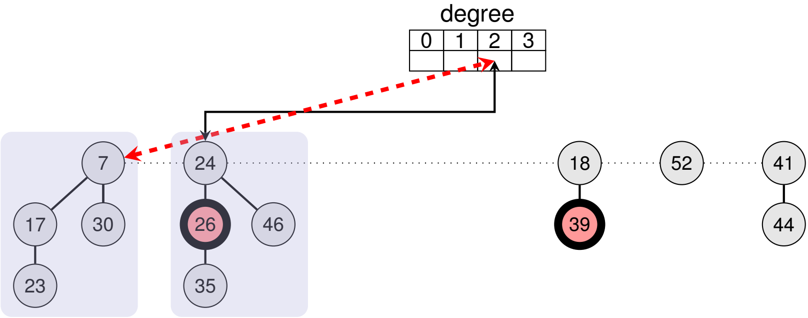

Fibonacci Heap: EXTRACT-MIN

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- Delete min ✓
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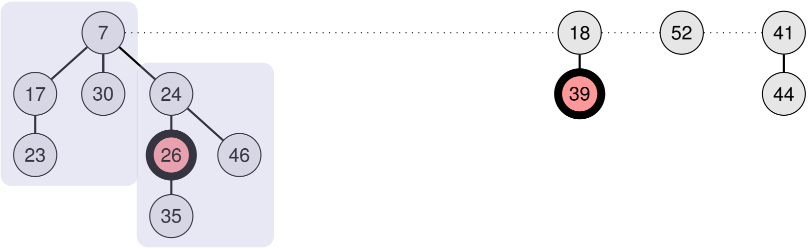
- Consolidate so that no roots have the same degree (# children)

degree

0	1	2	3



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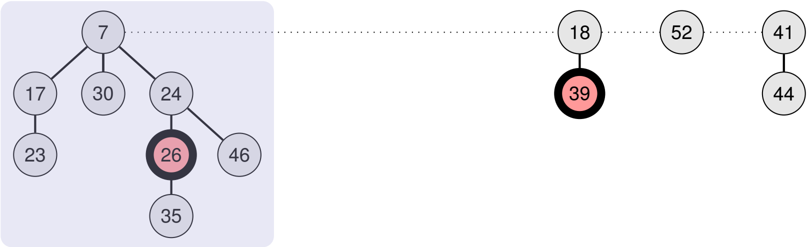

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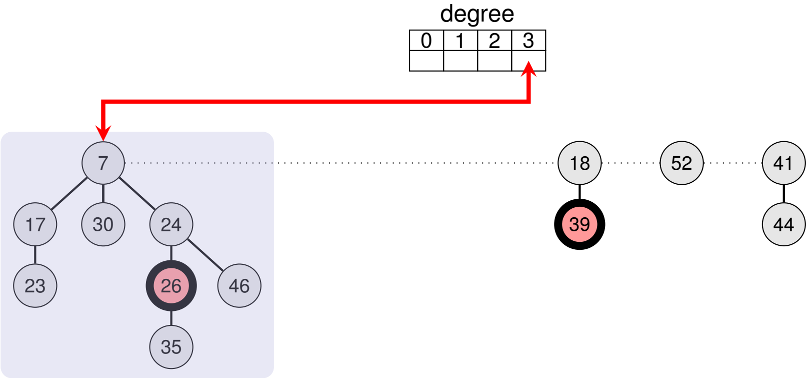

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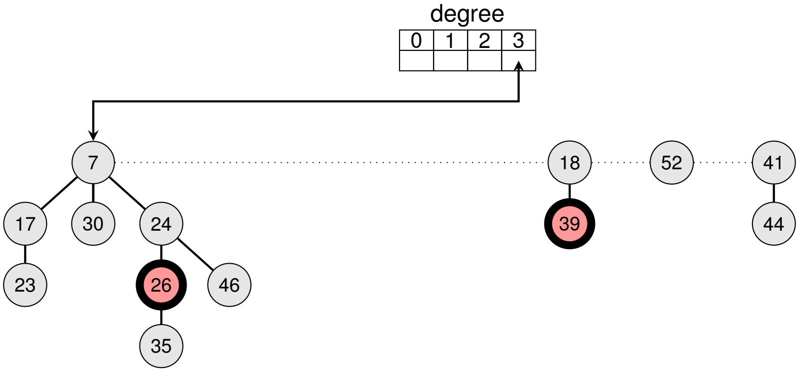

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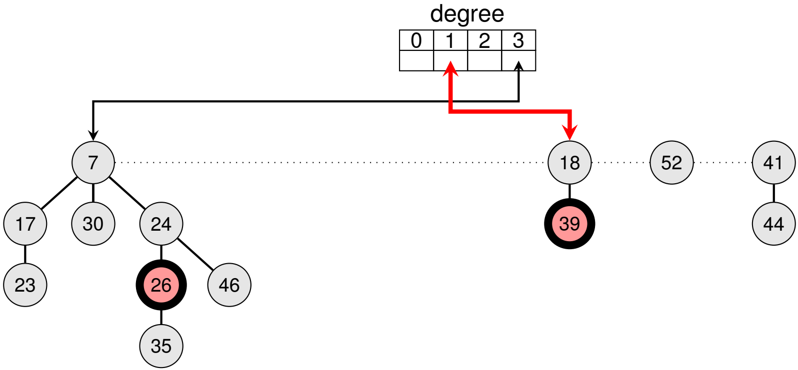

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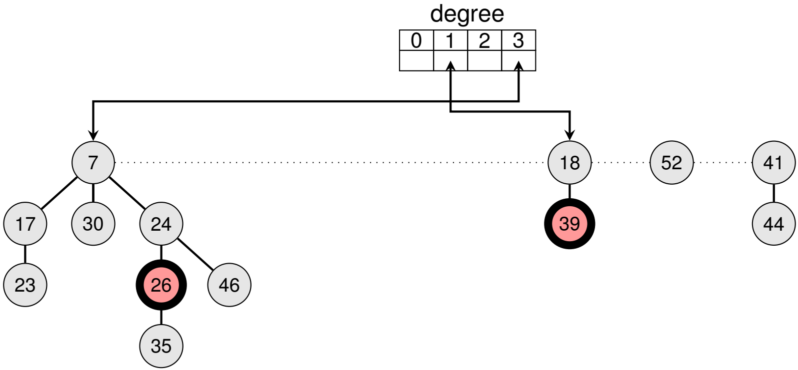

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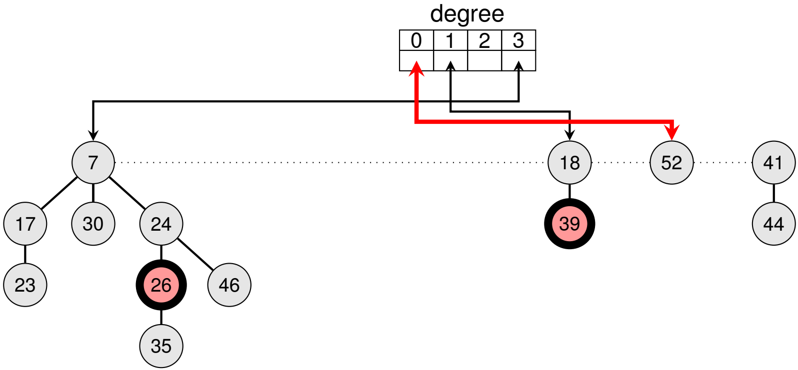

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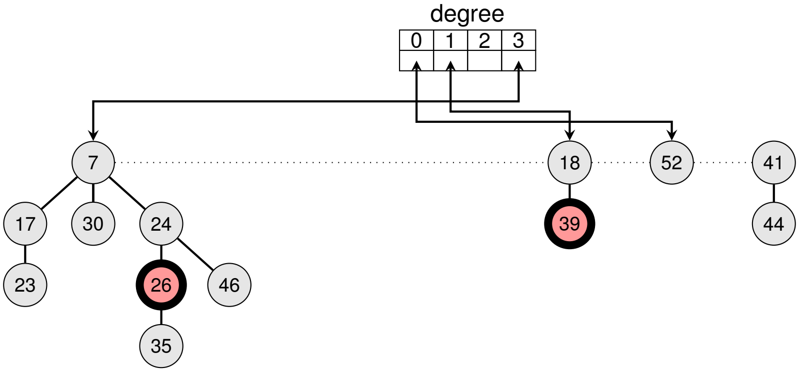

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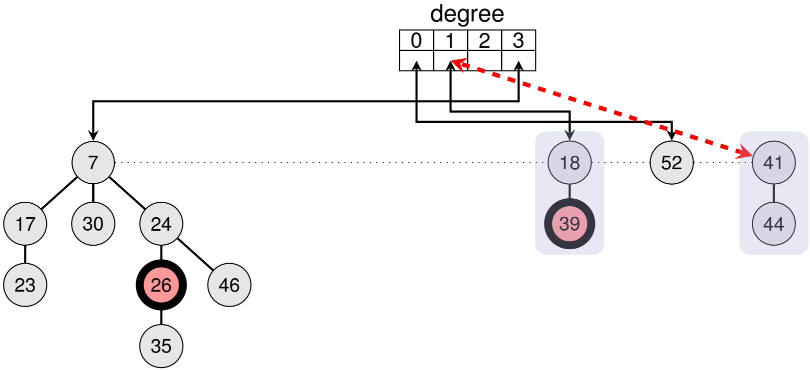

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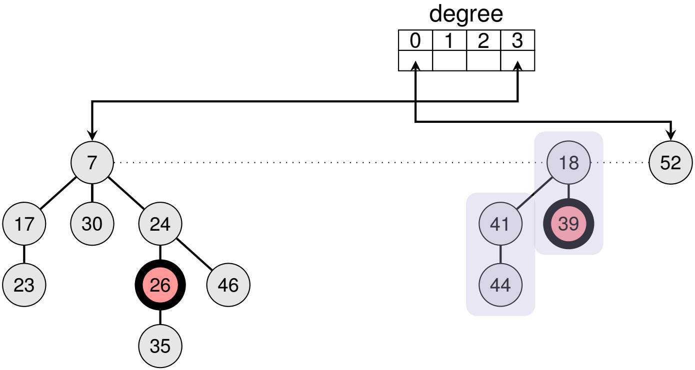

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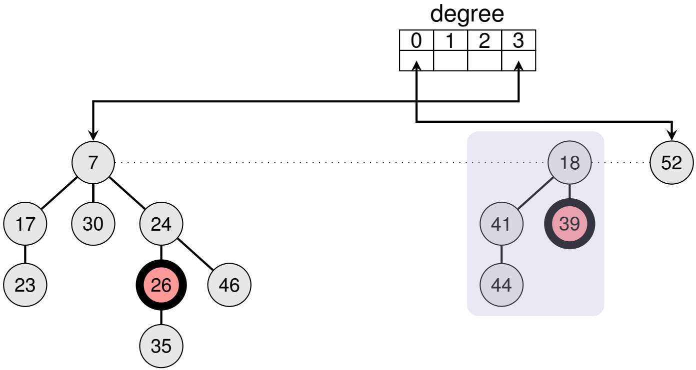

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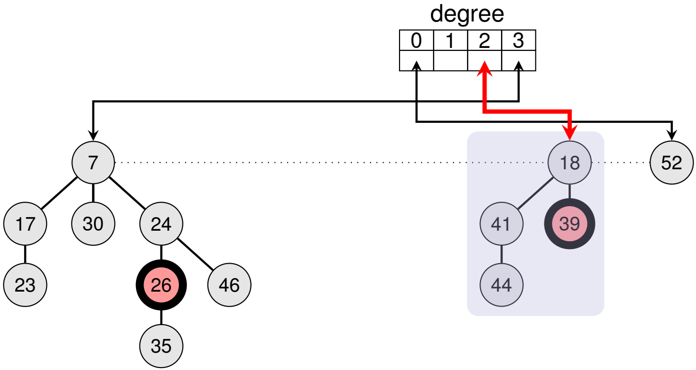

Fibonacci Heap: EXTRACT-MIN

EXTRACT-MIN

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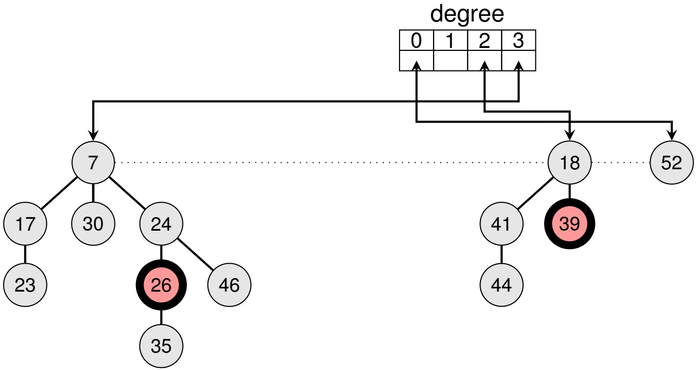
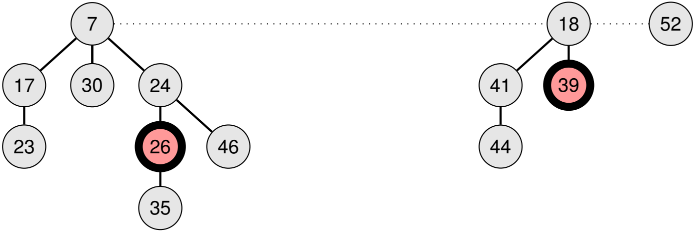
- Consolidate so that no roots have the same degree (# children) ✓



Fibonacci Heap: EXTRACT-MIN

EXTRACT-MIN

- Delete min ✓
- Meld childen into root list and unmark them ✓
- Consolidate so that no roots have the same degree (# children) ✓
- Update minimum



Fibonacci Heap: EXTRACT-MIN

EXTRACT-MIN

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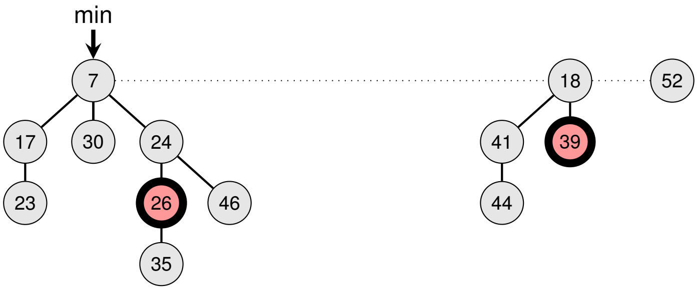
- Consolidate so that no roots have the same degree (# children) ✓
- Update minimum ✓



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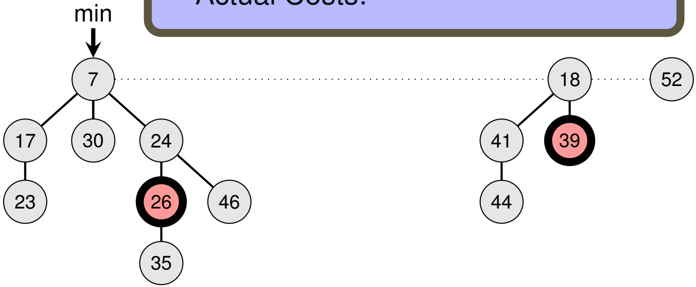
- Update minimum ✓

Actual Costs:



Fibonacci Heap: EXTRACT-MIN

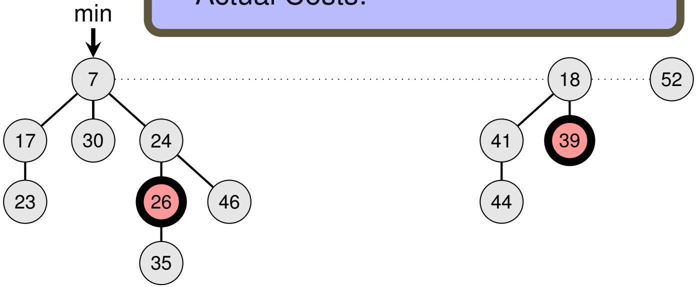
EXTRACT-MIN

- Delete min ✓
- Meld childen into root list and unmark them ✓
- Consolidate so that no roots have the same degree (# children) ✓
- Update minimum ✓

Every root becomes child of another root at most once!

$d(n)$ is the maximum degree of a root in any Fibonacci heap of size n

Actual Costs:



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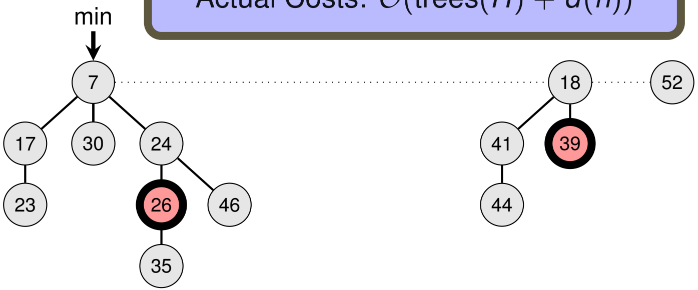
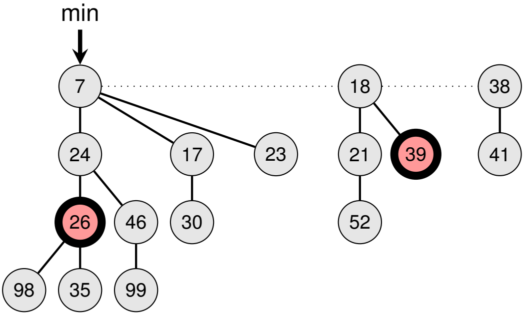
Actual Costs: $\mathcal{O}(\text{trees}(H) + d(n))$



Fibonacci Heap: DECREASE-KEY

DECREASE-KEY of node x

- Decrease the key of x (given by a pointer)



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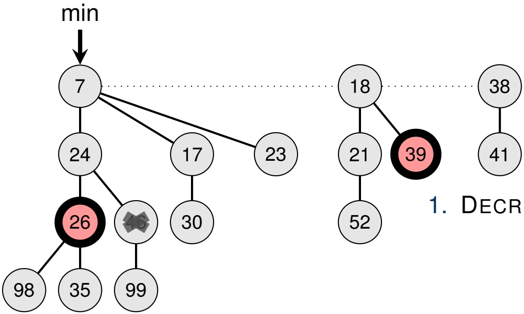

1. DECREASE-KEY 46 $\rightsquigarrow$ 15

DECREASE-KEY of node x

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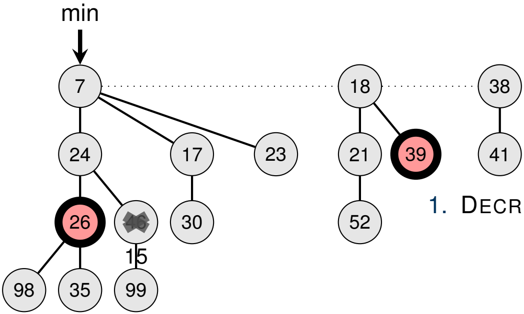
$\Rightarrow$ Cut tree rooted at x, unmark x, meld into root list



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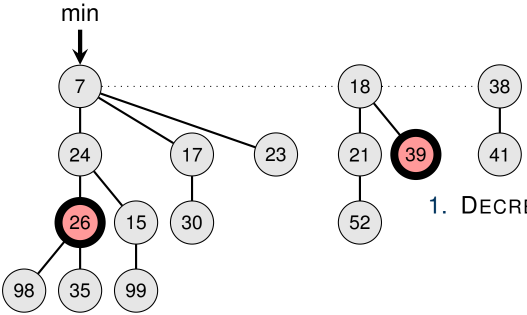

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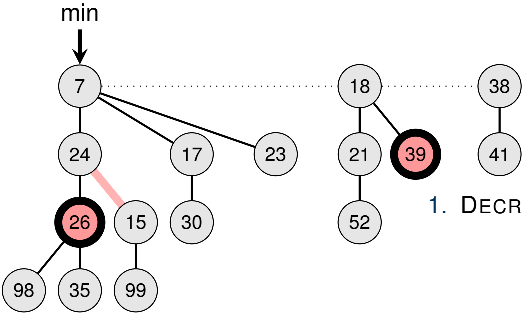

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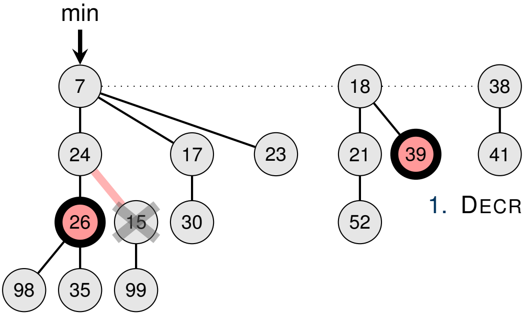

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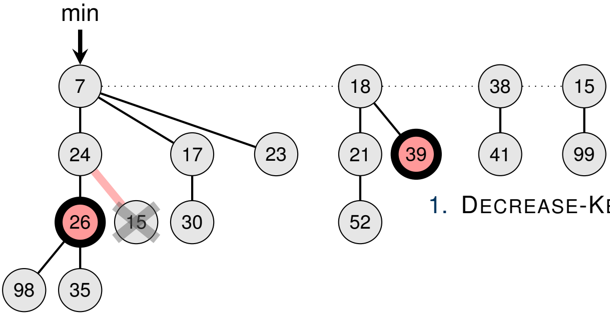

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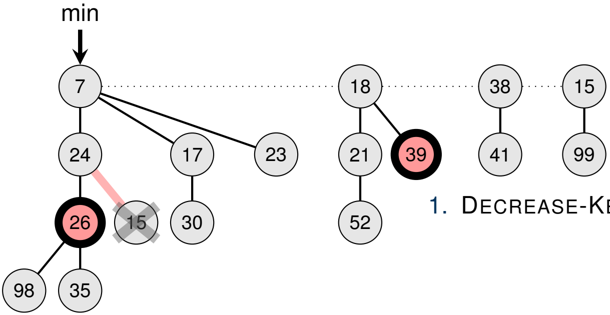

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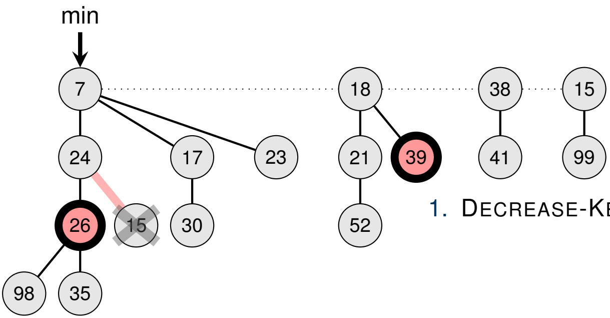

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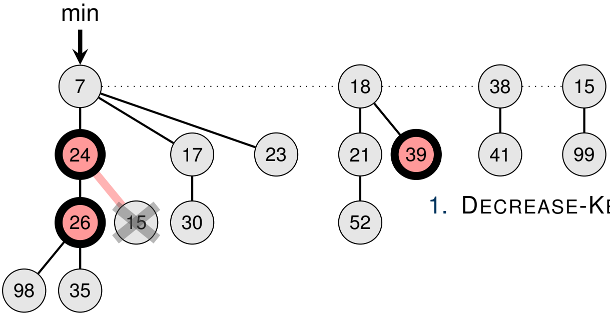

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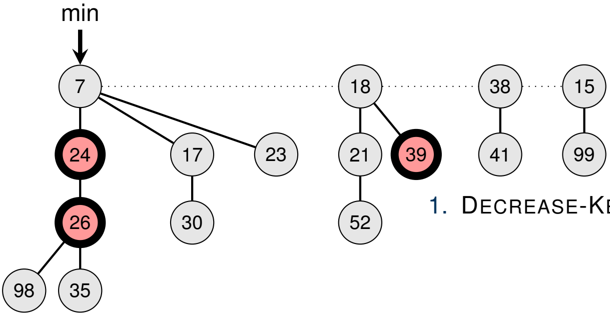

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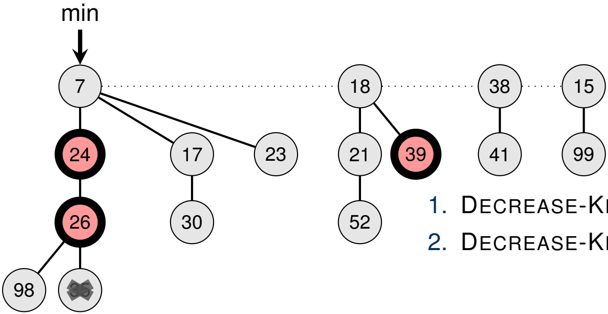

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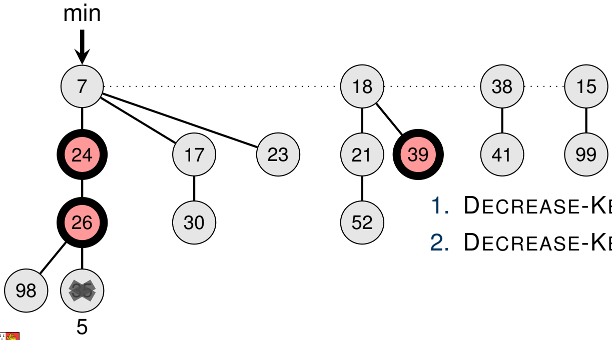

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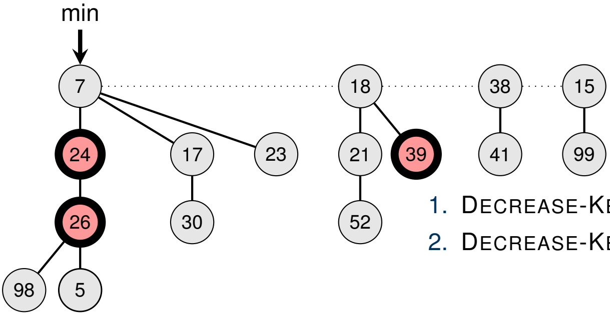

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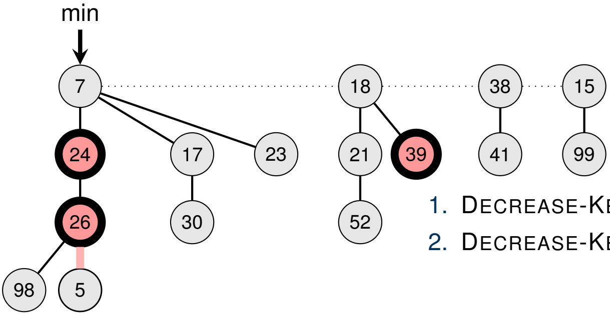

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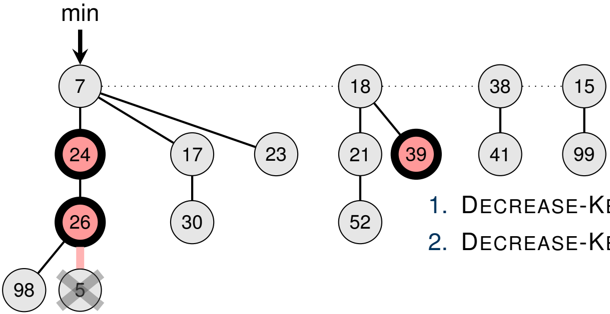

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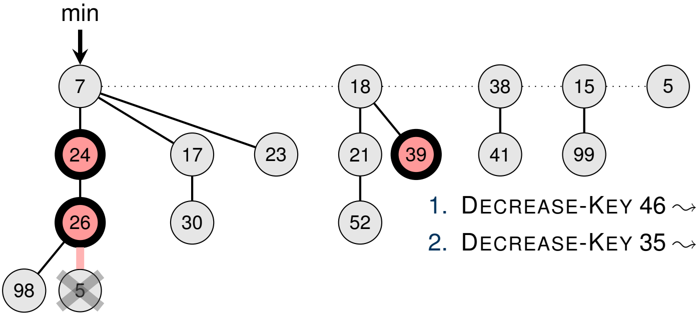

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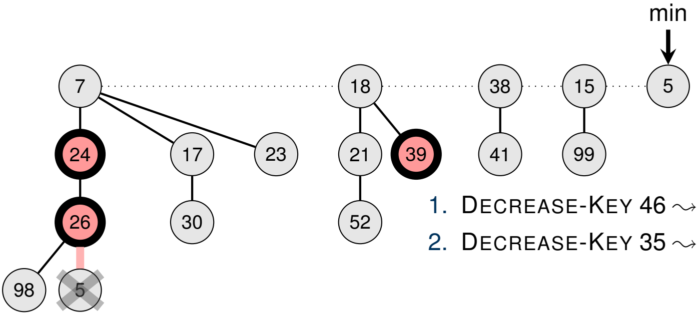

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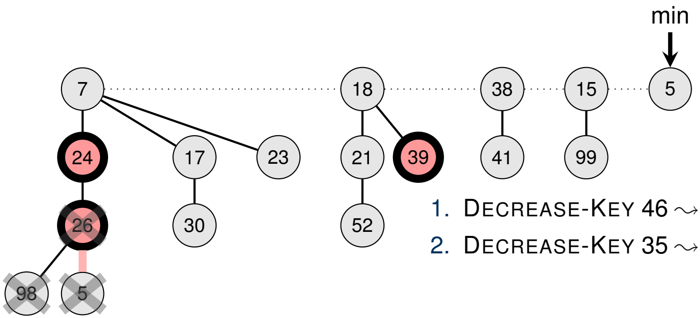

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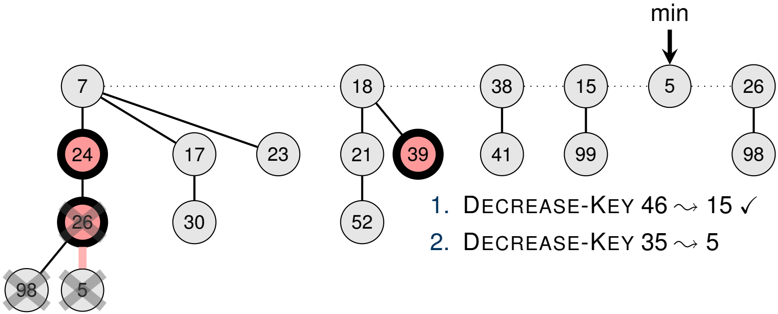

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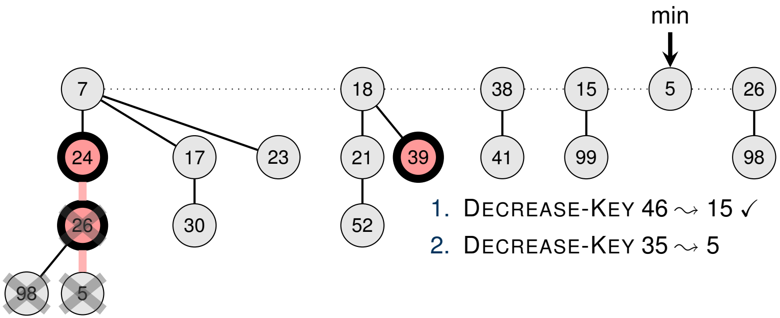

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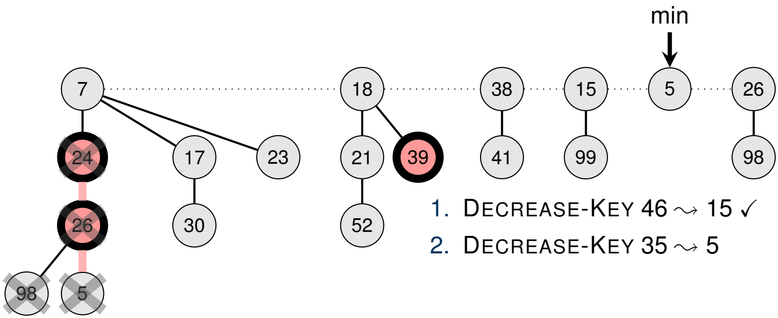

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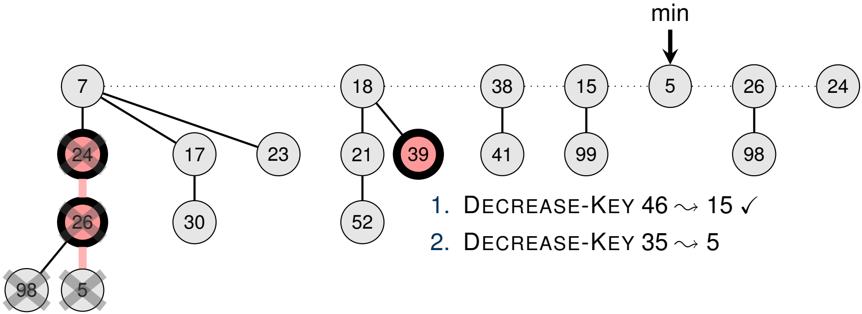

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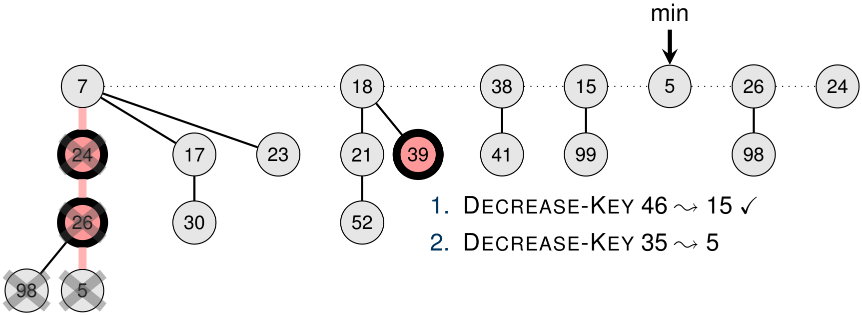

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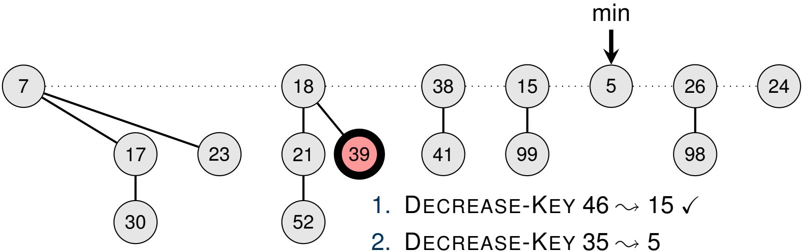

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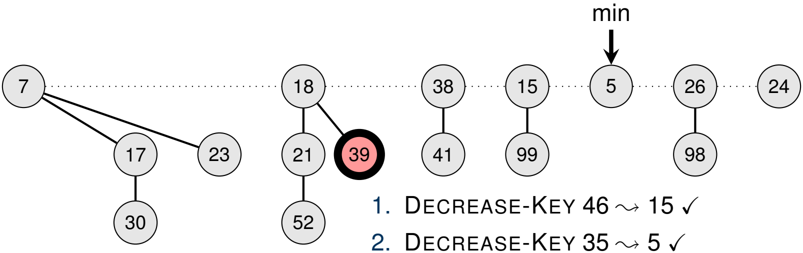

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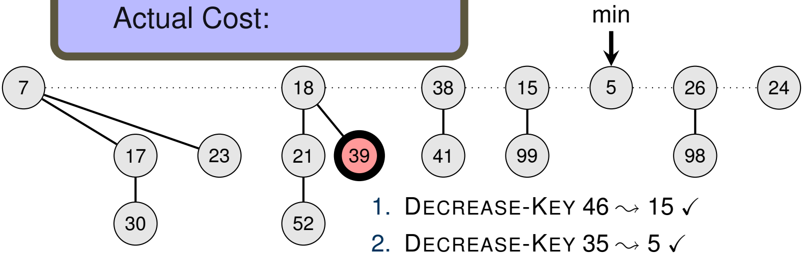

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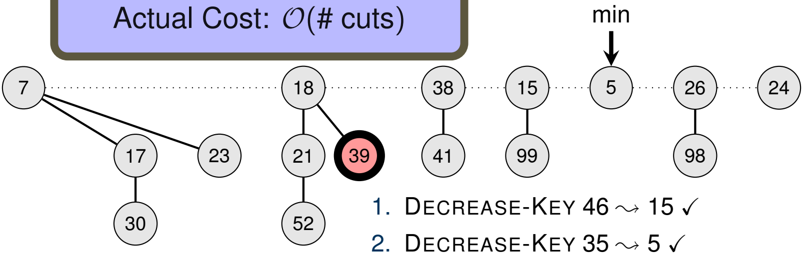
Actual Cost: $\mathcal{O}(\text{\# cuts})$



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Outline

Recap of INSERT, EXTRACT-MIN and DECREASE-KEY

Glimpse at the Analysis

Amortized Analysis

Bounding the Maximum Degree

Amortized Analysis via Potential Method

- INSERT: actual $\mathcal{O}(1)$
- EXTRACT-MIN: actual $\mathcal{O}(\text{trees}(H) + d(n))$
- DECREASE-KEY: actual $\mathcal{O}(\text{\# cuts}) \leq \mathcal{O}(\text{marks}(H))$

Amortized Analysis via Potential Method

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$$\boxed{\Phi(H) = \text{trees}(H) + 2 \cdot \text{marks}(H)}$$

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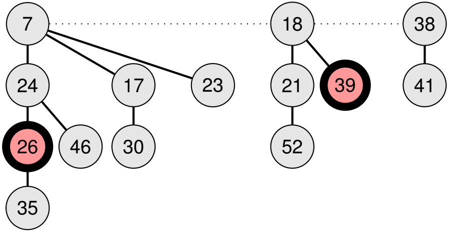

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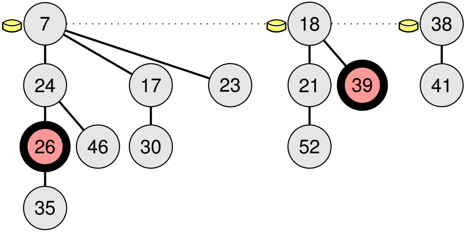

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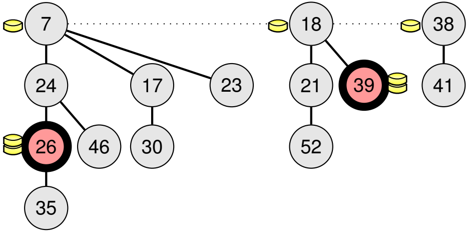
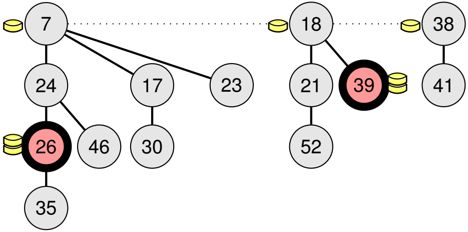


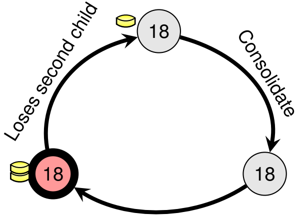
Amortized Analysis via Potential Method

- INSERT: actual $\mathcal{O}(1)$ amortized $\mathcal{O}(1)$
- EXTRACT-MIN: actual $\mathcal{O}(\text{trees}(H) + d(n))$ amortized $\mathcal{O}(d(n))$
- DECREASE-KEY: actual $\mathcal{O}(\#\text{ cuts}) \leq \mathcal{O}(\text{marks}(H))$ amortized $\mathcal{O}(1)$

$$\Phi(H) = \text{trees}(H) + 2 \cdot \text{marks}(H)$$



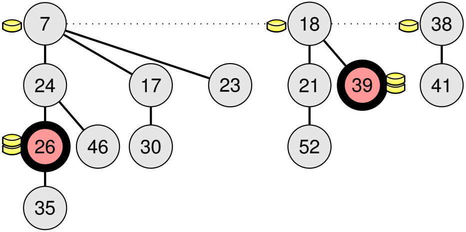
Lifecycle of a node



Amortized Analysis via Potential Method

- INSERT: actual $\mathcal{O}(1)$ amortized $\mathcal{O}(1)$ ✓
- EXTRACT-MIN: actual $\mathcal{O}(\text{trees}(H) + d(n))$ amortized $\mathcal{O}(d(n))$?
- DECREASE-KEY: actual $\mathcal{O}(\#\text{ cuts}) \leq \mathcal{O}(\text{marks}(H))$ amortized $\mathcal{O}(1)$?

$$\boxed{\Phi(H) = \text{trees}(H) + 2 \cdot \text{marks}(H)}$$



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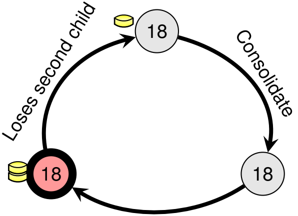

Outline

Recap of INSERT, EXTRACT-MIN and DECREASE-KEY

Glimpse at the Analysis

Amortized Analysis

Bounding the Maximum Degree

Amortized Analysis of DECREASE-KEY

Actual Cost

- DECREASE-KEY: $\mathcal{O}(x+1)$, where x is the number of cuts.

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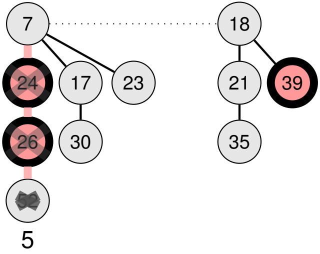
Change in Potential



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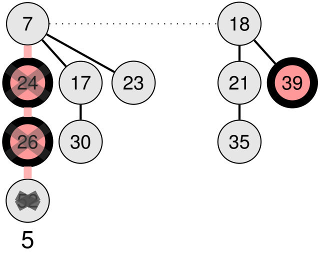
Change in Potential

- $\text{trees}(H') =$



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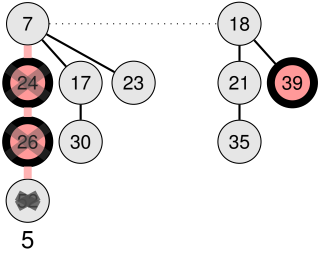
Change in Potential

- $\text{trees}(H') = \text{trees}(H) + x$



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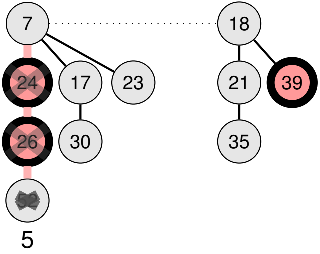
$$\Phi(H) = \text{trees}(H) + 2 \cdot \text{marks}(H)$$

Change in Potential

- $\text{trees}(H') = \text{trees}(H) + x$
- $\text{marks}(H') \leq$



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- $\text{trees}(H') = \text{trees}(H) + x$
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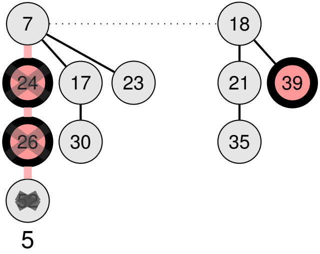

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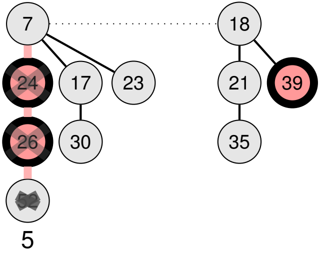
$\Rightarrow \Delta\Phi \leq x + 2 \cdot (-x + 2) = 4 - x.$



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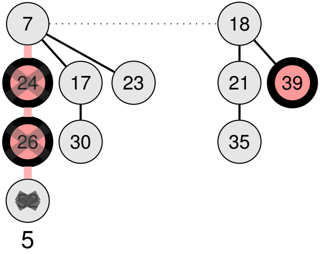

Amortized Cost

$$\widetilde{c}_i = c_i + \Delta\Phi$$

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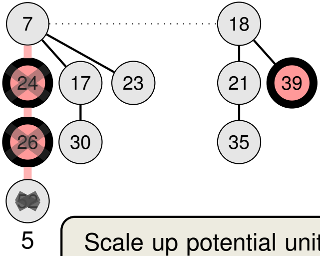

Scale up potential units

Amortized Cost

$$\widetilde{c}_i = c_i + \Delta\Phi \leq \mathcal{O}(x+1) + 4 - x$$

Amortized Analysis of DECREASE-KEY

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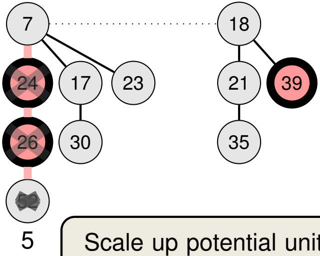

Scale up potential units

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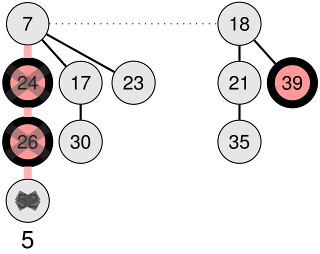
$$\Phi(H) = \text{trees}(H) + 2 \cdot \text{marks}(H)$$

First Coin $\rightsquigarrow$ pays cut
Second Coin $\rightsquigarrow$ increase of trees(H)



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Amortized Cost

$$\widetilde{c}_i = c_i + \Delta\Phi \leq \mathcal{O}(x+1) + 4 - x = \mathcal{O}(1)$$

Amortized Analysis of EXTRACT-MIN

Actual Cost

- EXTRACT-MIN: $\mathcal{O}(\text{trees}(H) + d(n))$

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Change in Potential

Amortized Analysis of Extract-Min

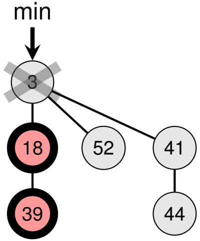
Actual Cost

- Extract-Min: $\mathcal{O}(\text{trees}(H) + d(n))$

$$\Phi(H) = \text{trees}(H) + 2 \cdot \text{marks}(H)$$

Change in Potential

- $\text{marks}(H') \ ? \ \text{marks}(H)$



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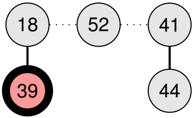

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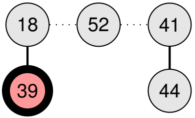
Change in Potential

- $\text{marks}(H') \leq \text{marks}(H)$



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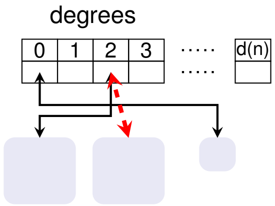

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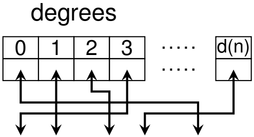

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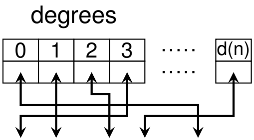
Change in Potential

- $\text{marks}(H') \leq \text{marks}(H)$
- $\text{trees}(H') \leq d(n) + 1$



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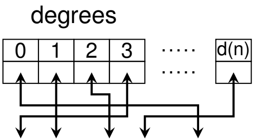
- $\text{marks}(H') \leq \text{marks}(H)$
- $\text{trees}(H') \leq \ d(n) + 1$

$\Rightarrow$ $\Delta\Phi \leq d(n) + 1 - \text{trees}(H)$



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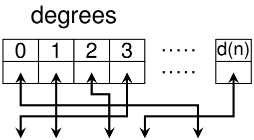

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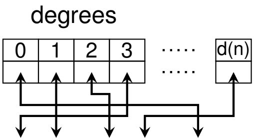

Amortized Cost

$$\widetilde{c_i} = c_i + \Delta\Phi \leq \mathcal{O}(\text{trees(H)} + d(n)) + d(n) + 1 - \text{trees(H)}$$

Amortized Analysis of EXTRACT-MIN

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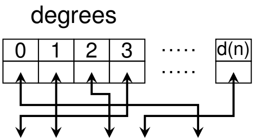

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$$\Phi(H) = \text{trees}(H) + 2 \cdot \text{marks}(H)$$

Change in Potential

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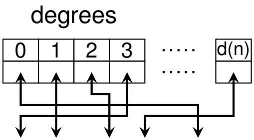

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How to bound $d(n)$?

Outline

Recap of INSERT, EXTRACT-MIN and DECREASE-KEY

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Bounding the Maximum Degree

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Every tree is a binomial tree $\Rightarrow d(n) \leq \log_2 n$.

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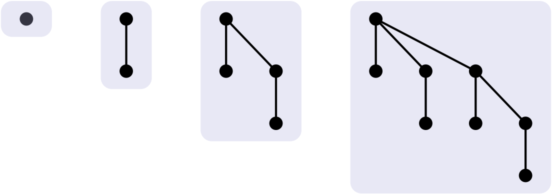

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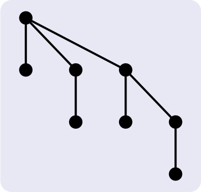

$d = 3, n = 2^3$

Bounding the Maximum Degree

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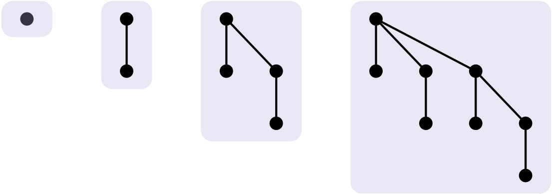

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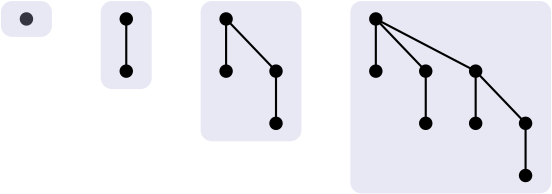

Fibonacci Heap

Not all trees are binomial trees, but still $d(n) \leq \log_\varphi n$, where $\varphi \approx 1.62$.

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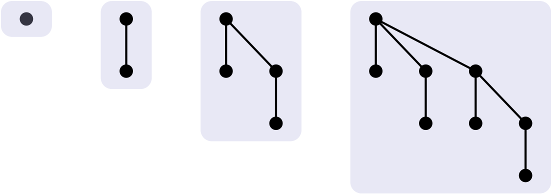

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Skip Analysis

Lower Bounding Degrees of Children

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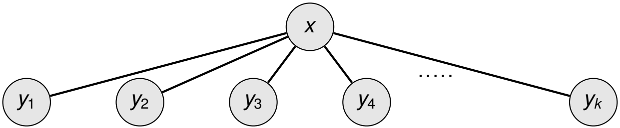
- Let $y_1, y_2, \ldots, y_k$ be the children in the order of attachment



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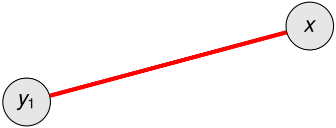

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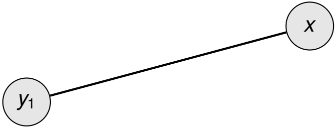

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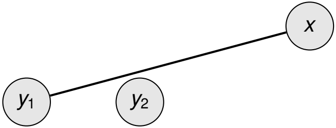

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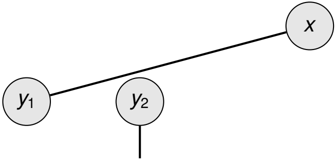

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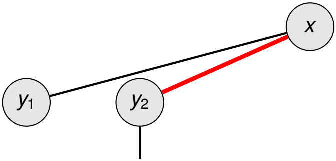

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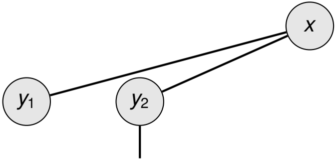

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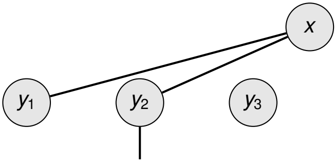

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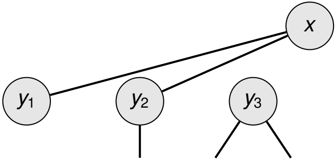

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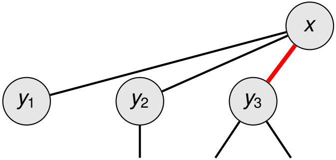

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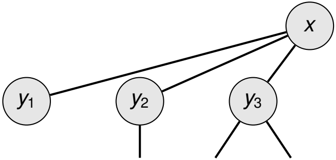

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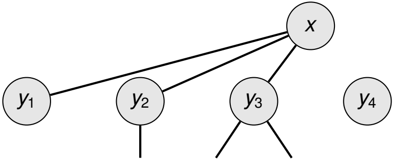

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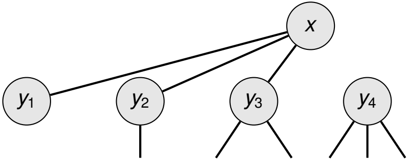

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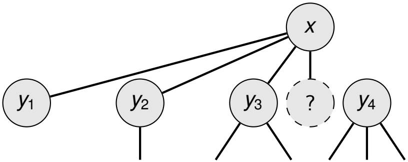

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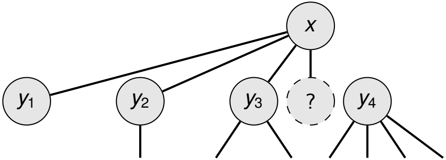

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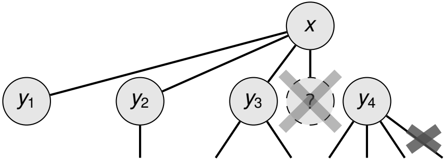

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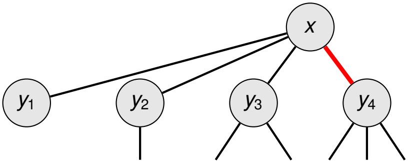

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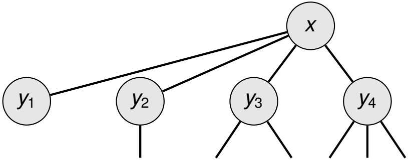

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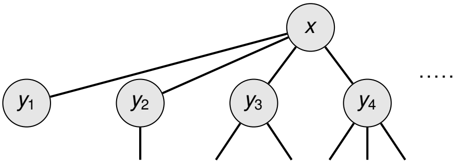

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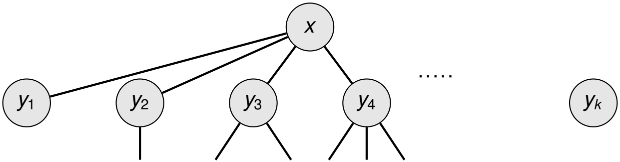

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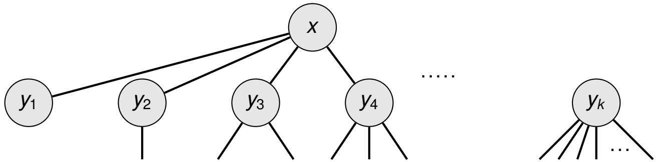

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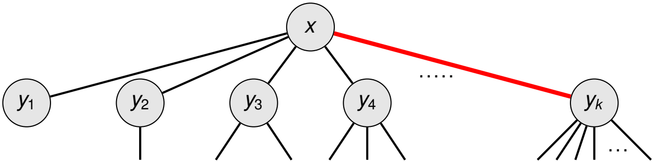

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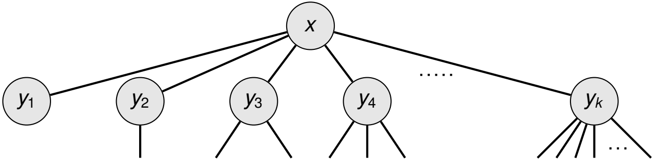

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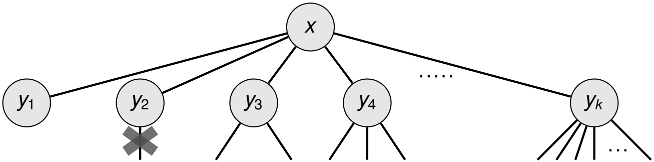

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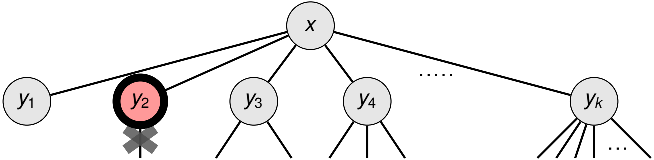

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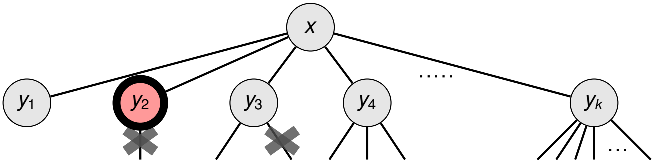

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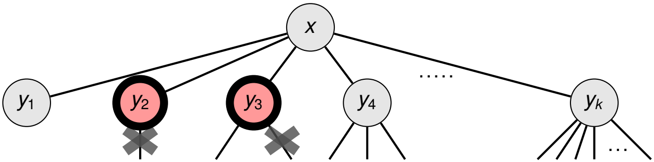

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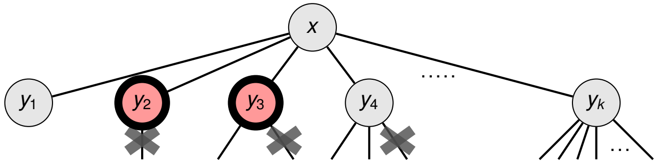

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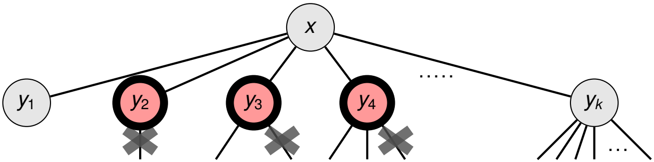

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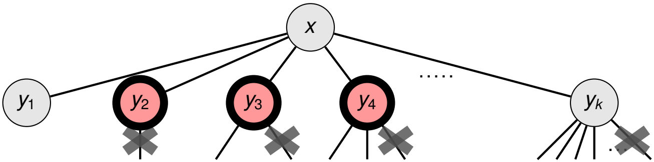

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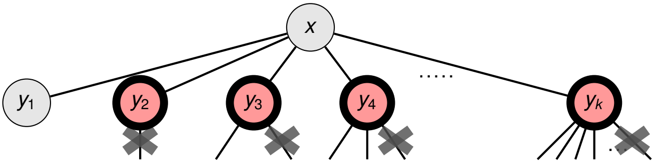

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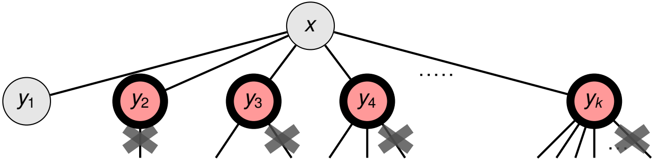

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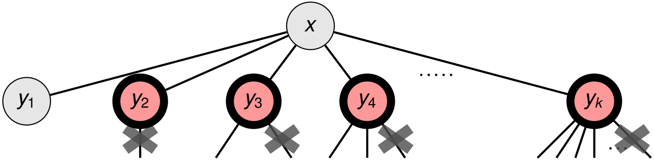
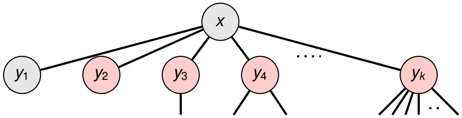
$\Rightarrow$ $\boxed{\forall 1 \leq i \leq k: \quad d_i \geq i - 2}$



From Degrees to Minimum Subtree Sizes



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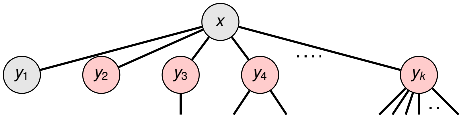

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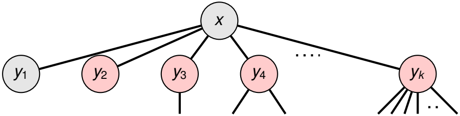
Definition

Let $N(k)$ be the minimum possible number of nodes of a subtree rooted at a node of degree k.

From Degrees to Minimum Subtree Sizes



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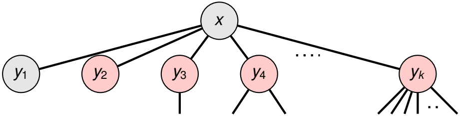

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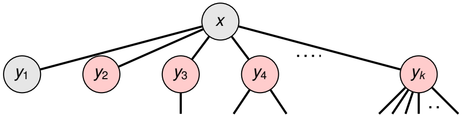

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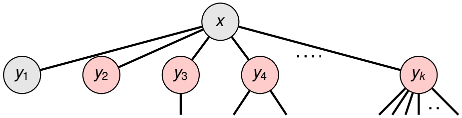

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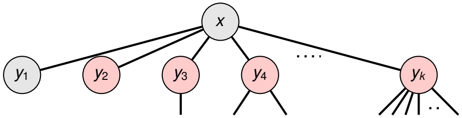

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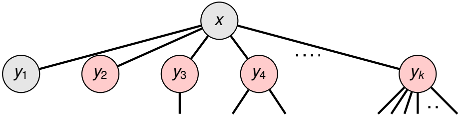

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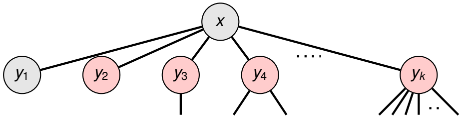

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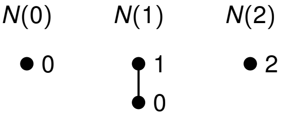

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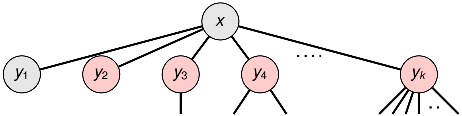

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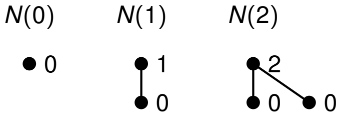

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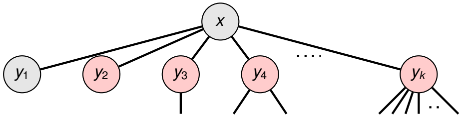

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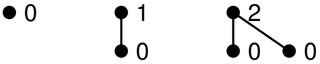

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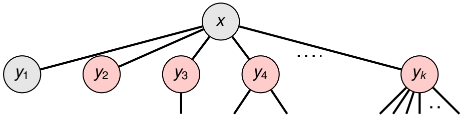

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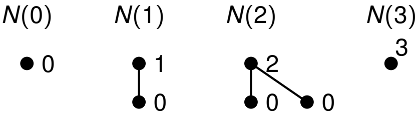

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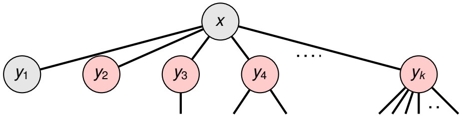

$$\forall 1 \leq i \leq k: \quad d_i \geq i - 2$$

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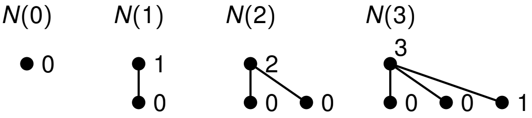

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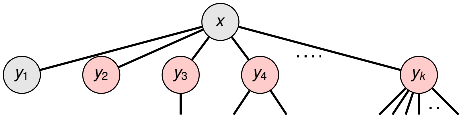

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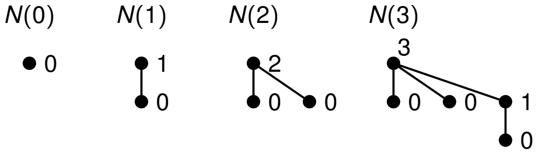

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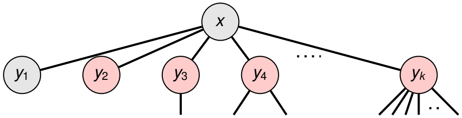

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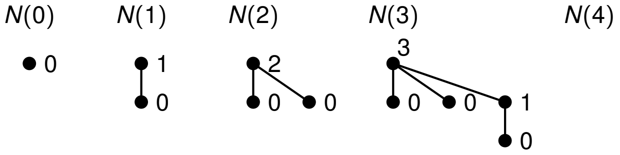

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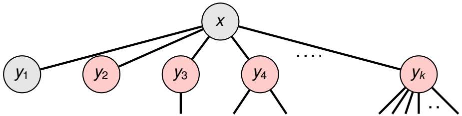

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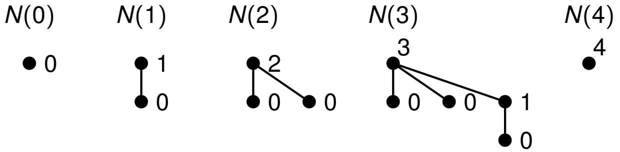

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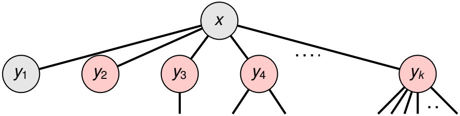

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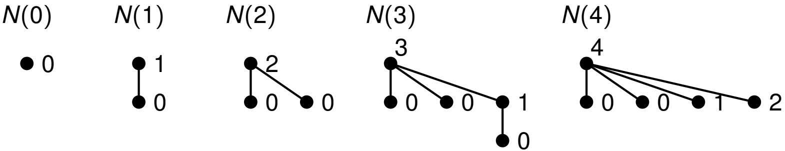

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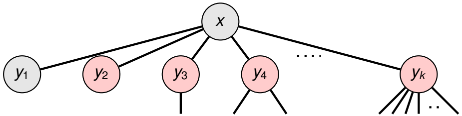

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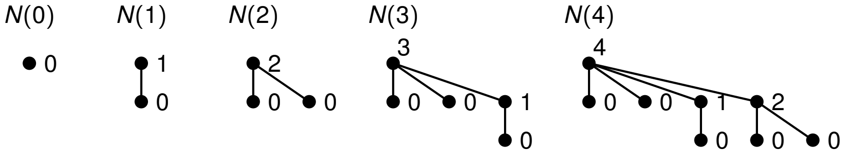

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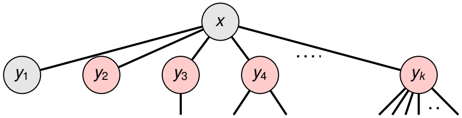

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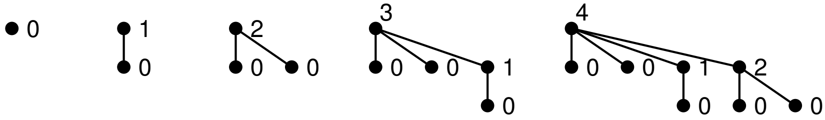

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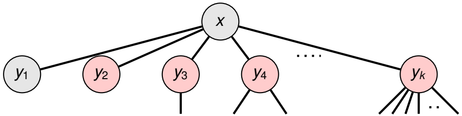

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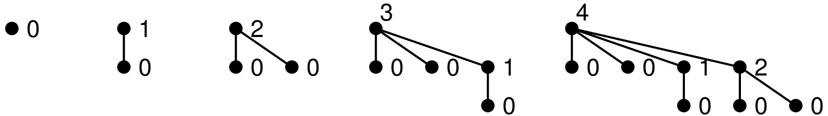

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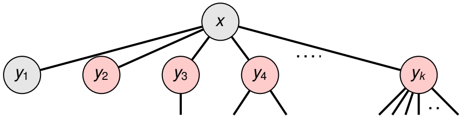

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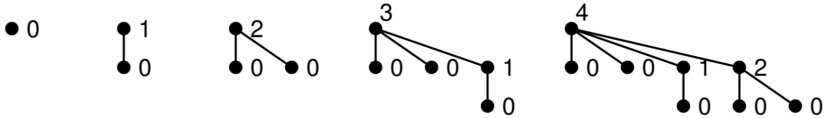

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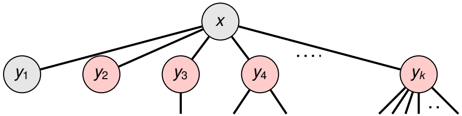

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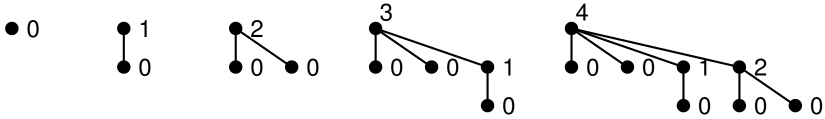

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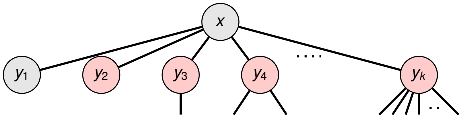

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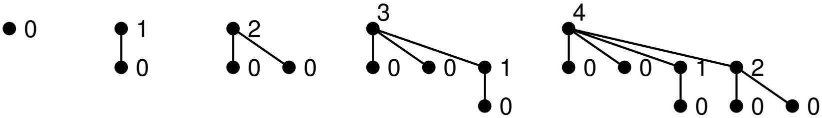

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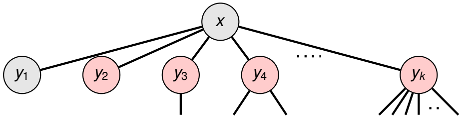

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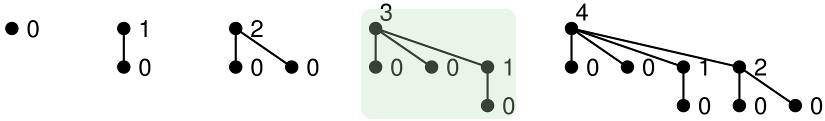

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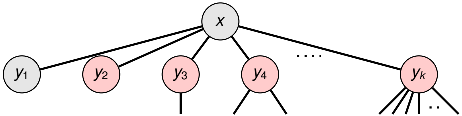

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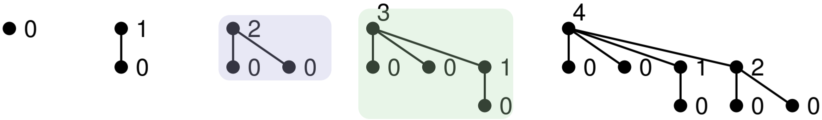

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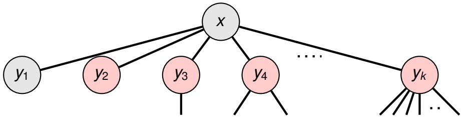

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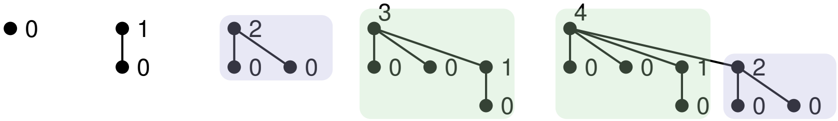

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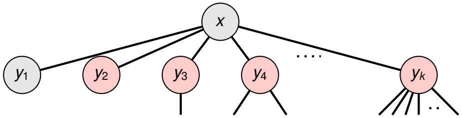

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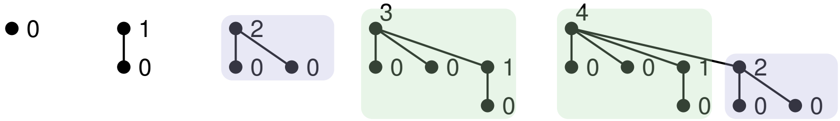

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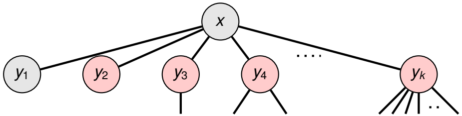

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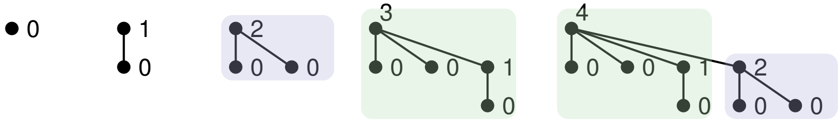

From Minimum Subtree Sizes to Fibonacci Numbers

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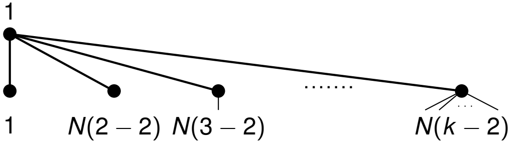

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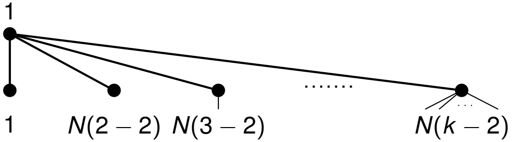

$$
\begin{aligned}
N(k) &= 1 + 1 + N(2-2) + N(3-2) + \cdots + N(k-2) \\
&= 1 + 1 + \sum_{\ell=0}^{k-2} N(\ell) \\
&= 1 + 1 + \sum_{\ell=0}^{k-3} N(\ell) + N(k-2) \\
&= N(k-1) + N(k-2) \\
&= F(k+1) + F(k) = F(k+2) \qquad \square
\end{aligned}
$$

Exponential Growth of Fibonacci Numbers

Lemma 19.3

For all integers $k \geq 0$, the $(k+2)$nd Fib. number satisfies $F(k+2) \geq \varphi^k$, where $\varphi = (1+\sqrt{5})/2 = 1.61803\ldots$.

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&= \varphi^{k-2} \cdot \varphi^2 && (\varphi^2 = \varphi + 1) \\
&= \varphi^k && \square
\end{aligned}$$

Putting the Pieces Together

Amortized Analysis

- INSERT: amortized cost $\mathcal{O}(1)$
- EXTRACT-MIN: amortized cost $\mathcal{O}(d(n))$
- DECREASE-KEY: amortized cost $\mathcal{O}(1)$

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$$N(k) = F(k+2)$$

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$$n \geq N(k) = F(k+2) \geq \varphi^k$$

Putting the Pieces Together

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- INSERT: amortized cost $\mathcal{O}(1)$
- EXTRACT-MIN: amortized cost $\mathcal{O}(d(n))$
- DECREASE-KEY: amortized cost $\mathcal{O}(1)$

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\begin{aligned}
& n \geq N(k) = F(k+2) \geq \varphi^k \\
\Rightarrow \quad & \log_\varphi n \geq k
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What if we don't have marked nodes?

- INSERT: actual $\mathcal{O}(1)$
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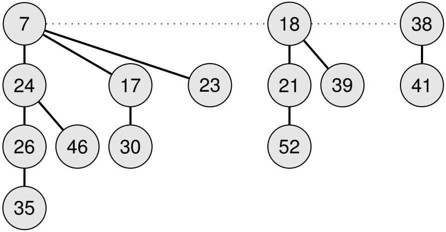

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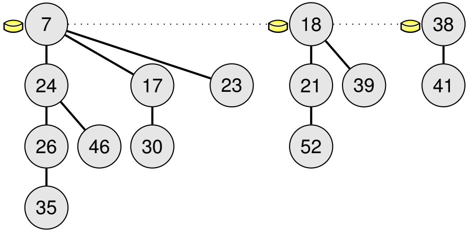

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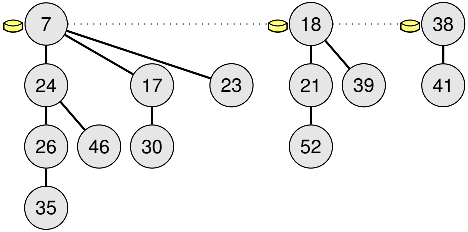

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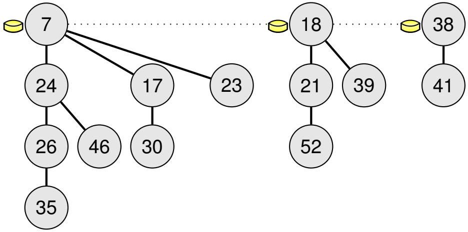

Summary

Operation	Linked list	Binary heap	Binomial heap	Fibon. heap
MAKE-HEAP	$\mathcal{O}(1)$	$\mathcal{O}(1)$	$\mathcal{O}(1)$	$\mathcal{O}(1)$
<u>INSERT</u>	$\mathcal{O}(1)$	$\mathcal{O}(\log n)$	$\mathcal{O}(\log n)$	$\mathcal{O}(1)$
MINIMUM	$\mathcal{O}(n)$	$\mathcal{O}(1)$	$\mathcal{O}(\log n)$	$\mathcal{O}(1)$
<u>EXTRACT-MIN</u>	$\mathcal{O}(n)$	$\mathcal{O}(\log n)$	$\mathcal{O}(\log n)$	$\mathcal{O}(\log n)$
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Summary

Can we perform EXTRACT-MIN in $o(\log n)$?

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If this was possible, then there would be a sorting algorithm with runtime $o(n \log n)$!

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Crucial for many applications including shortest paths and minimum spanning trees!

Recent Studies

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$\Rightarrow$ marked bit is not redundant!

Outlook: A More Efficient Priority Queue for fixed Universe

Operation	Fibonacci heap amortized cost	Van Emde Boas Tree actual cost
INSERT	$\mathcal{O}(1)$	$\mathcal{O}(\log\log u)$
MINIMUM	$\mathcal{O}(1)$	$\mathcal{O}(1)$
EXTRACT-MIN	$\mathcal{O}(\log n)$	$\mathcal{O}(\log\log u)$
MERGE/UNION	$\mathcal{O}(1)$	-
DECREASE-KEY	$\mathcal{O}(1)$	$\mathcal{O}(\log\log u)$
DELETE	$\mathcal{O}(\log n)$	$\mathcal{O}(\log\log u)$
SUCC	-	$\mathcal{O}(\log\log u)$
PRED	-	$\mathcal{O}(\log\log u)$
MAXIMUM	-	$\mathcal{O}(1)$

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SUCC	-	$\mathcal{O}(\log\log u)$
PRED	-	$\mathcal{O}(\log\log u)$
MAXIMUM	-	$\mathcal{O}(1)$

all this requires key values to be in a universe of size u!