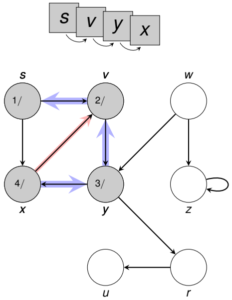


6.1 & 6.2: Graph Searching

Frank Stajano Thomas Sauerwald

Lent 2016

UNIVERSITY OF CAMBRIDGE

Outline

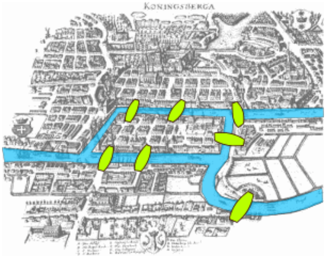
Introduction to Graphs and Graph Searching

Breadth-First Search

Depth-First Search

Topological Sort

Origin of Graph Theory



Source: Wikipedia

Seven Bridges at Königsberg 1737

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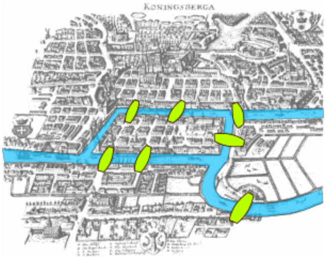

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Is there a tour which crosses each bridge **exactly once**?

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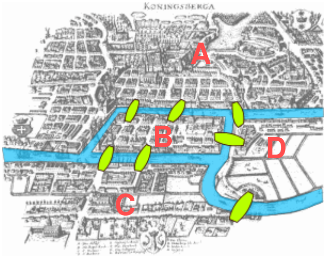

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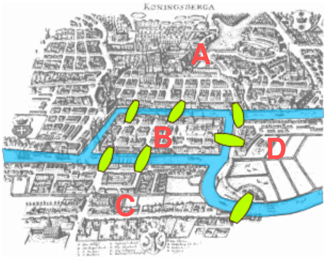

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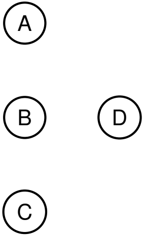

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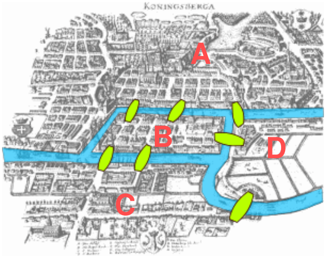

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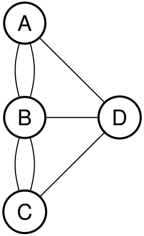

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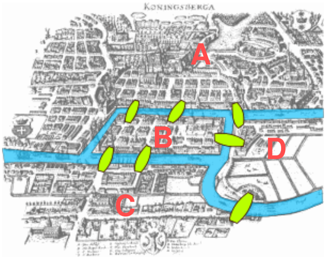

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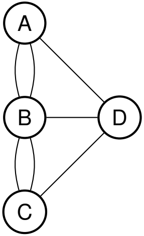

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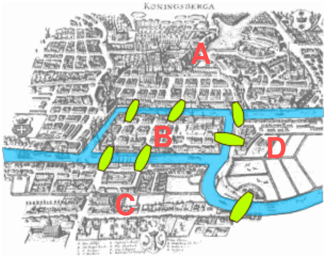

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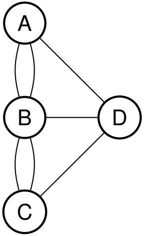

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$\rightsquigarrow$ 1B course: Complexity Theory

What is a Graph?

Directed Graph

A graph $G = (V, E)$ consists of:

- V: the set of vertices
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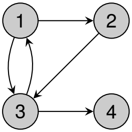

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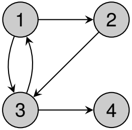

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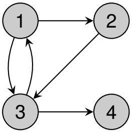

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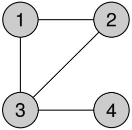

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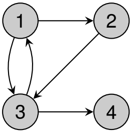

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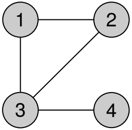

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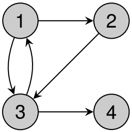

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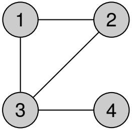

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Paths and Connectivity

- A sequence of edges between two vertices forms a path

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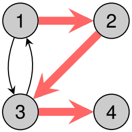

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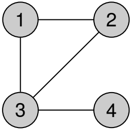

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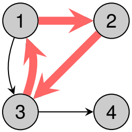
Path $p = (1, 2, 3, 1)$, which is a cycle



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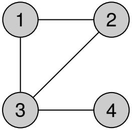

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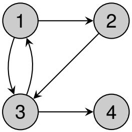

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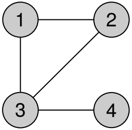

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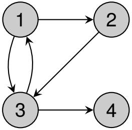

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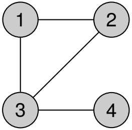

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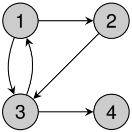

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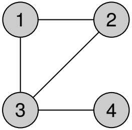

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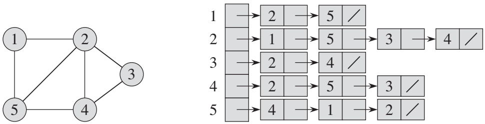
Later: edge-weighted graphs $G = (V, E, w)$

Representations of Directed and Undirected Graphs



(a) (b)

	1	2	3	4	5
1	0	1	0	0	1
2	1	0	1	1	1
3	0	1	0	1	0
4	0	1	1	0	1
5	1	1	0	1	0

(c)

Figure 22.1 Two representations of an undirected graph. **(a)** An undirected graph G with 5 vertices and 7 edges. **(b)** An adjacency-list representation of G. **(c)** The adjacency-matrix representation of G.

Representations of Directed and Undirected Graphs

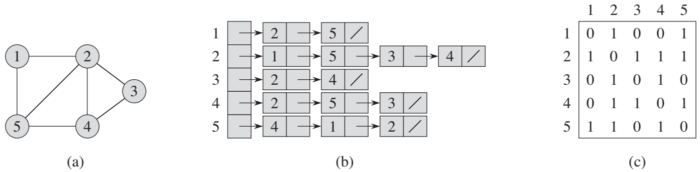


Figure 22.1 Two representations of an undirected graph. **(a)** An undirected graph G with 5 vertices and 7 edges. **(b)** An adjacency-list representation of G. **(c)** The adjacency-matrix representation of G.

Most times we will use the adjacency-list representation!

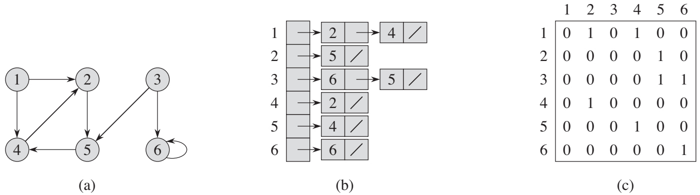


Figure 22.2 Two representations of a directed graph. **(a)** A directed graph G with 6 vertices and 8 edges. **(b)** An adjacency-list representation of G. **(c)** The adjacency-matrix representation of G.

Overview

Amortized Analysis

Fibonacci Heaps

Disjoint Sets

Overview

- Amortized Analysis
- Fibonacci Heaps
- Disjoint Sets
- Graphs, DFS/BFS, Topological Sort
- Minimum Spanning Trees
- Single-Source/All-Pairs Shortest Paths
- Maximum Flow, Bipartite Matchings
- Geometric Algorithms

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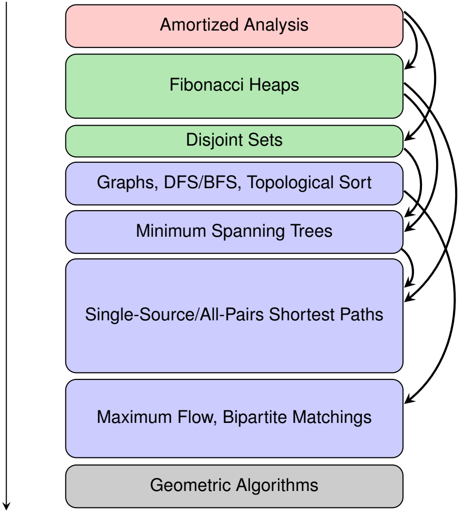
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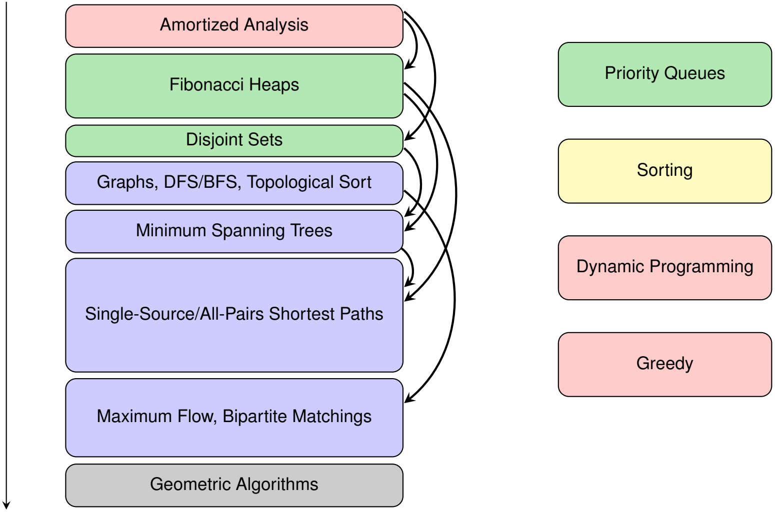
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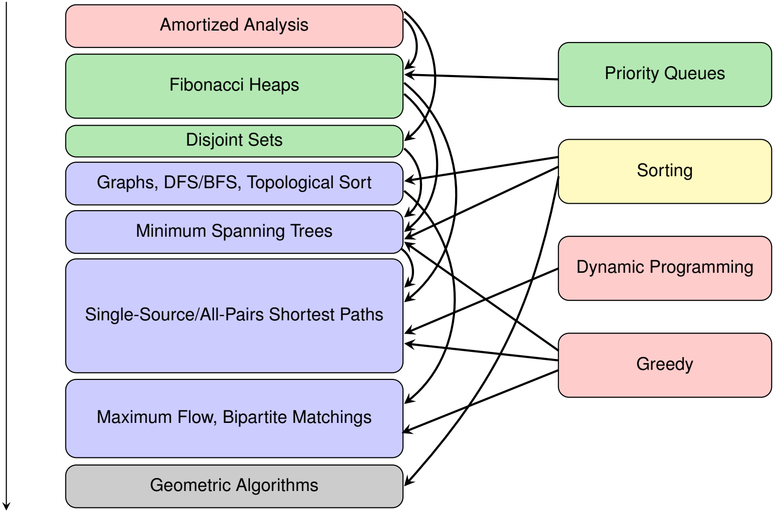
Dynamic Programming
Greedy

Overview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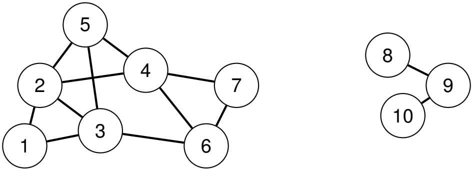
Graph Searching



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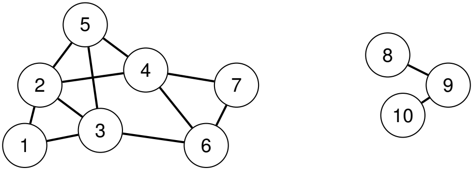
- useful for identifying connected components, computing the diameter etc.

Graph Searching



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- Two strategies: Breadth-First-Search and Depth-First-Search

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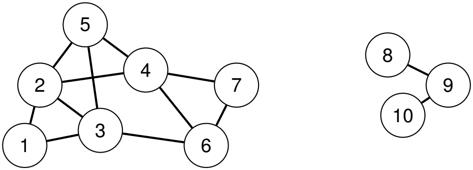

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- Two strategies: Breadth-First-Search and Depth-First-Search

Measure time complexity in terms of the size of V and E (often write just V instead of $|V|$, and E instead of $|E|$)

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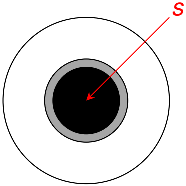
Introduction to Graphs and Graph Searching

Breadth-First Search

Depth-First Search

Topological Sort

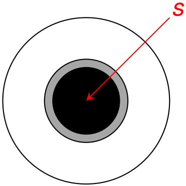
Breadth-First Search: Basic Ideas



Basic Idea

- Given an undirected/directed graph $G = (V, E)$ and source vertex s

Breadth-First Search: Basic Ideas



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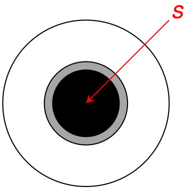

Basic Idea

- Given an undirected/directed graph $G = (V, E)$ and source vertex s
- BFS sends out a wave from $s \rightsquigarrow$ compute distances/shortest paths
- Vertex Colours:

  White = Unvisited

  Grey = Visited, but not all neighbors (=adjacent vertices)

  Black = Visited and all neighbors

Breadth-First-Search: Pseudocode

```
def bfs(G,s)
    Run BFS on the given graph G
    starting from source s

   assert(s in G.vertices())

# Initialize graph and queue
for v in G.vertices():
      v.predecessor = None
      v.d = Infinity # .d = distance from s
      v.colour = "white"
Q = Queue()

# Visit source vertex
s.d = 0
s.colour = "grey"
Q.insert(s)

# Visit the adjacents of each vertex in Q
while not Q.isEmpty():
    u = Q.extract()
    assert (u.colour == "grey")
    for v in u.adjacent()
        if v.colour = "white"
            v.colour = "grey"
            v.d = u.d+1
            v.predecessor = u
            Q.insert(v)
    u.colour = "black"
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         v.colour = "grey"
         v.d = u.d+1
         v.predecessor = u
         Q.insert(v)
   u.colour = "black"
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- From any vertex, visit all adjacent vertices before going any deeper

Breadth-First-Search: Pseudocode

```
def bfs(G,s)
    Run BFS on the given graph G
    starting from source s

  assert(s in G.vertices())

# Initialize graph and queue
for v in G.vertices():
    v.predecessor = None
    v.d = Infinity # .d = distance from s
    v.colour = "white"
Q = Queue()

# Visit source vertex
s.d = 0
s.colour = "grey"
Q.insert(s)

# Visit the adjacents of each vertex in Q
while not Q.isEmpty():
    u = Q.extract()
    assert (u.colour == "grey")
    for v in u.adjacent()
        if v.colour = "white"
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            v.d = u.d+1
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Assuming that all executions of the FOR-loop for u takes $O(|u.adj|)$ (**adjacency list model!**)

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$$\sum_{u \in V} |u.adj| = 2|E|$$

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- Vertex Colours:

  White = Unvisited

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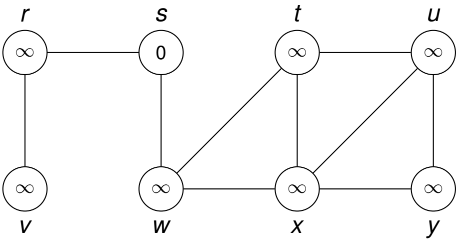
  Black = Visited and all neighbors
- Runtime $O(V + E)$

Assuming that all executions of the FOR-loop for u takes $O(|u.adj|)$ (**adjacency list model!**)

$$\sum_{u \in V} |u.adj| = 2|E|$$

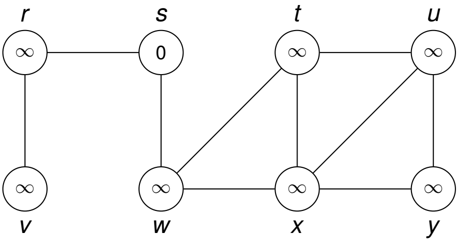
Execution of BFS (Figure 22.3)

Queue:



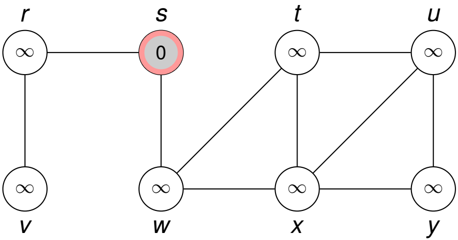
Execution of BFS (Figure 22.3)

Queue: *s*



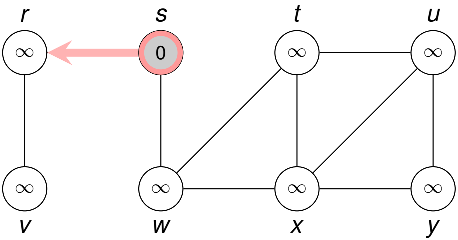
Execution of BFS (Figure 22.3)

Queue: ~~s~~



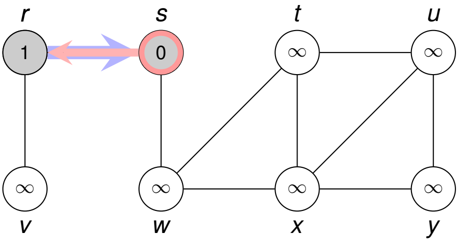
Execution of BFS (Figure 22.3)

Queue: ~~s~~



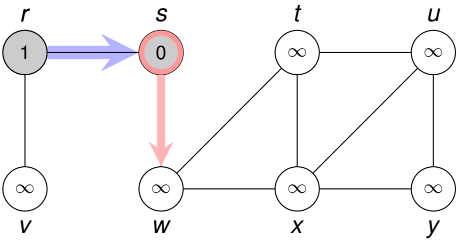
Execution of BFS (Figure 22.3)

Queue: ~~s~~ r



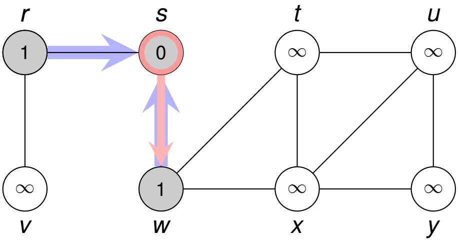
Execution of BFS (Figure 22.3)

Queue: ~~*s*~~ *r*



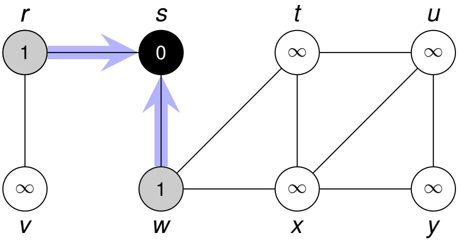
Execution of BFS (Figure 22.3)

Queue: ~~*s*~~ *r* *w*



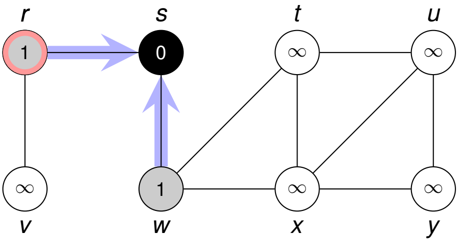
Execution of BFS (Figure 22.3)

Queue: ~~*s*~~ *r* *w*



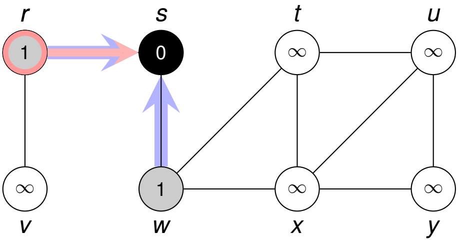
Execution of BFS (Figure 22.3)

Queue: ~~s~~ ~~r~~ *w*



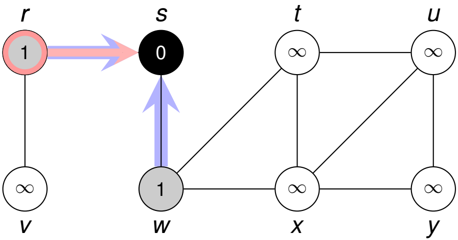
Execution of BFS (Figure 22.3)

Queue: ~~*s*~~ ~~*r*~~ *w*



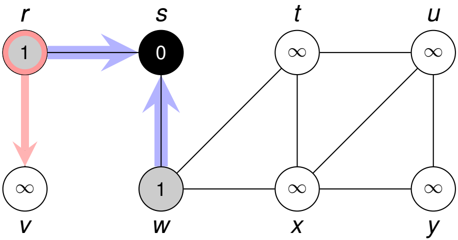
Execution of BFS (Figure 22.3)

Queue: ~~s~~ ~~r~~ *w*



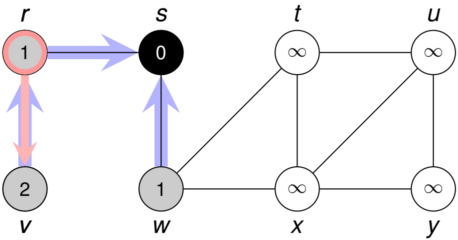
Execution of BFS (Figure 22.3)

Queue: ~~*s*~~ ~~*r*~~ *w*



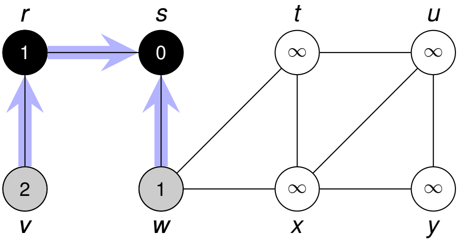
Execution of BFS (Figure 22.3)

Queue: ~~s~~ ~~r~~ w v



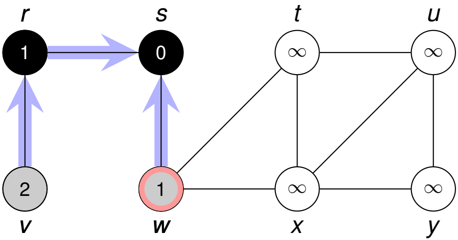
Execution of BFS (Figure 22.3)

Queue: ~~*s*~~ ~~*r*~~ *w* *v*



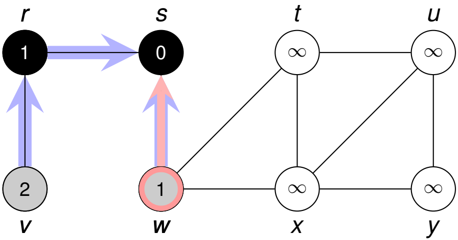
Execution of BFS (Figure 22.3)

Queue: ~~*s*~~ ~~*r*~~ ~~*w*~~ *v*



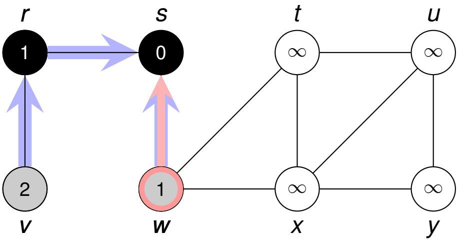
Execution of BFS (Figure 22.3)

Queue: ~~s~~ ~~r~~ ~~w~~ v



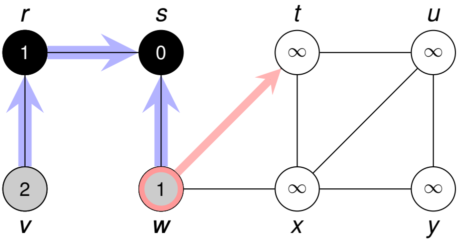
Execution of BFS (Figure 22.3)

Queue: ~~s~~ ~~r~~ ~~w~~ v



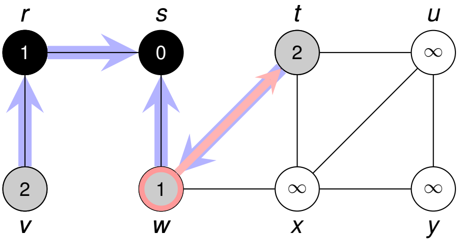
Execution of BFS (Figure 22.3)

Queue: ~~s~~ ~~r~~ ~~w~~ v



Execution of BFS (Figure 22.3)

Queue: ~~s~~ ~~r~~ ~~w~~ v t



Execution of BFS (Figure 22.3)

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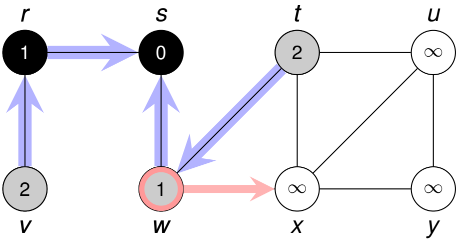
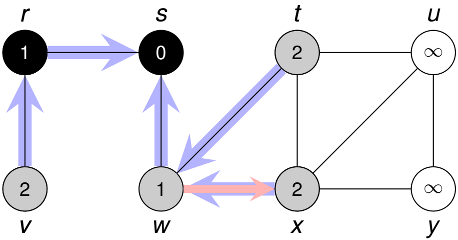


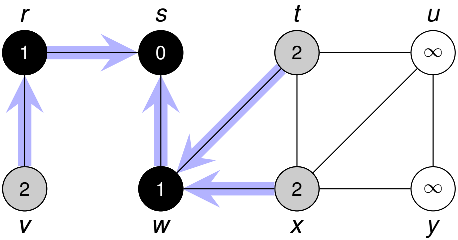
Execution of BFS (Figure 22.3)

Queue: ~~s~~ ~~r~~ ~~w~~ *v* *t* *x*



Execution of BFS (Figure 22.3)

Queue: ~~*s*~~ ~~*r*~~ ~~*w*~~ *v* *t* *x*



Execution of BFS (Figure 22.3)

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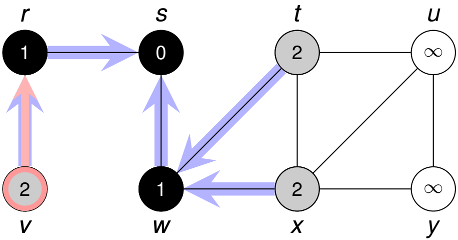

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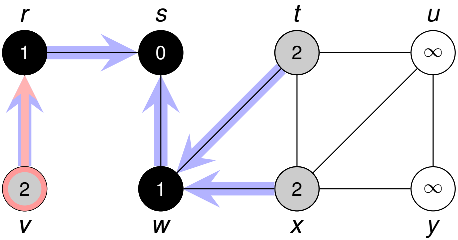

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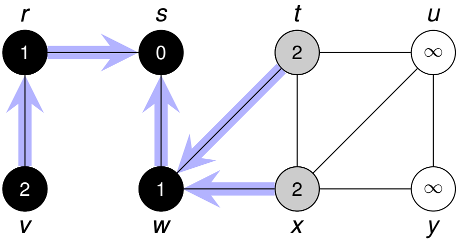

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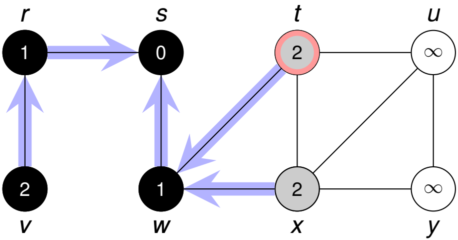

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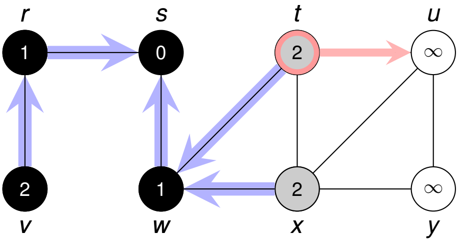

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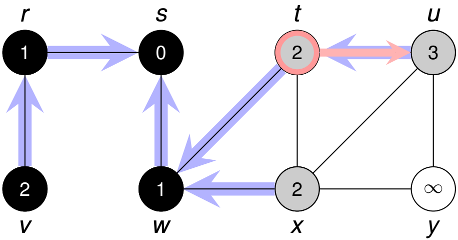

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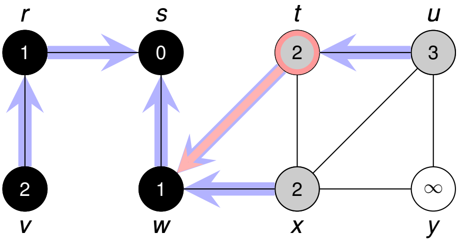

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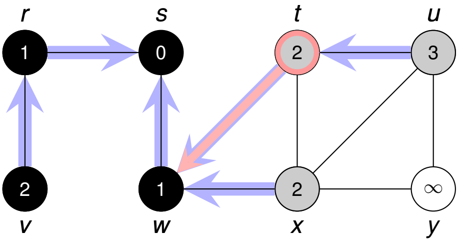

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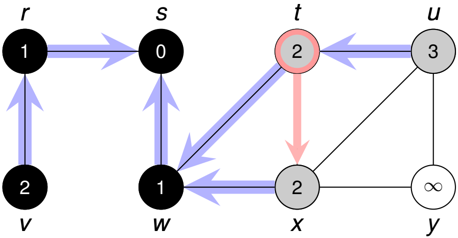

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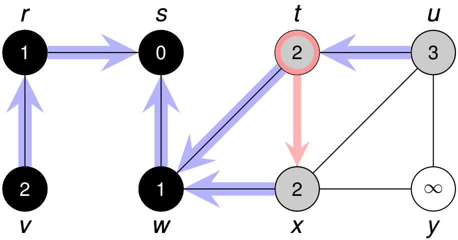

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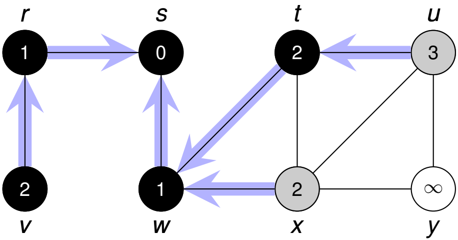

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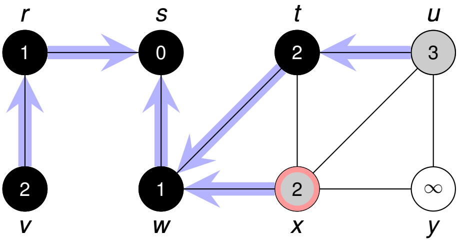

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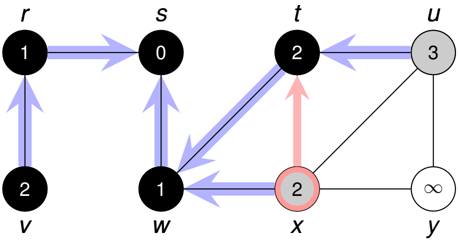

Execution of BFS (Figure 22.3)

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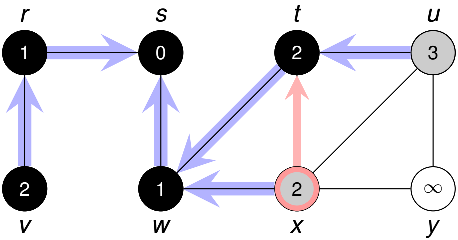

Execution of BFS (Figure 22.3)

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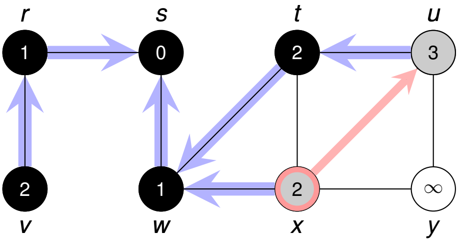
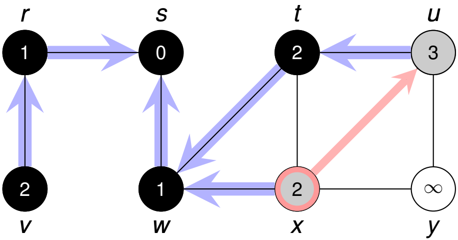


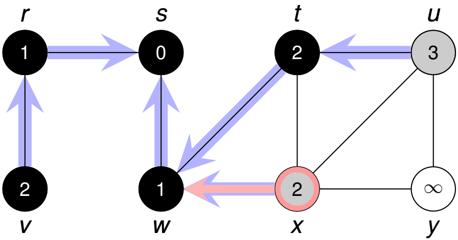
Execution of BFS (Figure 22.3)

Queue: ~~*s*~~ ~~*r*~~ ~~*w*~~ ~~*v*~~ ~~*t*~~ ~~*x*~~ *u*



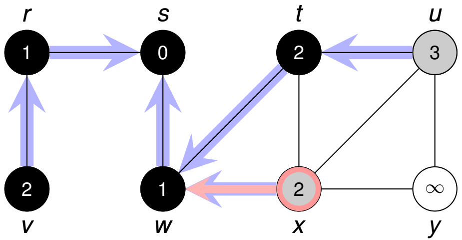
Execution of BFS (Figure 22.3)

Queue: ~~s~~ ~~r~~ ~~w~~ ~~v~~ ~~t~~ ~~x~~ u



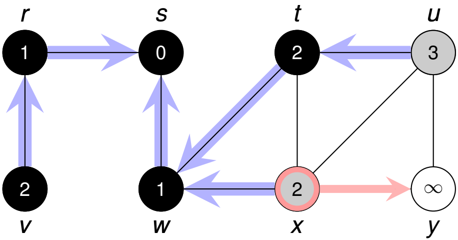
Execution of BFS (Figure 22.3)

Queue: ~~s~~ ~~r~~ ~~w~~ ~~v~~ ~~t~~ ~~x~~ u



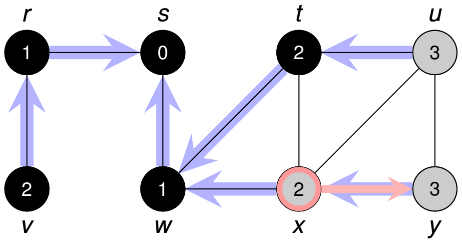
Execution of BFS (Figure 22.3)

Queue: ~~*s*~~ ~~*r*~~ ~~*w*~~ ~~*v*~~ ~~*t*~~ ~~*x*~~ *u*



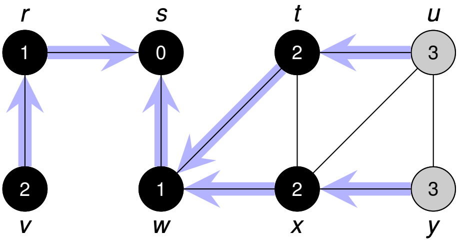
Execution of BFS (Figure 22.3)

Queue: ~~s~~ ~~r~~ ~~w~~ ~~v~~ ~~t~~ ~~x~~ *u* *y*



Execution of BFS (Figure 22.3)

Queue: ~~*s*~~ ~~*r*~~ ~~*w*~~ ~~*v*~~ ~~*t*~~ ~~*x*~~ *u* *y*



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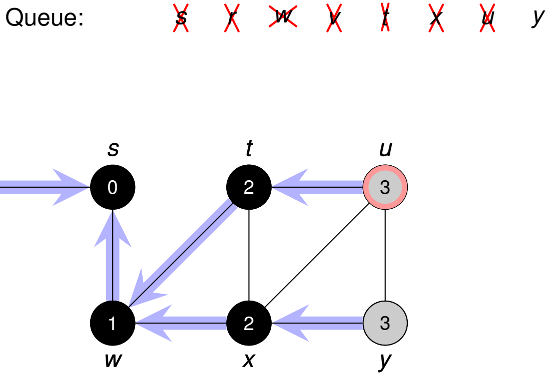

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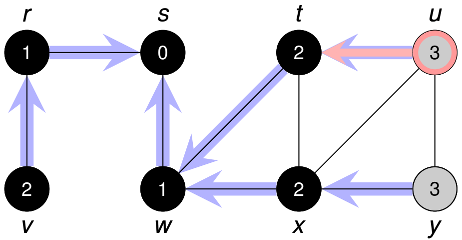

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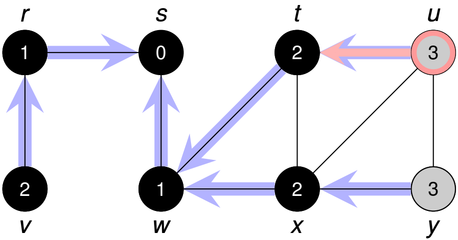

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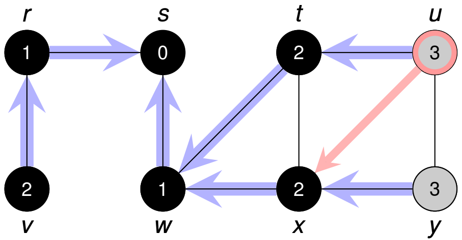

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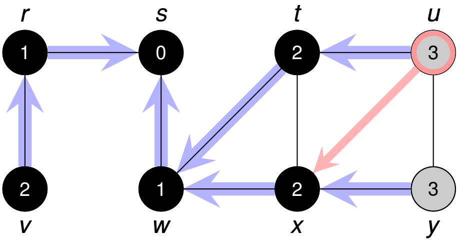

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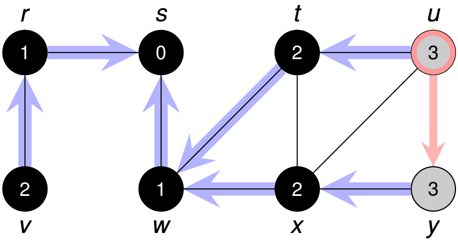

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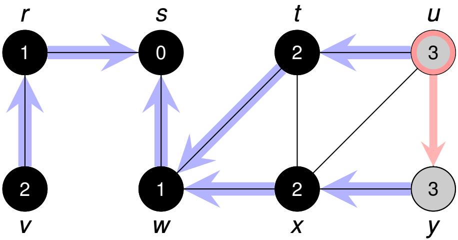

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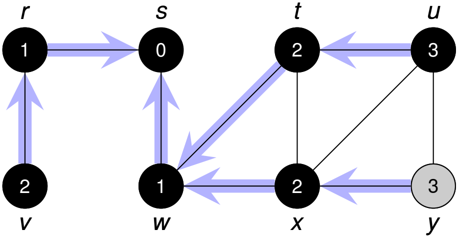

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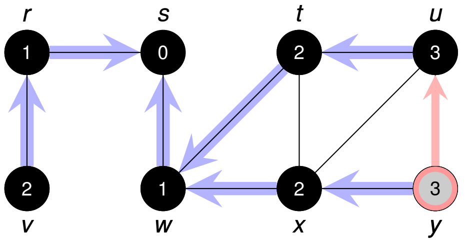

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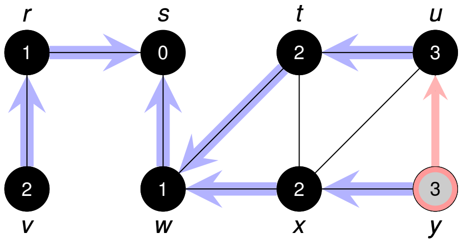

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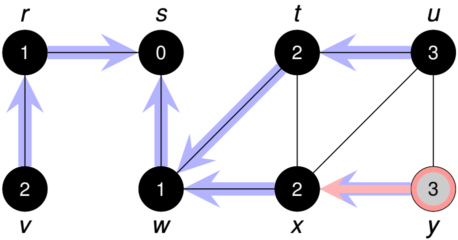

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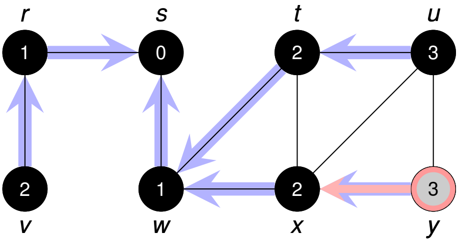

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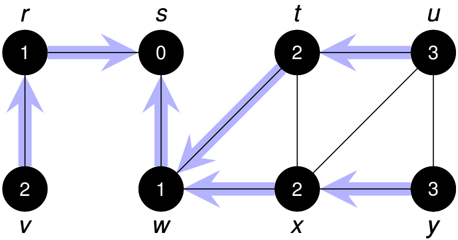


Outline

Introduction to Graphs and Graph Searching

Breadth-First Search

Depth-First Search

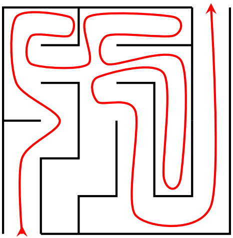
Topological Sort

Depth-First Search: Basic Ideas

Basic Idea

- Given an undirected/directed graph $G = (V, E)$ and source vertex s

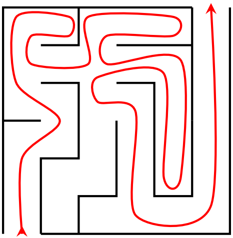
Depth-First Search: Basic Ideas



Basic Idea

- Given an undirected/directed graph $G = (V, E)$ and source vertex s
- As soon as we discover a vertex, explore from it $\rightsquigarrow$ Solving Mazes

Depth-First Search: Basic Ideas



Basic Idea

- Given an undirected/directed graph $G = (V, E)$ and source vertex s
- As soon as we discover a vertex, explore from it $\rightsquigarrow$ Solving Mazes
- Two time stamps for every vertex: Discovery Time, Finishing Time

Depth-First-Search: Pseudocode

```
def dfs(G,s):
    Run DFS on the given graph G
    starting from the given source s

  assert(s in G.vertices())

  # Initialize graph
  for v in G.vertices():
      v.predecessor = None
      v.colour = "white"
  dfsRecurse(G,s)
```

```
def dfsRecurse(G,s):
  s.colour = "grey"
  s.d = time() # .d = discovery time
  for v in s.adjacent()
      if v.colour = "white"
          v.predecessor = s
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- We always go deeper before visiting other neighbors

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- We always go deeper before visiting other neighbors
- Discovery and Finish times, *.d* and *.f*

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White = Unvisited

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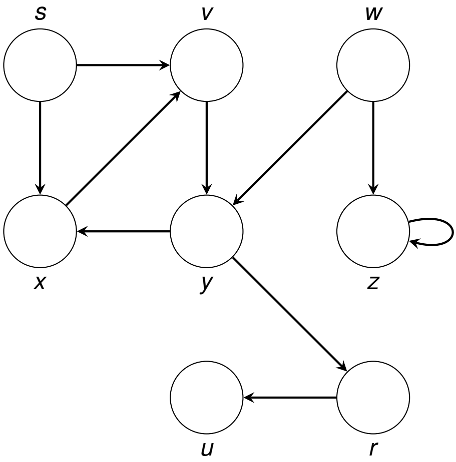
- Runtime $O(V + E)$

Execution of DFS



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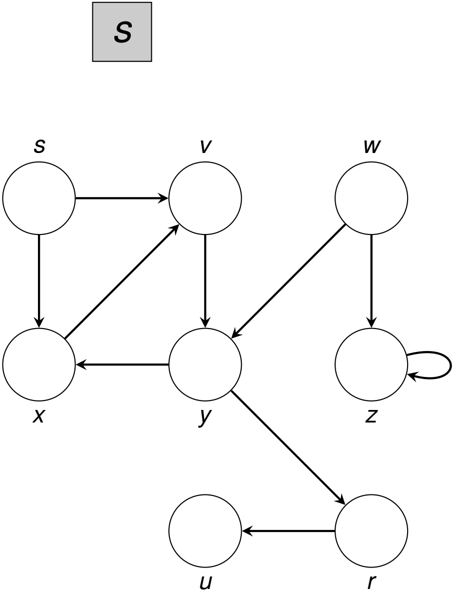

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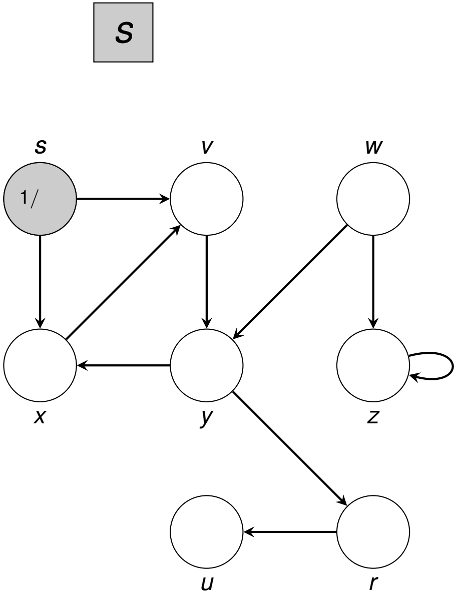

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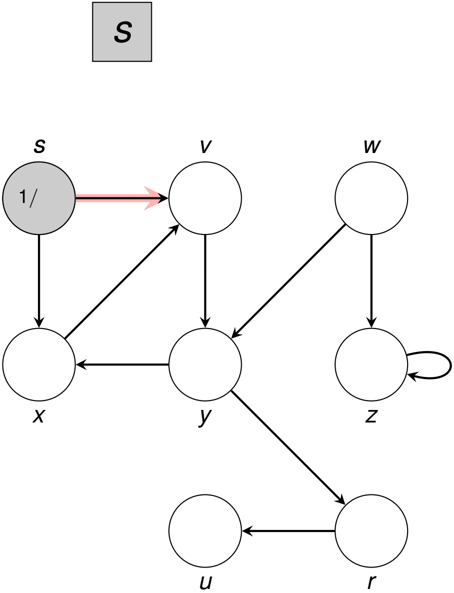

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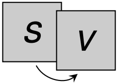

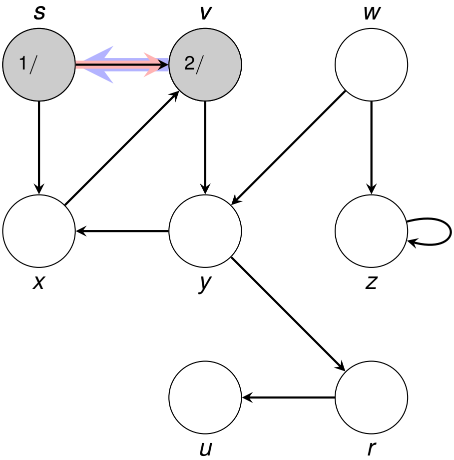

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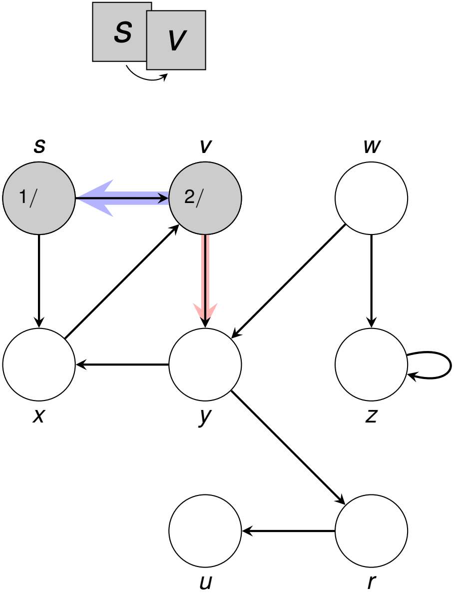

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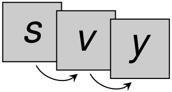

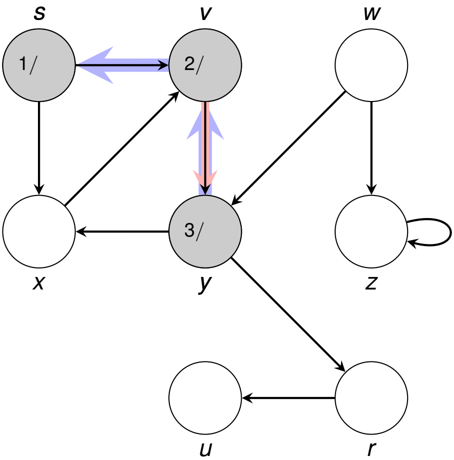

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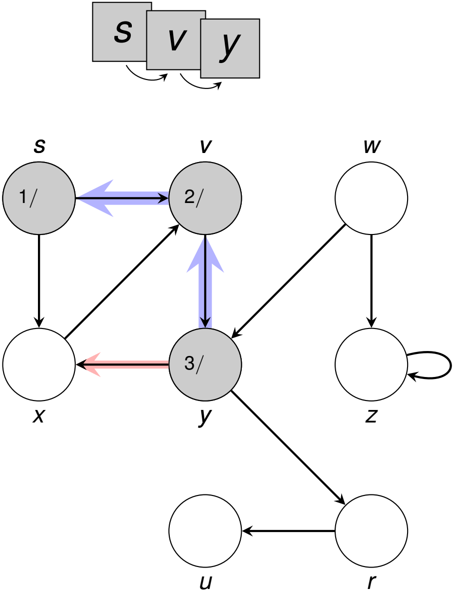

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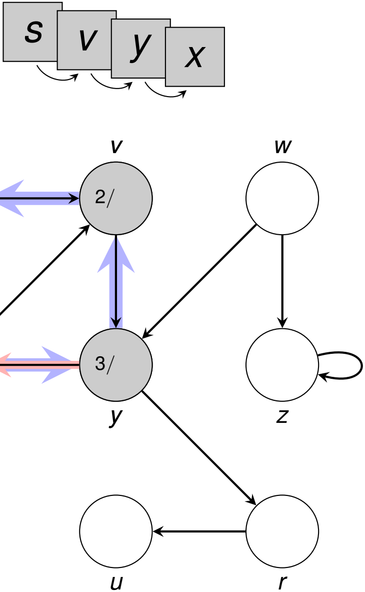

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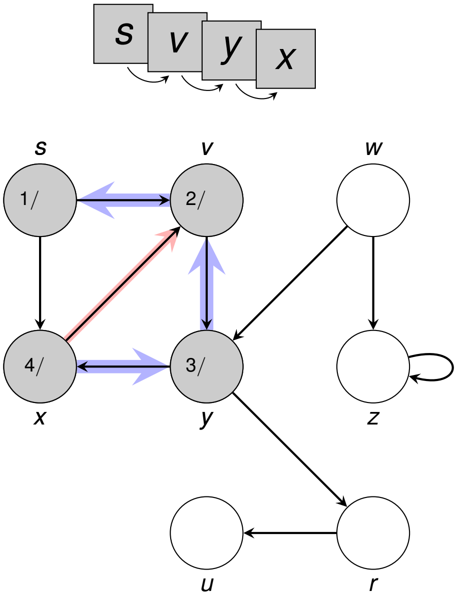

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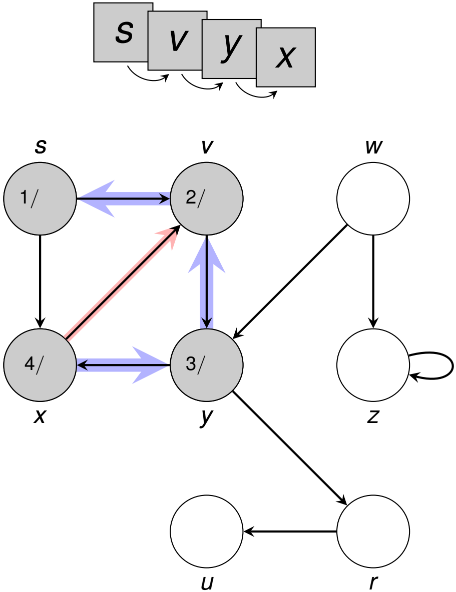

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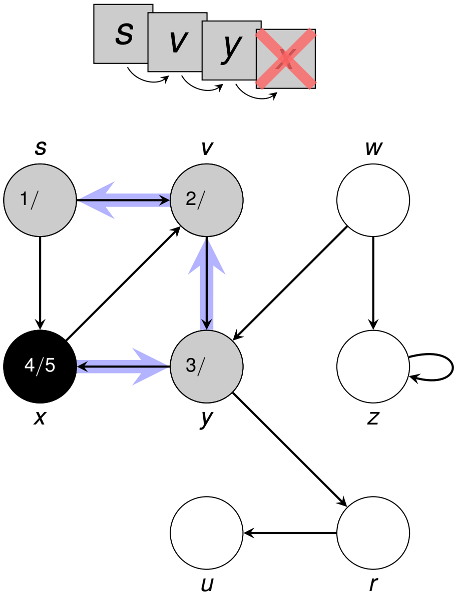

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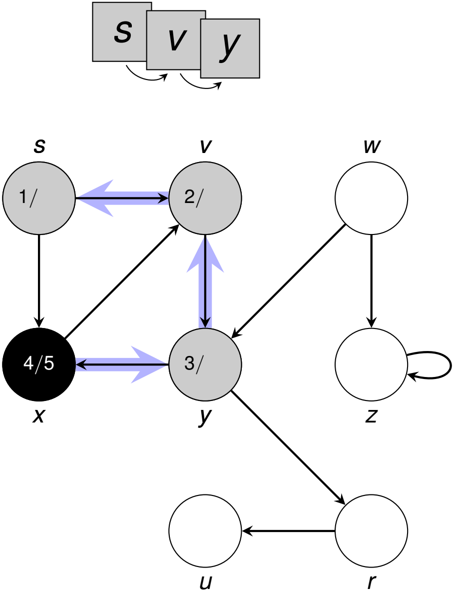

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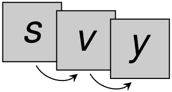

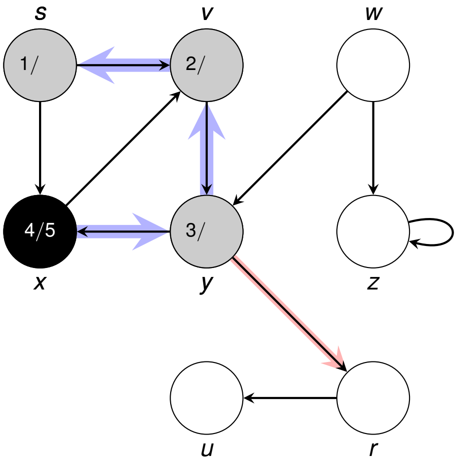

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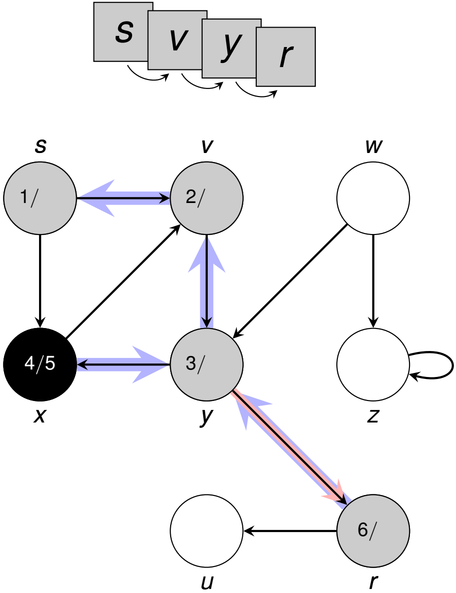

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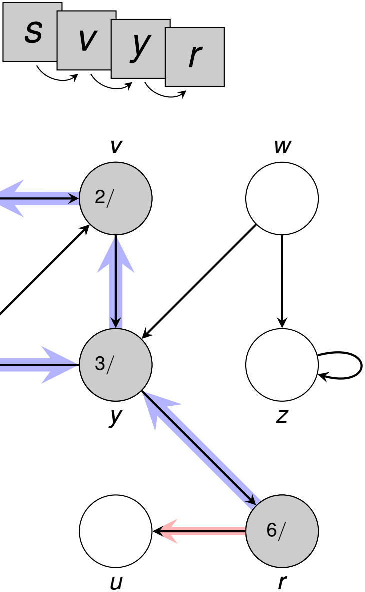

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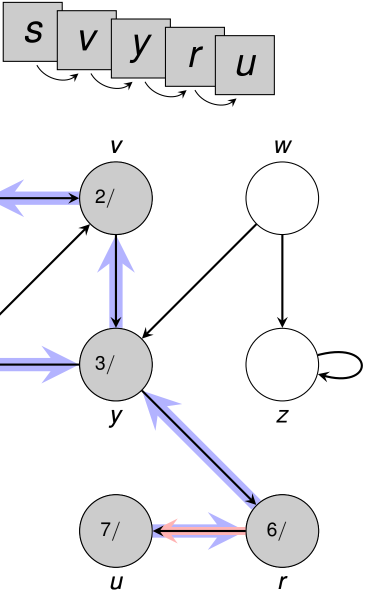

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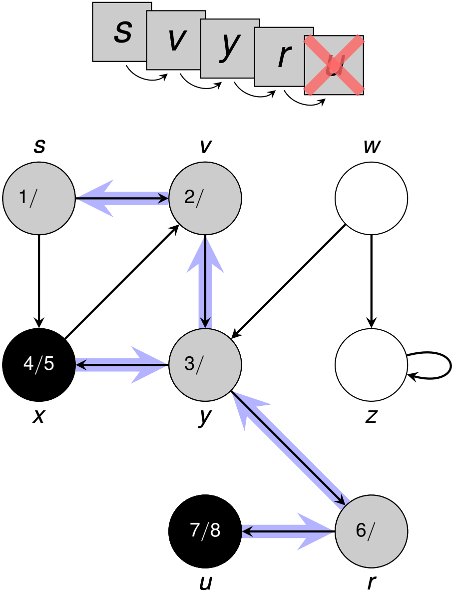

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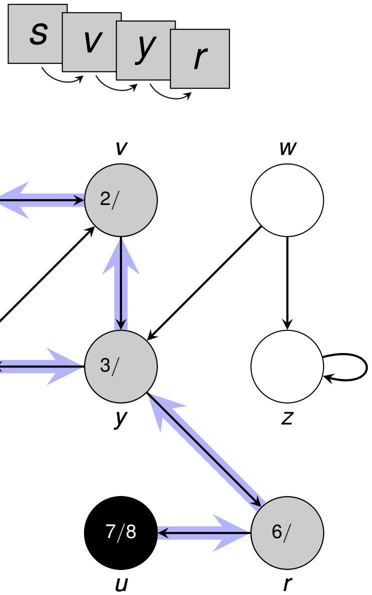

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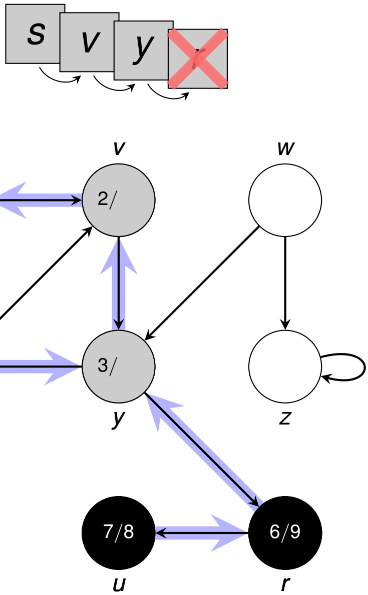

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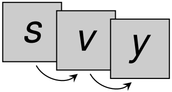

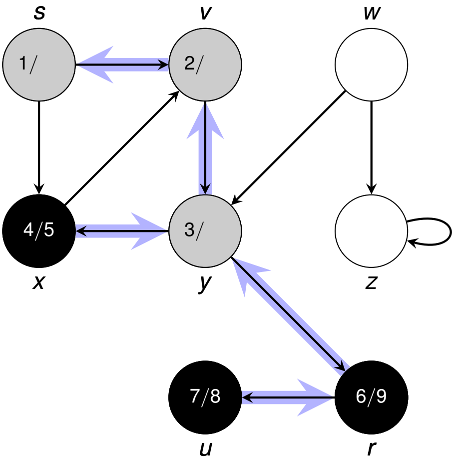

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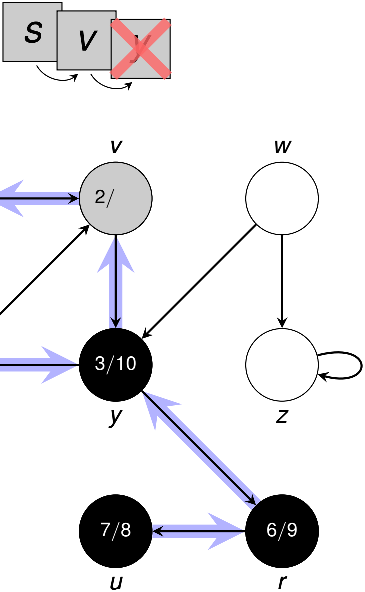

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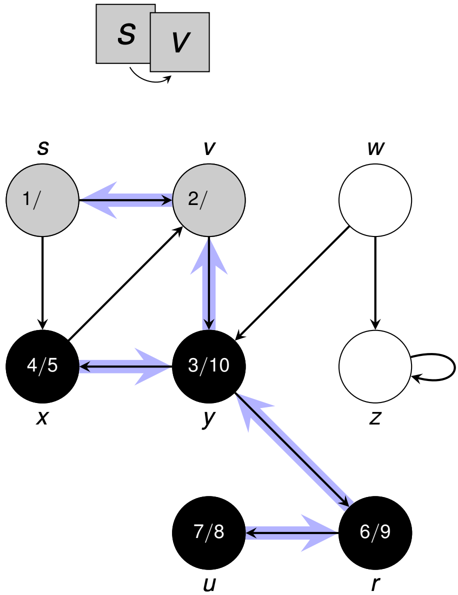

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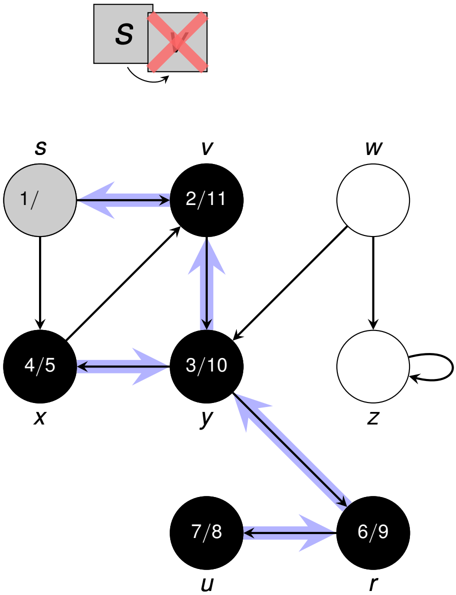

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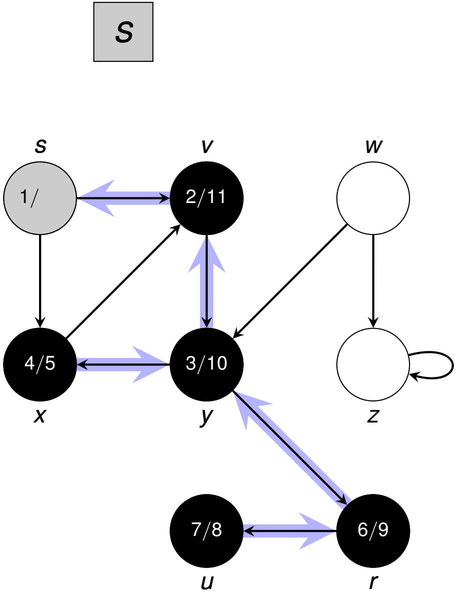

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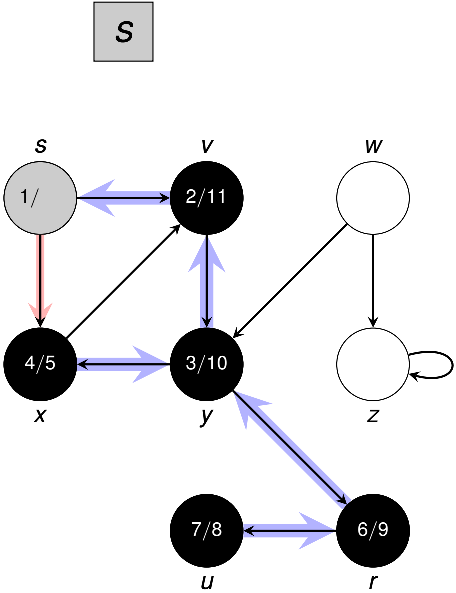

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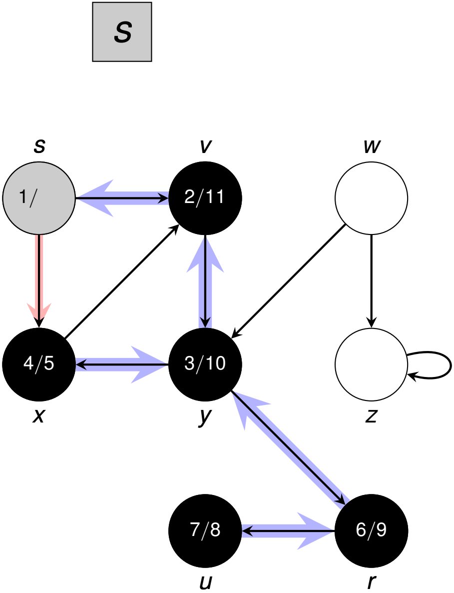

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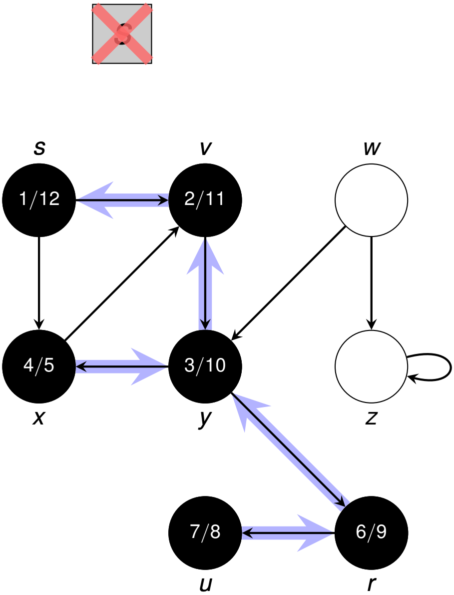

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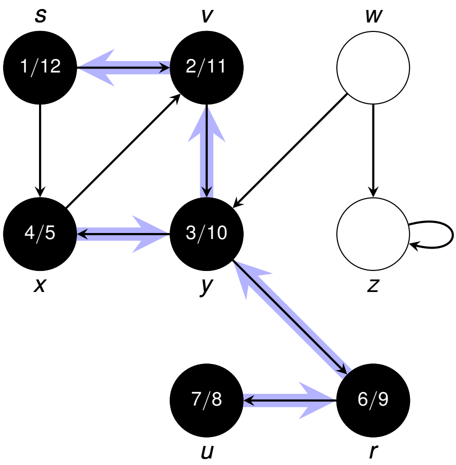

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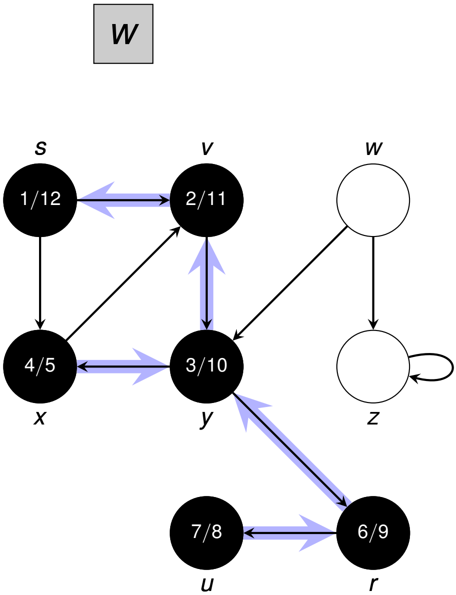

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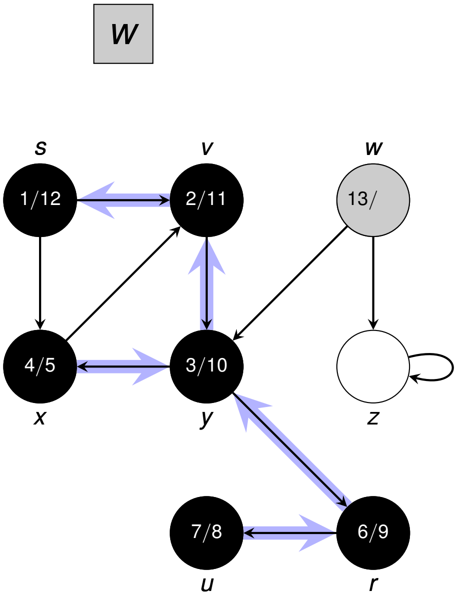

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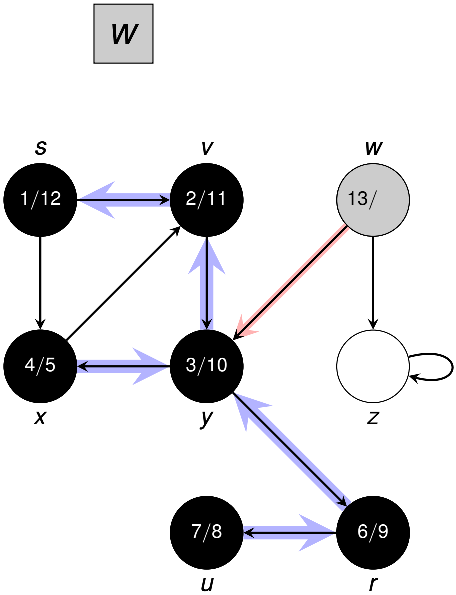

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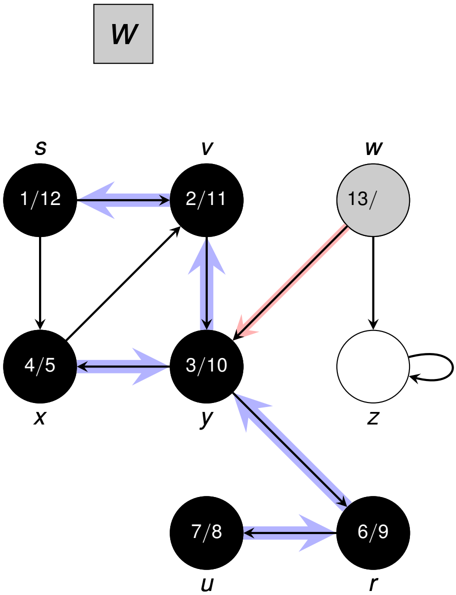

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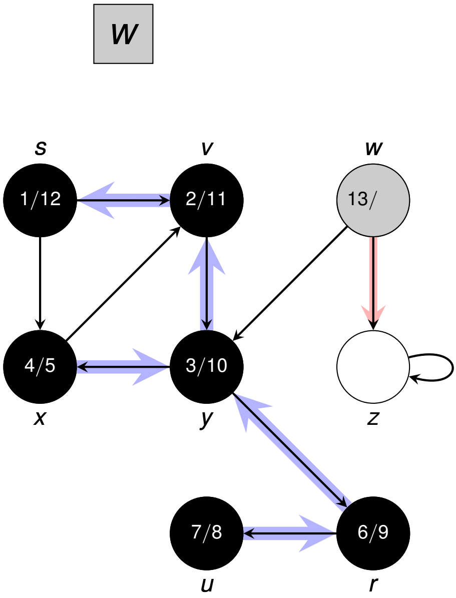

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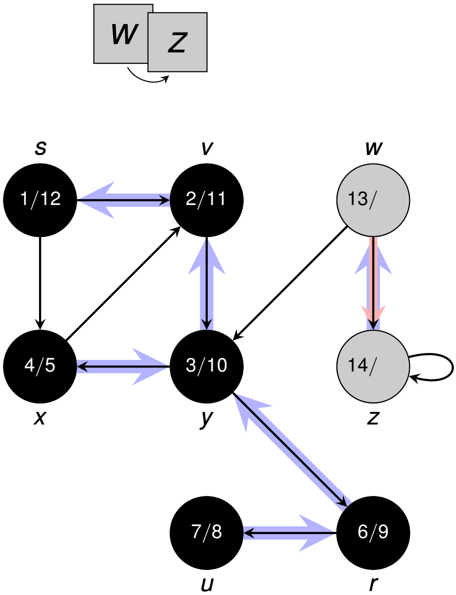
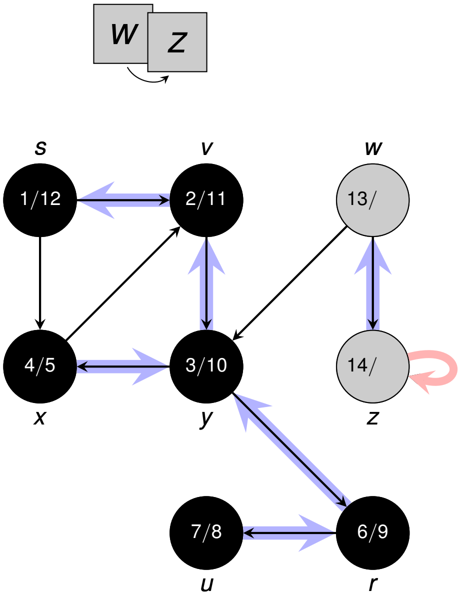


Execution of DFS



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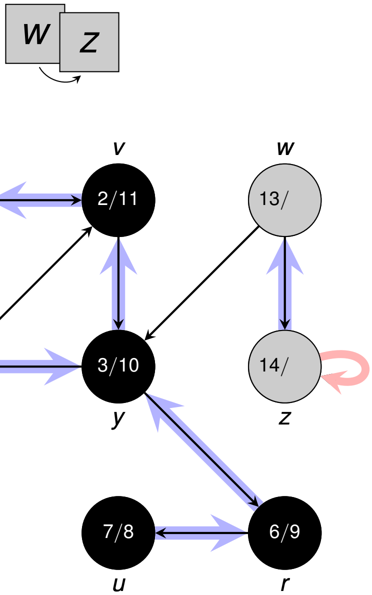

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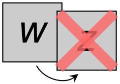
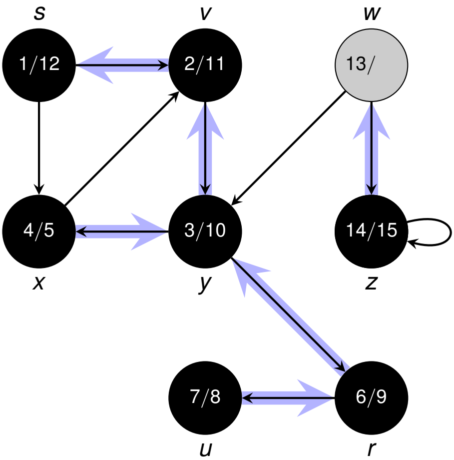
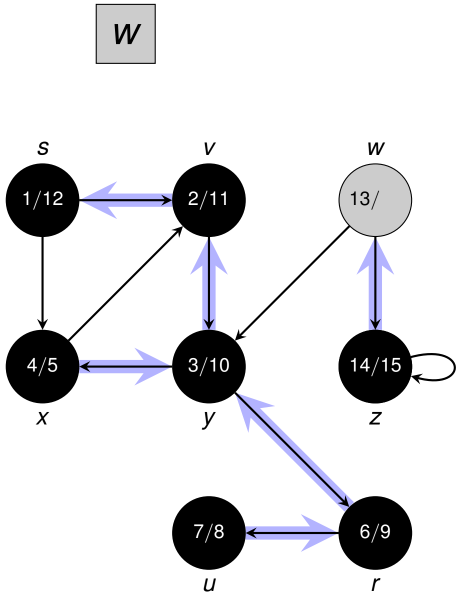




Execution of DFS



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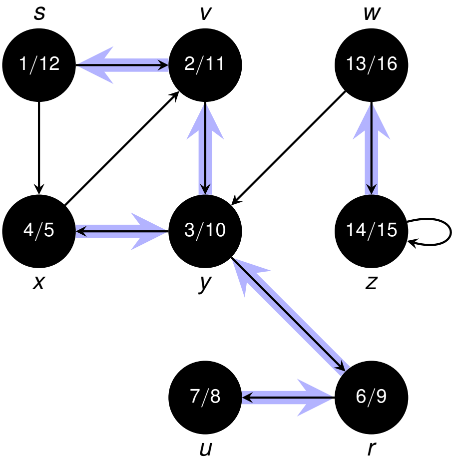

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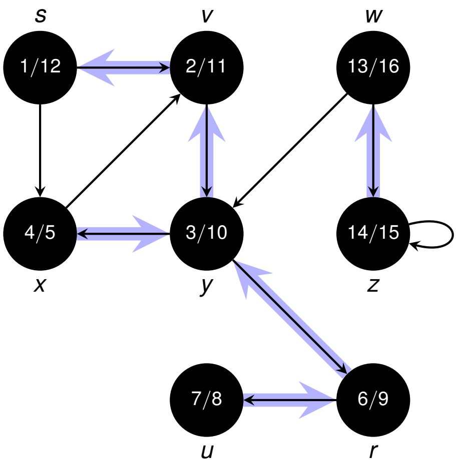
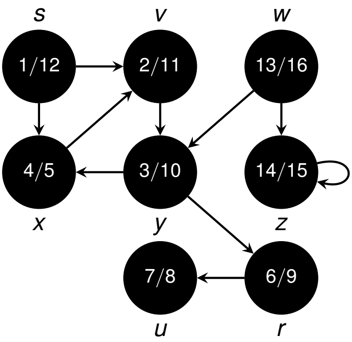
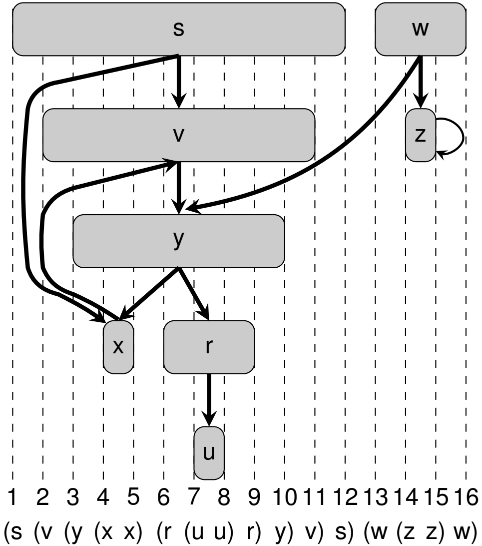


Paranthesis Theorem (Theorem 22.7)





Outline

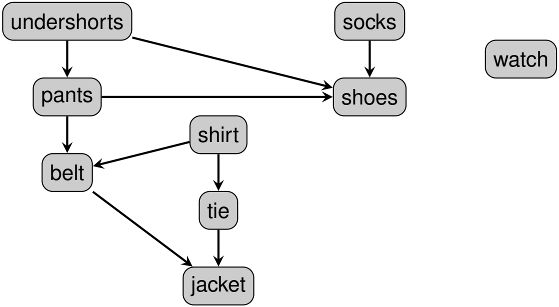
Introduction to Graphs and Graph Searching

Breadth-First Search

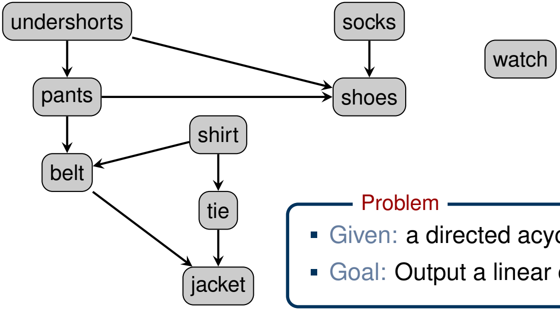
Depth-First Search

Topological Sort

Topological Sort



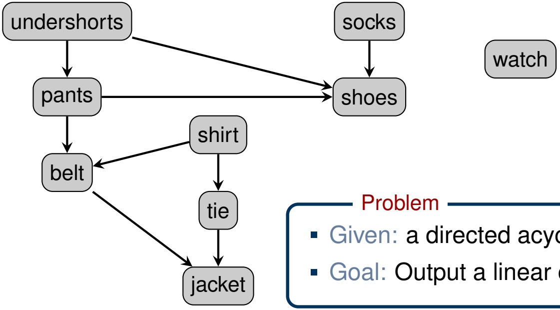
Topological Sort



Problem

- Given: a directed acyclic graph (DAG)
- Goal: Output a linear ordering of all vertices

Topological Sort



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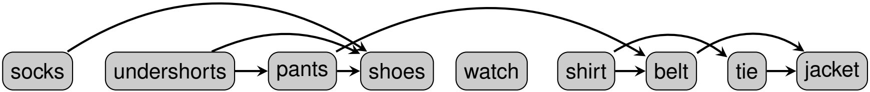

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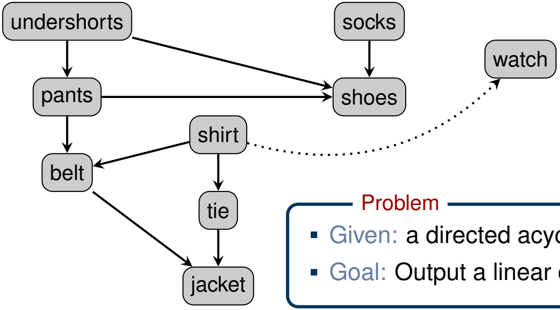

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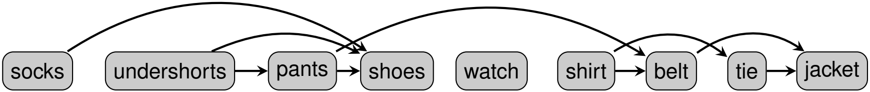

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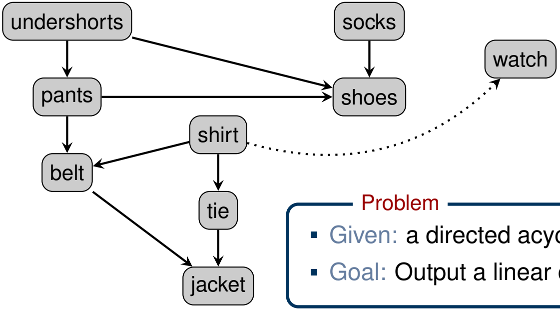

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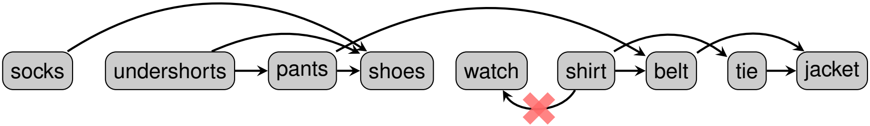
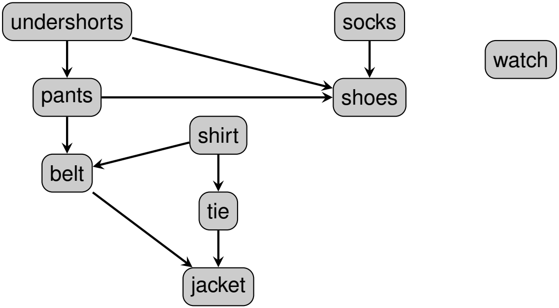


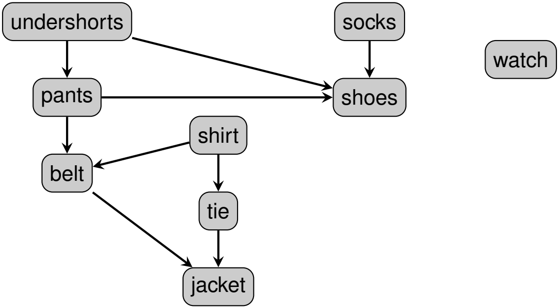
Solving Topological Sort



Knuth's Algorithm (1968)

- Perform DFS's so that all vertices are visited
- Output vertices in decreasing order of their finishing time

Solving Topological Sort



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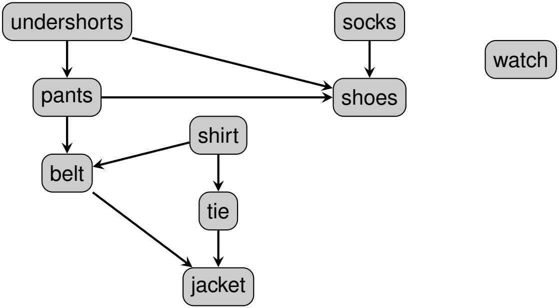


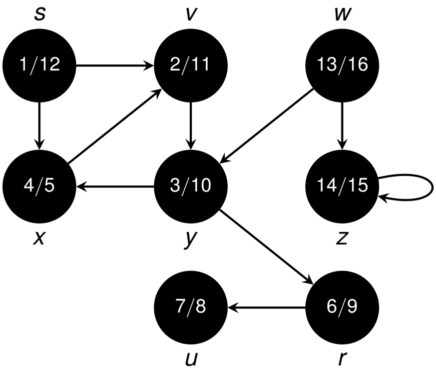
Knuth's Algorithm (1968)

- Perform DFS's so that all vertices are visited
- Output vertices in decreasing order of their finishing time

Runtime $O(V + E)$

Don't need to sort the vertices – use DFS directly!

Execution of Knuth's Algorithm



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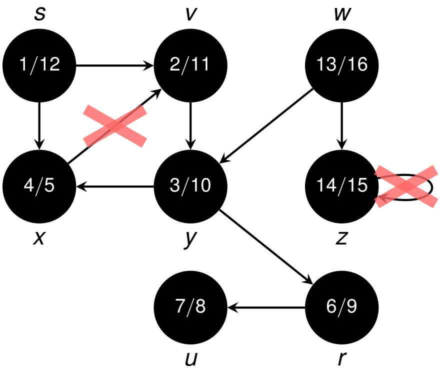

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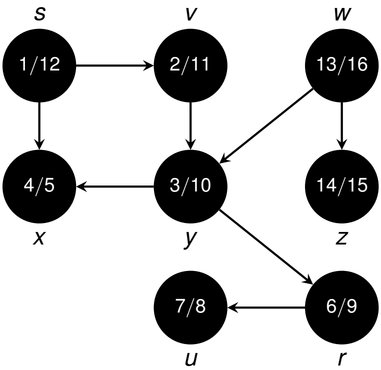

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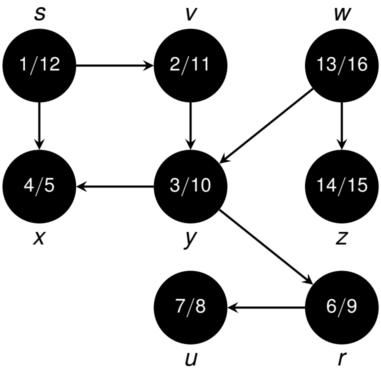

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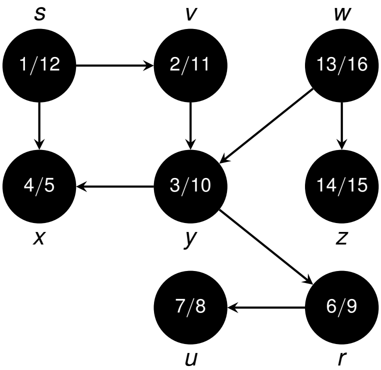

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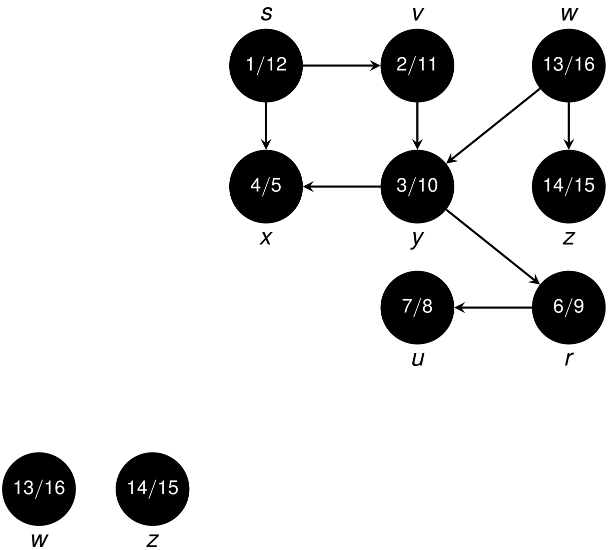

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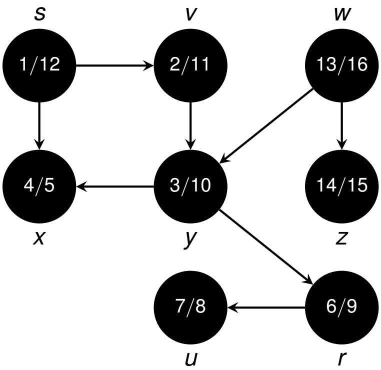

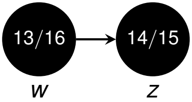

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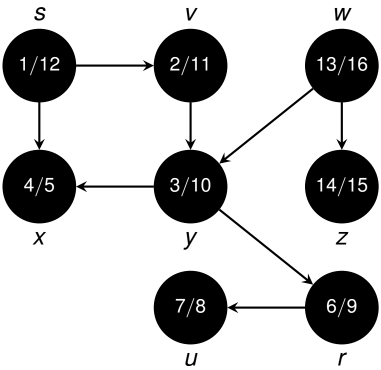

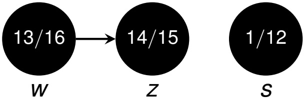

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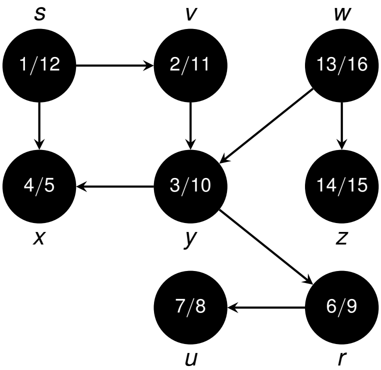

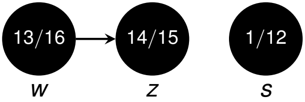

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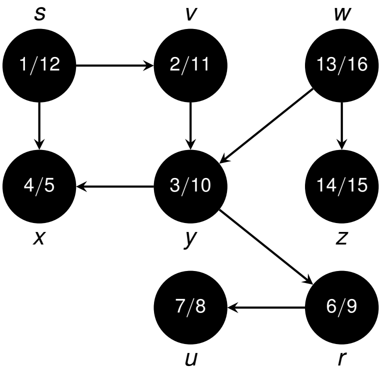

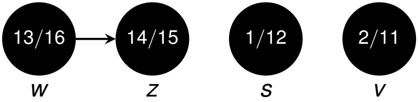

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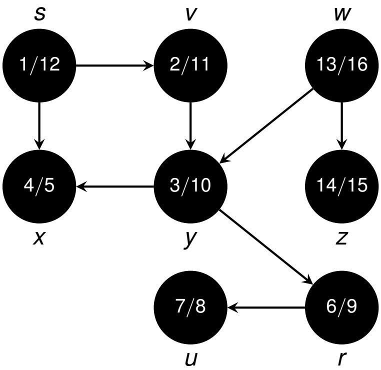

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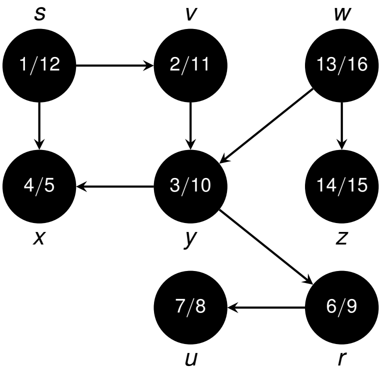

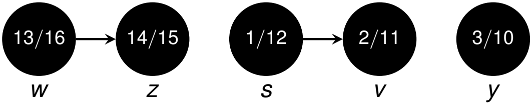

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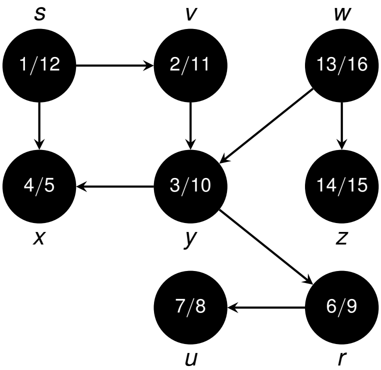

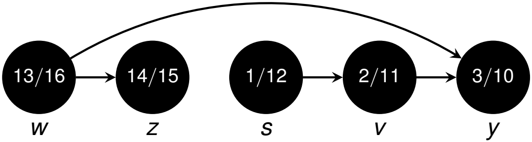

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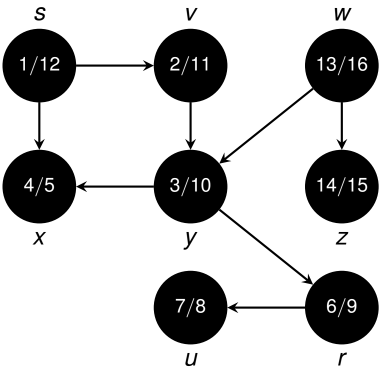

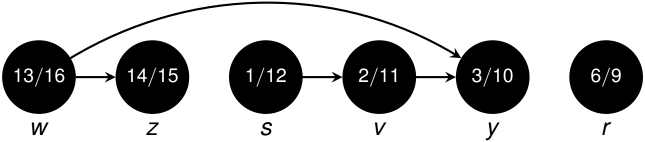

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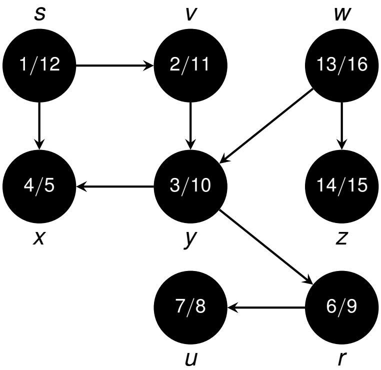

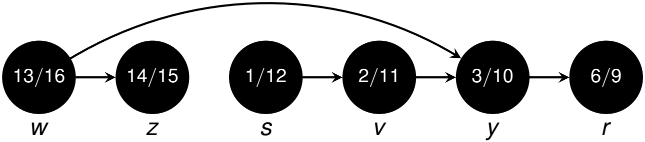

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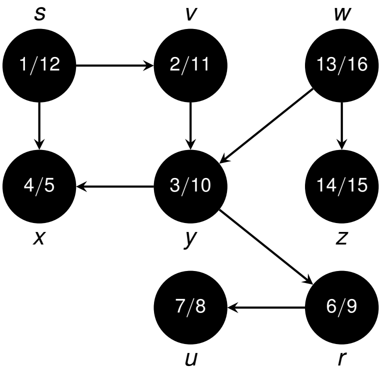

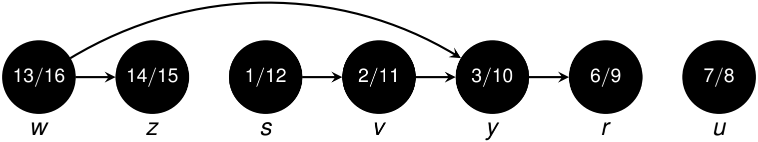

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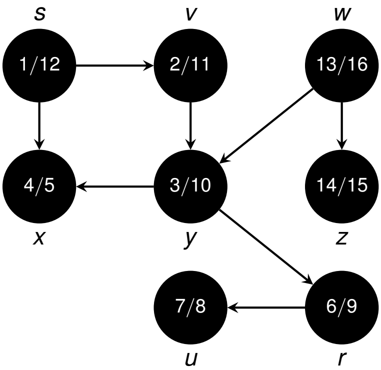

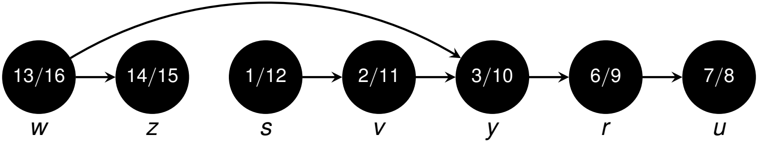

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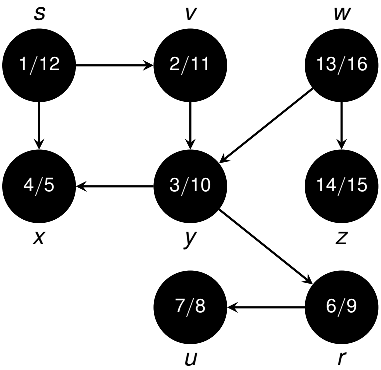

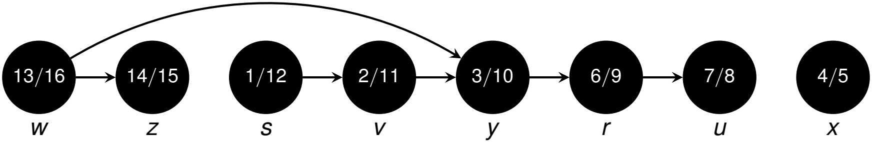

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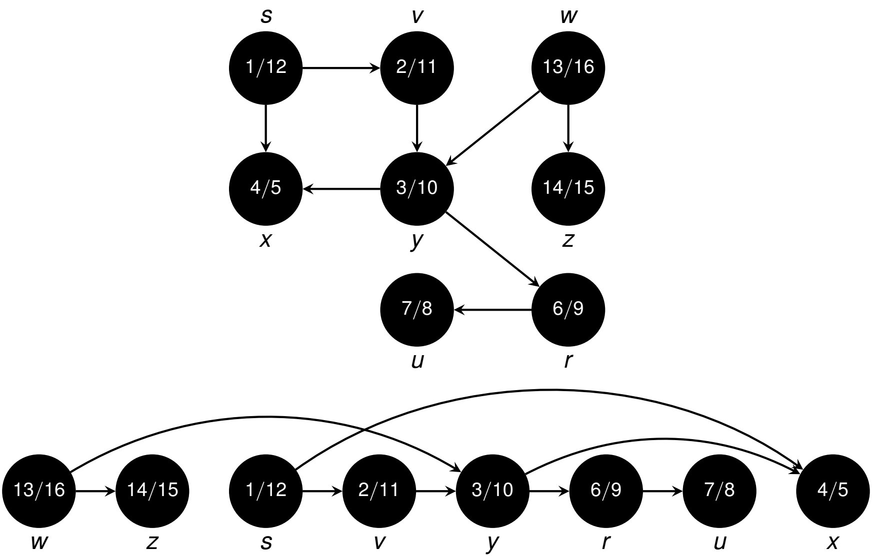
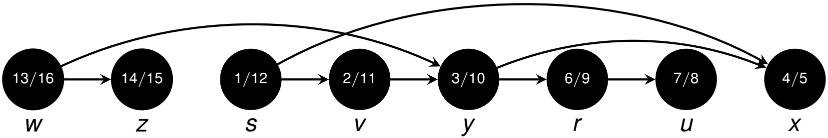


Correctness of Topological Sort using DFS



Theorem 22.12

If the input graph is a DAG, then the algorithm computes a linear order.

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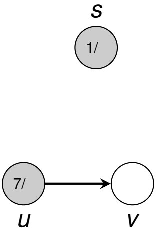
- Consider any edge $(u, v) \in E(G)$ being explored,



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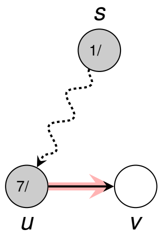
Proof:

- Consider any edge $(u, v) \in E(G)$ being explored,
  $\Rightarrow$ u is grey and we have to show that $v.f < u.f$



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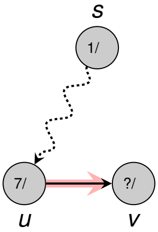

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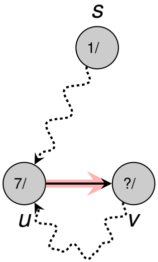

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  1. *If v is grey, then there is a cycle*
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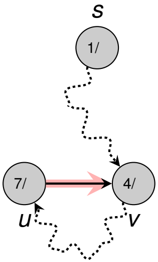

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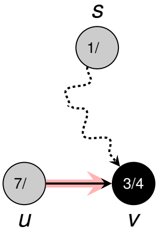

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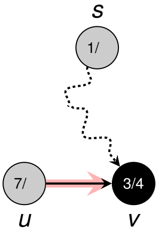

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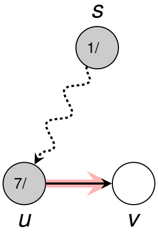
  3. *If v is white,*



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  1. *If v is grey, then there is a cycle*
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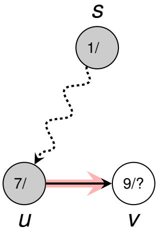
  2. *If v is black, then $v.f < u.f$.*
  3. *If v is white, we call DFS(v) and $v.f < u.f$.*



Correctness of Topological Sort using DFS

Theorem 22.12

If the input graph is a DAG, then the algorithm computes a linear order.

Proof:

- Consider any edge $(u, v) \in E(G)$ being explored,
  $\Rightarrow$ u is grey and we have to show that $v.f < u.f$

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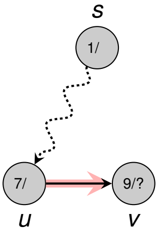

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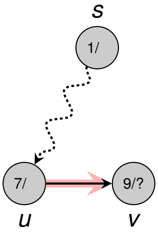

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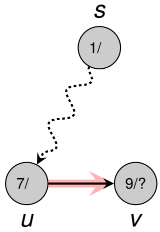

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Summary of Graph Searching

Breadth-First-Search

- vertices are processed by a queue
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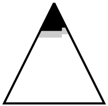
  $\rightsquigarrow$ similar idea used later in Prim's and Dijkstra's algorithm
- Runtime $\mathcal{O}(V + E)$



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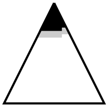

Depth-First-Search

- vertices are processed by recursive calls ($\approx$ stack)
- discovery and finishing times
- application: Topogical Sorting of DAGs
- Runtime $\mathcal{O}(V + E)$