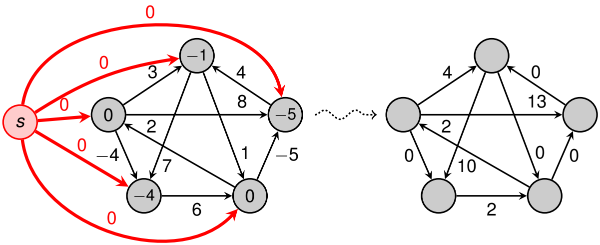

6.5: All-Pairs Shortest Paths

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Thomas Sauerwald

Lent 2016

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Outline

All-Pairs Shortest Path

APSP via Matrix Multiplication

Johnson's Algorithm

Formalising the Problem

All-Pairs Shortest Path Problem

- Given: directed graph $G = (V, E)$, $V = \{1, 2, \ldots, n\}$, with edge weights represented by a matrix W:

$$w_{i,j} = \begin{cases} \text{weight of edge } (i,j) & \text{for an edge } (i,j) \in E, \\ \infty & \text{if there is no edge from } i \text{ to } j, \\ 0 & \text{if } i = j. \end{cases}$$

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Here we will only compute the weight of the shortest path without keeping track of the edges of the path!

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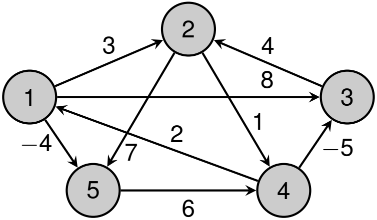
Example of Shortest Path via Matrix Multiplication (Figure 25.1)



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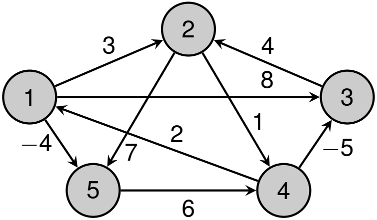

$$L^{(1)} = W = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

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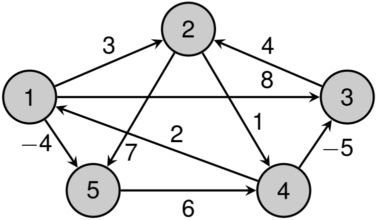

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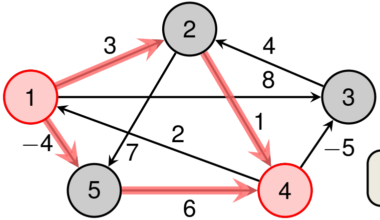

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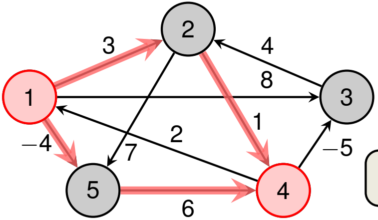

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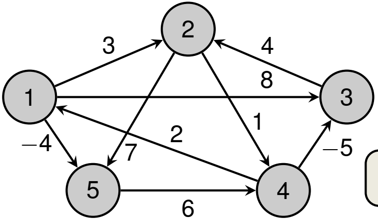

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$$L^{(3)} = \begin{pmatrix} 0 & 3 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}$$

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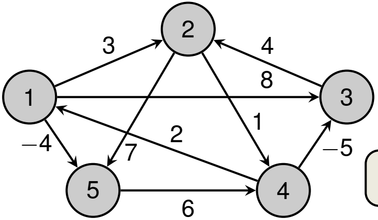

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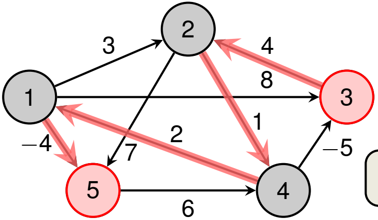

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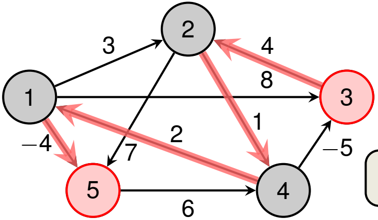

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$$+ \Leftrightarrow \times$$

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$L^{(m)}$ can be computed in $\mathcal{O}(n^3)$

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Computing $L^{(n-1)}$ efficiently

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- For, say, $n = 738$, we subsequently compute

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- Since we don't need the intermediate matrices, a more efficient way is

$$L^{(1)}, L^{(2)}, L^{(4)}, \ldots, L^{(512)}, L^{(1024)} = L$$

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$$L^{(1)}, L^{(2)}, L^{(3)}, L^{(4)}, \ldots, L^{(737)} = L$$

- Since we don't need the intermediate matrices, a more efficient way is

$$L^{(1)}, L^{(2)}, L^{(4)}, \ldots, L^{(512)}, L^{(1024)} = L$$

Takes $\mathcal{O}(\log n \cdot n^3)$.

Computing $L^{(n-1)}$ efficiently

$$\ell_{i,j}^{(m)} = \min_{1 \leq k \leq n} \left(\ell_{i,k}^{(m-1)} + w_{k,j} \right)$$

Takes $\mathcal{O}(n \cdot n^3) = \mathcal{O}(n^4)$

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We need $L^{(4)} = L^{(2)} \cdot L^{(2)} = L^{(3)} \cdot L^{(1)}$! (see Ex. 25.1-4)

Takes $\mathcal{O}(\log n \cdot n^3)$.

Outline

All-Pairs Shortest Path

APSP via Matrix Multiplication

Johnson's Algorithm

Johnson's Algorithm

Overview

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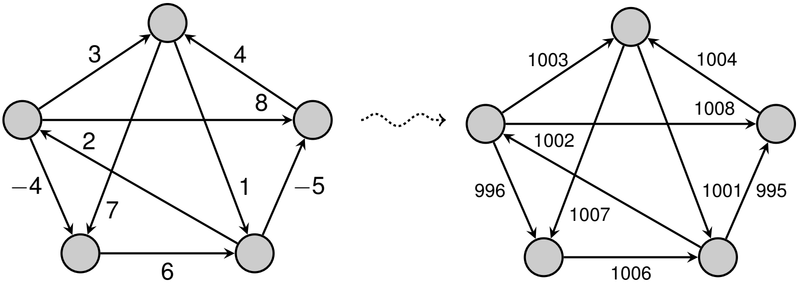

How Johnson's Algorithm works

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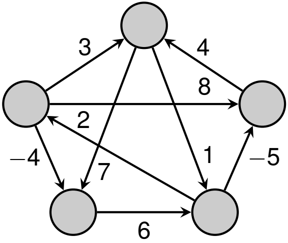
1. Add a new vertex s and directed edges (s, v), $v \in V$, with weight 0



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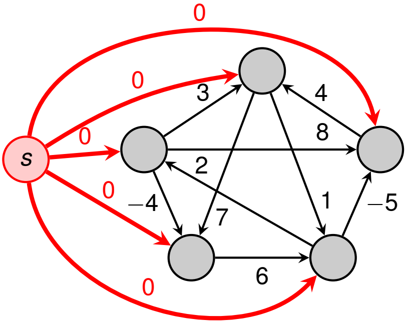

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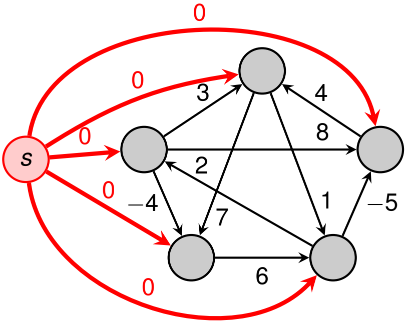

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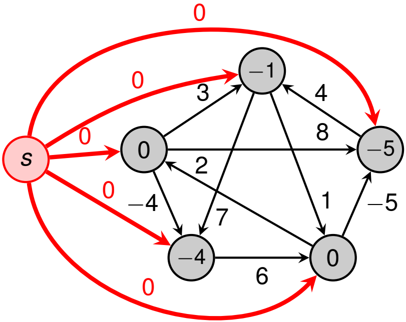

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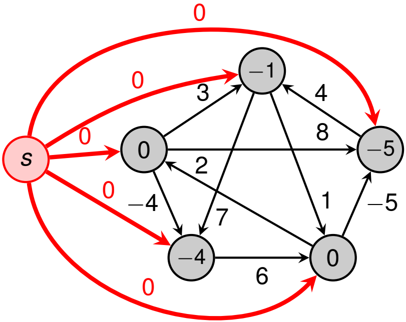

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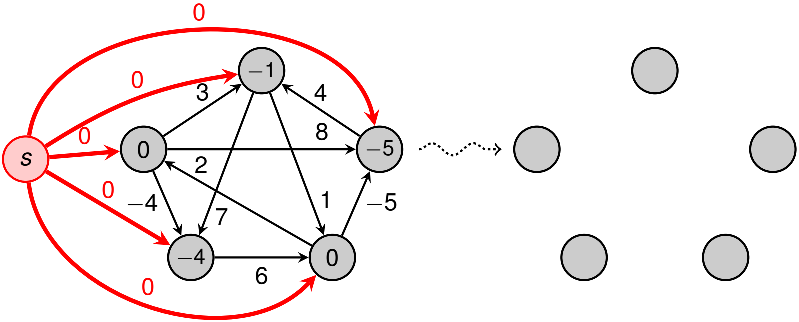

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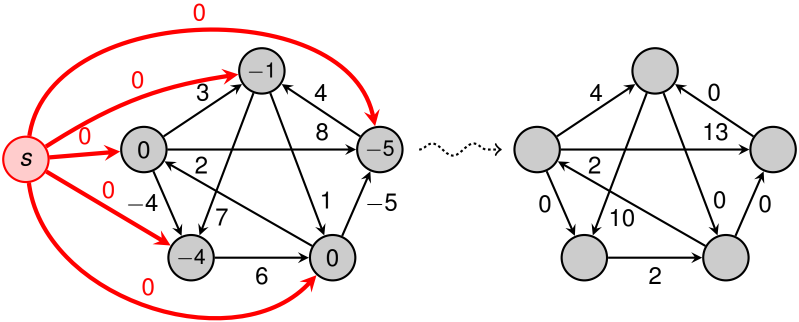

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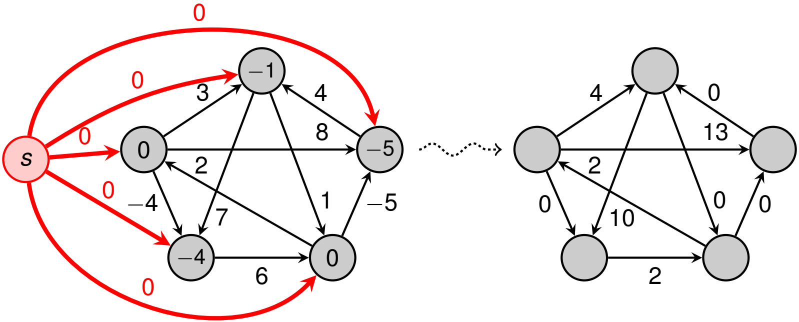

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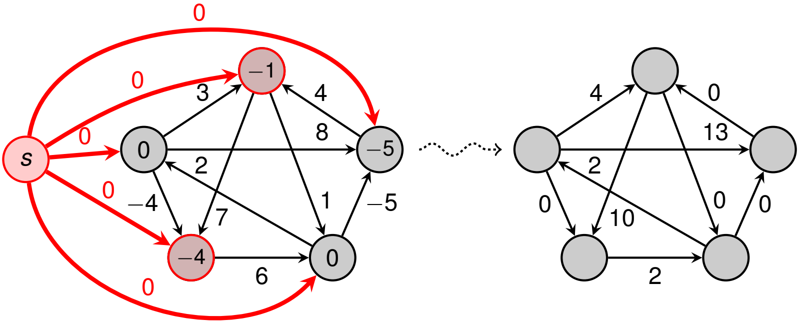

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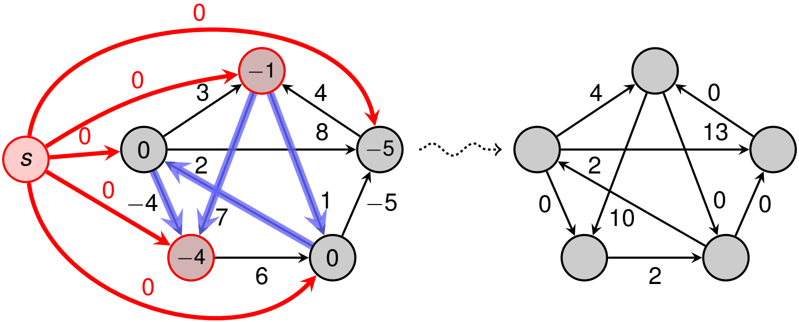

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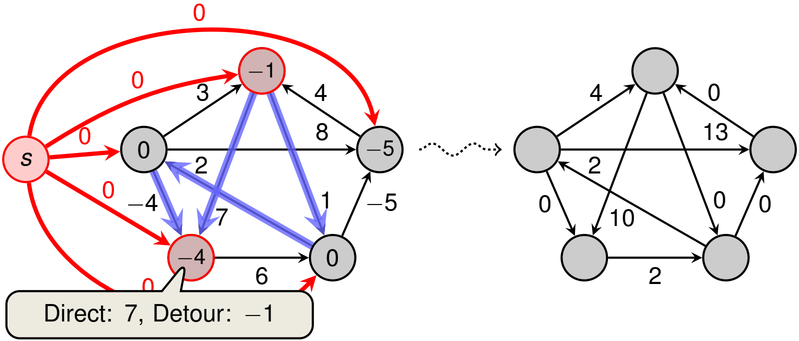

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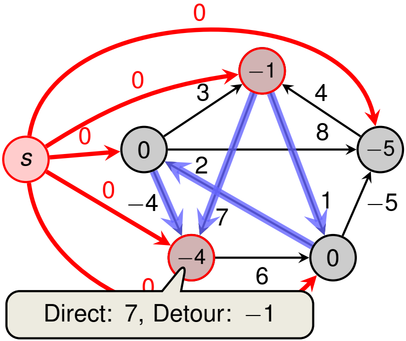

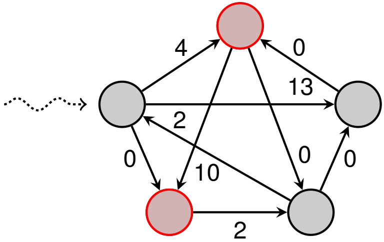

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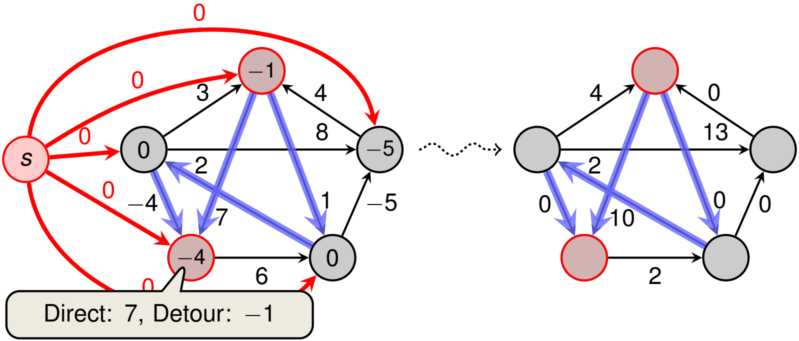

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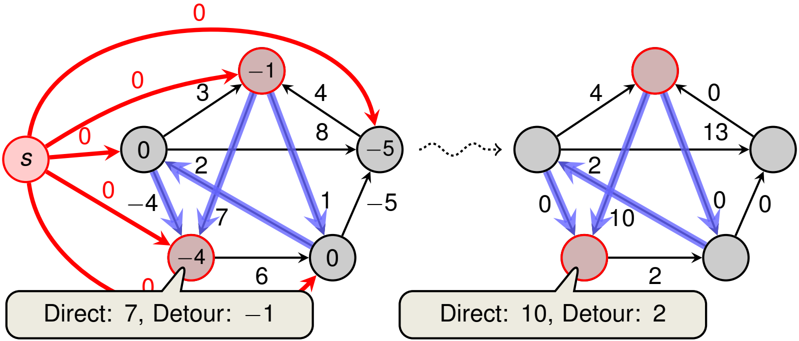

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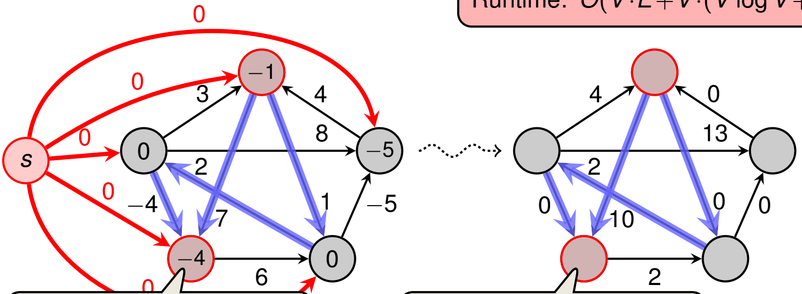
Runtime: $O(V \cdot E + V \cdot (V \log V + E))$



Direct: 7, Detour: −1

Direct: 10, Detour: 2

Correctness of Johnson's Algorithm

Theorem

For any graph $G = (V, E, w)$ without negative-weight cycles:

1. After reweighting, all edges are non-negative
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$$u.\delta + w(u,v) \geq v.\delta \qquad \text{(triangle inequality)}$$

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$$\begin{aligned} u.\delta + w(u,v) &\geq v.\delta \qquad \text{(triangle inequality)} \\ \Rightarrow \quad \widetilde{w}(u,v) + u.\delta + w(u,v) &\geq w(u,v) + u.\delta - v.\delta + v.\delta \\ \Rightarrow \quad \widetilde{w}(u,v) &\geq 0 \end{aligned}$$

$\square$

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Comparison of all Shortest-Path Algorithms

Algorithm	SSSP		APSP		negative weights
	sparse	dense	sparse	dense	
Bellman-Ford	V^2	V^3	V^3	V^4	✓
Dijkstra	$V \log V$	V^2	$V^2 \log V$	V^3	X
Matrix Mult.	–	–	$V^3 \log V$	$V^3 \log V$	(✓)
Johnson	–	–	$V^2 \log V$	V^3	✓

can handle negative weight edges,
but not negative weight cycles