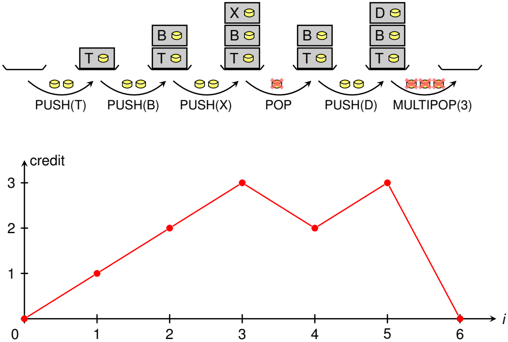


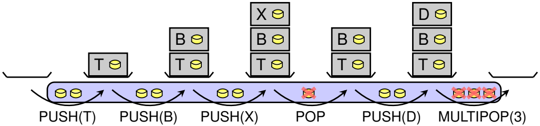
5.1: Amortized Analysis

Frank Stajano Thomas Sauerwald

Lent 2016

UNIVERSITY OF CAMBRIDGE

Use of Amortized Analysis



Amortized Analysis

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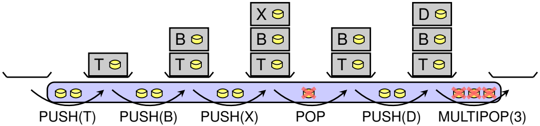

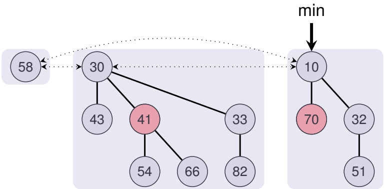

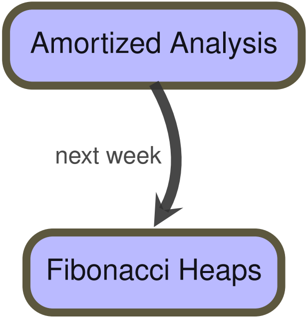

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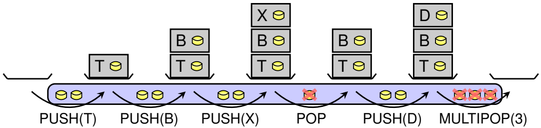

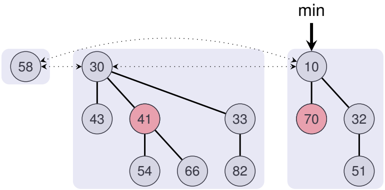

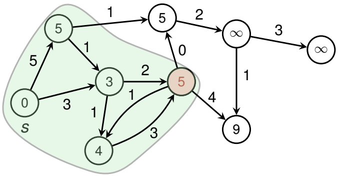

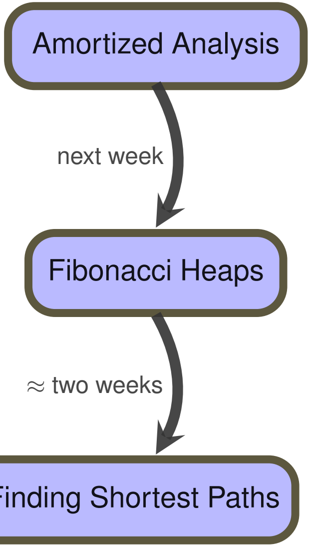

Motivating Example: Stack

Stack Operations

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- **`PUSH(S,x)`**

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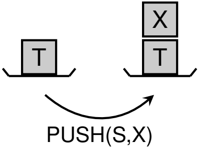
  - pushes object x onto stack S



Motivating Example: Stack

Stack Operations

- **`PUSH(S,x)`**
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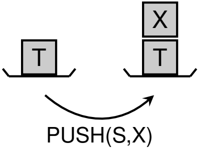

Motivating Example: Stack

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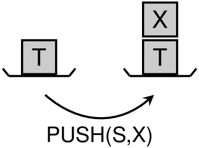

Motivating Example: Stack

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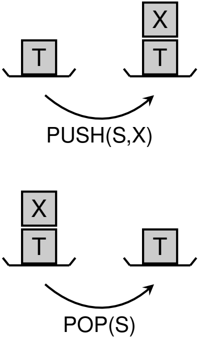

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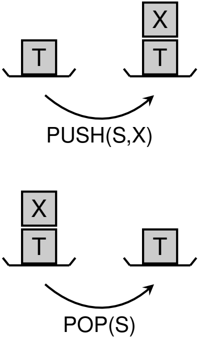

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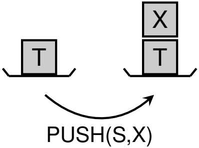

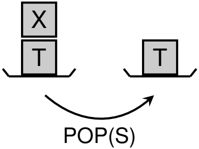

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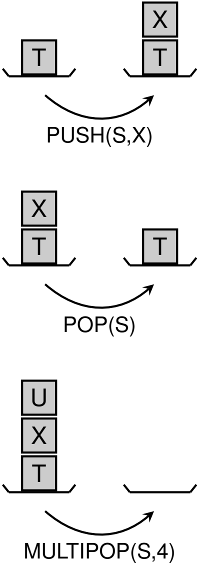

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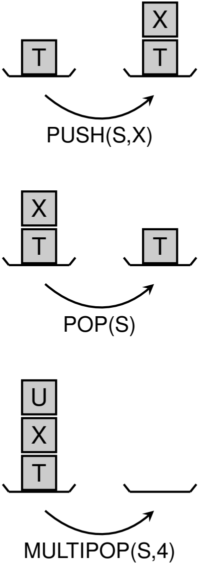

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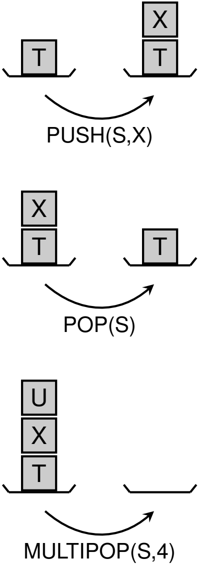
```
MULTIPOP(S,k)
while not S.empty() and k > 0
    POP(S)
    k = k - 1
```



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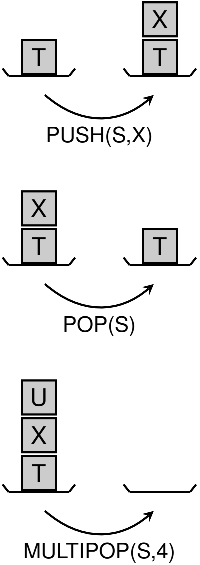

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Simple Worst-Case Bound (stack is initially empty):

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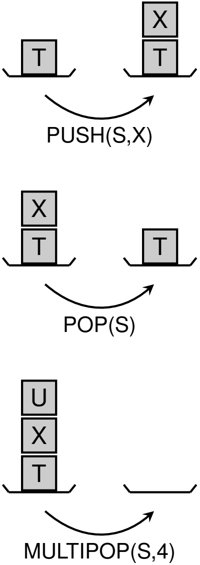

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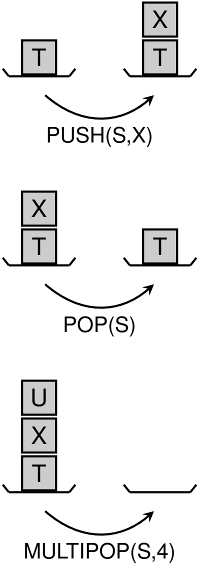

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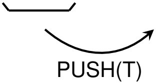

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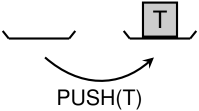

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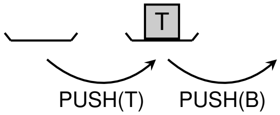

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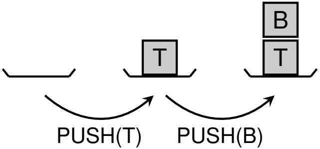

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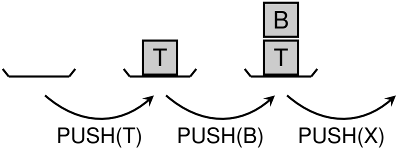

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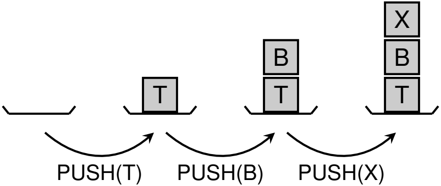

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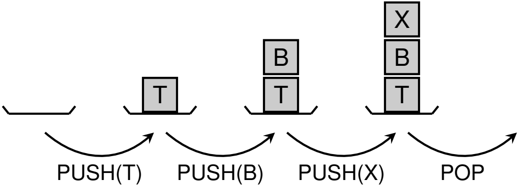

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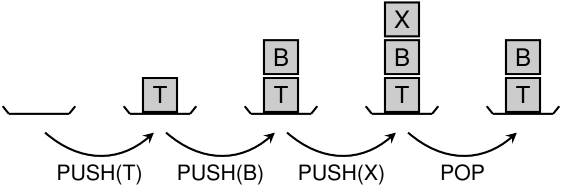

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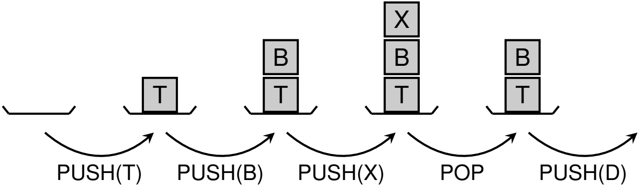

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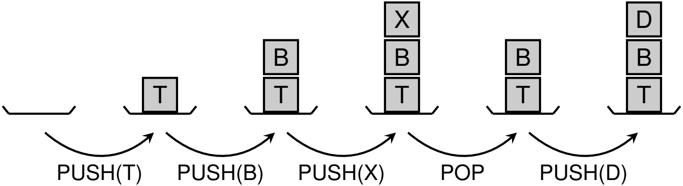

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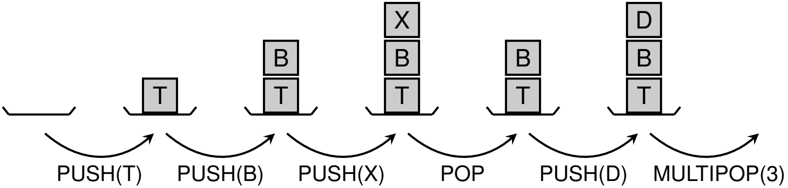

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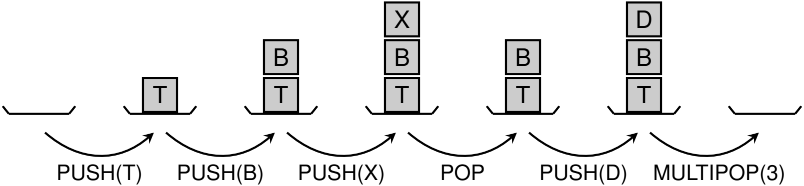

A new Analysis Tool: Amortized Analysis

Amortized Analysis

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Amortized Analysis

- analyse a sequence of operations

A new Analysis Tool: Amortized Analysis

Data structure operations (Heap, Stack, Queue etc.)

Amortized Analysis

- analyse a sequence of operations

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- analyse a sequence of operations
- show that average cost of an operation is small

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This is **not** average case analysis!

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- concrete techniques
  - Aggregate Analysis
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- amortized cost of each operation is the average $\frac{T(n)}{n}$

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Even though operations may be of different types/costs

Stack: Aggregate Analysis

Simple Worst-Case Bound:

- largest cost of an operation: n
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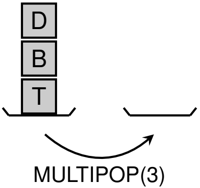

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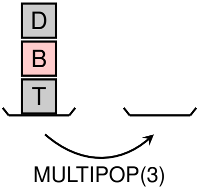

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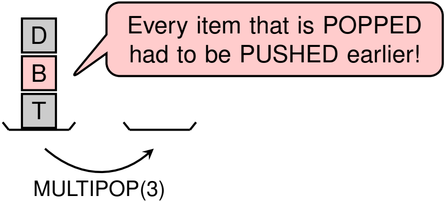

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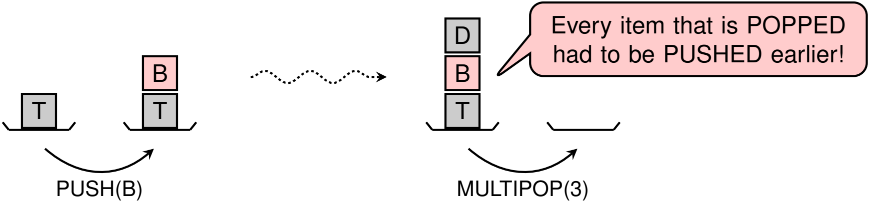

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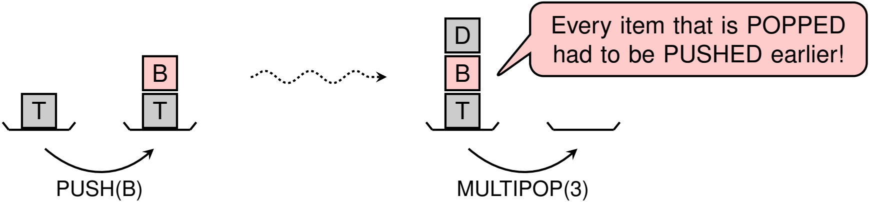

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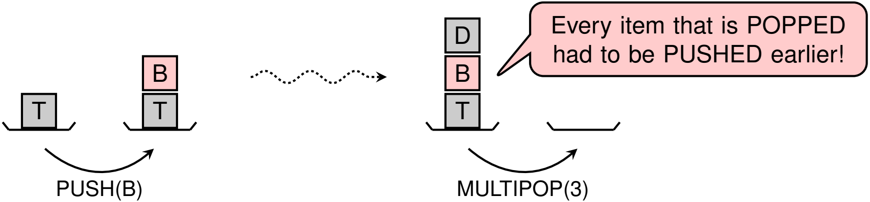

$T(n) \leq$

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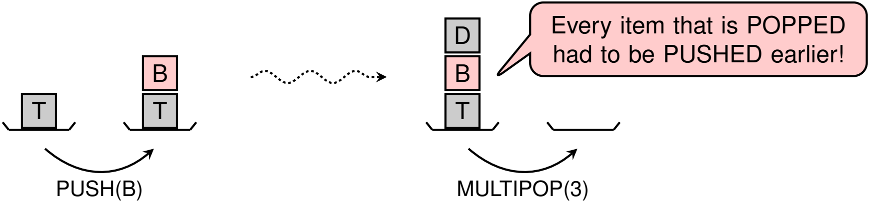

$$T(n) \leq T_{POP}(n) + T_{PUSH}(n)$$

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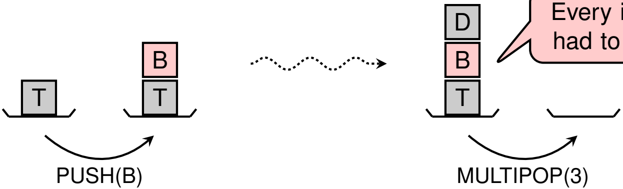

Every item that is POPPED had to be PUSHED earlier!

MULTIPOP(k) contributes $\min\{k, |S|\}$ to $T_{POP}(n)$

$$T(n) \leq T_{POP}(n) + T_{PUSH}(n)$$

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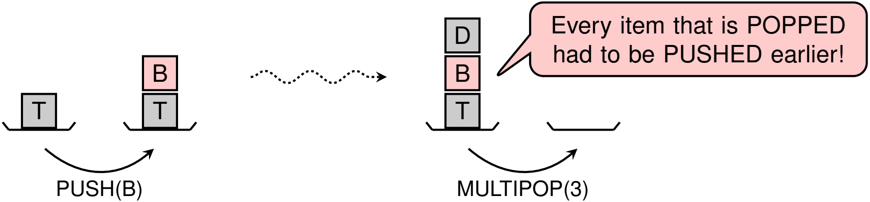

$$T(n) \leq T_{POP}(n) + T_{PUSH}(n) \leq 2 \cdot T_{PUSH}(n)$$

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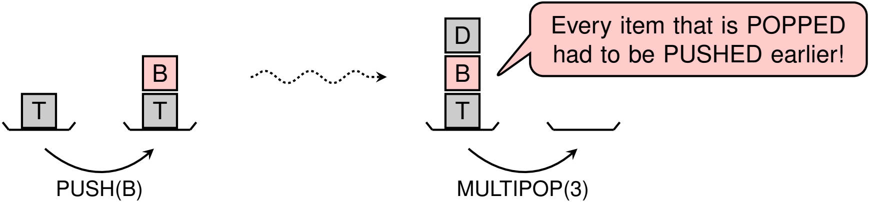

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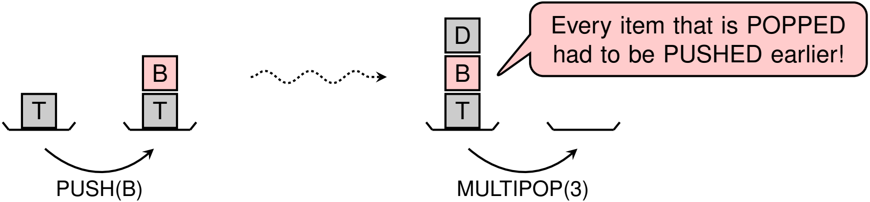

$$T(n) \leq T_{POP}(n) + T_{PUSH}(n) \leq 2 \cdot T_{PUSH}(n) \leq 2 \cdot n.$$

Aggregate Analysis: The amortized cost per operation is $\frac{T(n)}{n} \leq 2$

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$\rightsquigarrow$ store **(fictitious)** credit in the data structure to cover up for expensive operations

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- amount of potential energy stored
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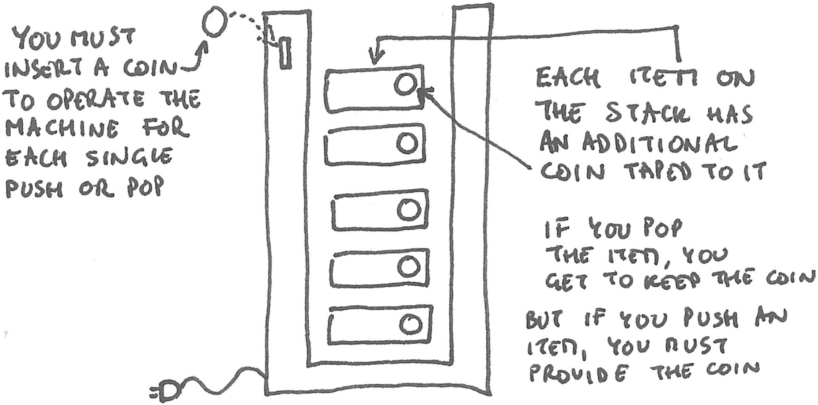
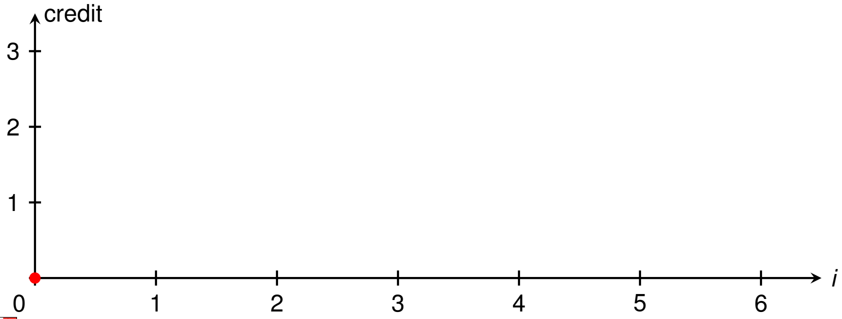
Stack as a coin-operated machine (p. 83)



Stack and Coins



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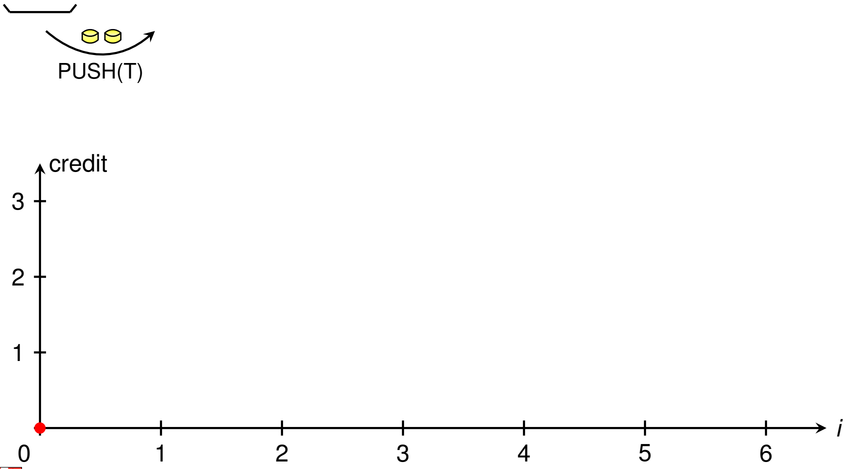

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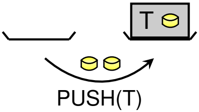

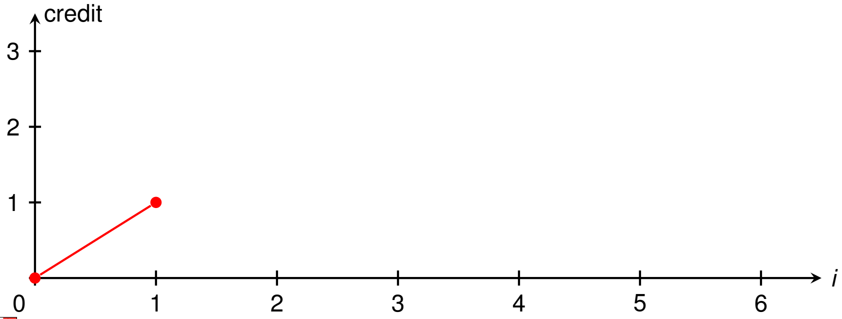

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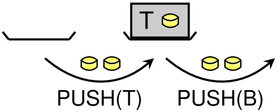

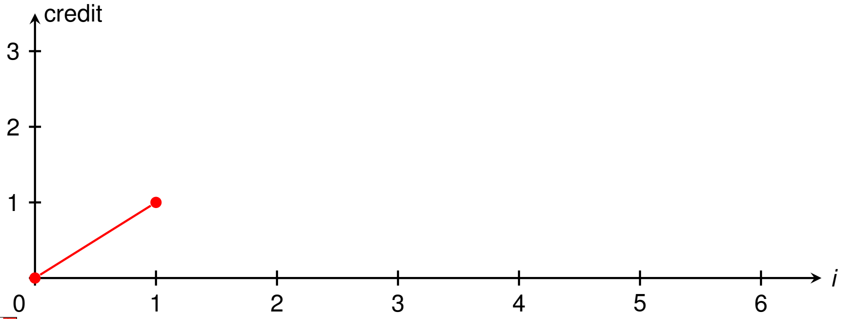

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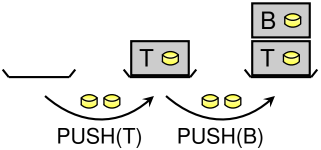

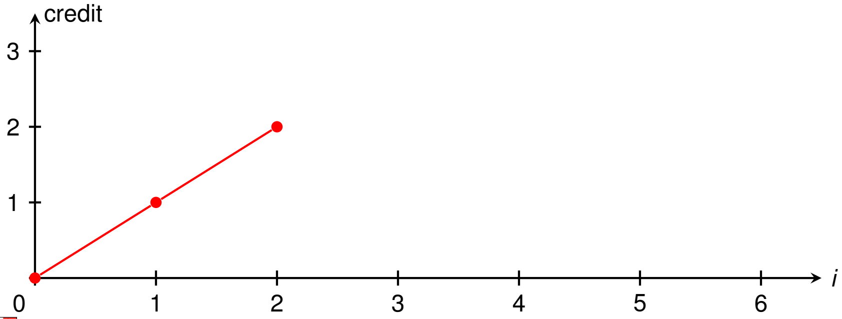

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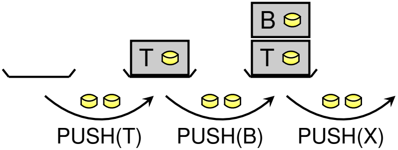

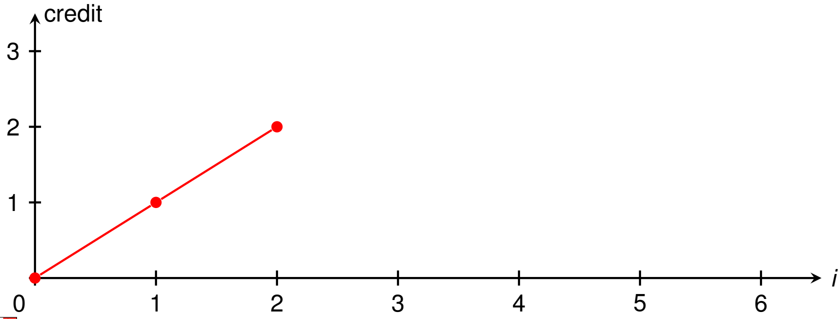

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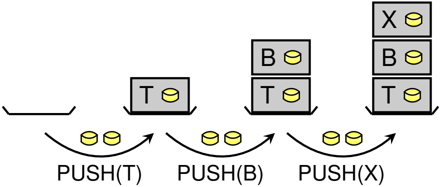

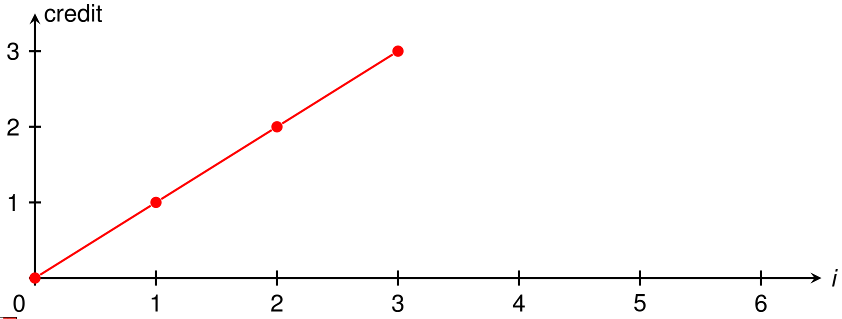

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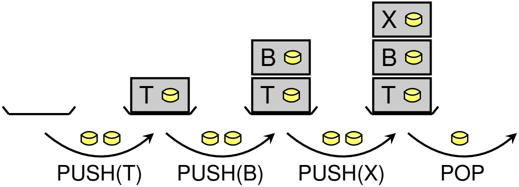

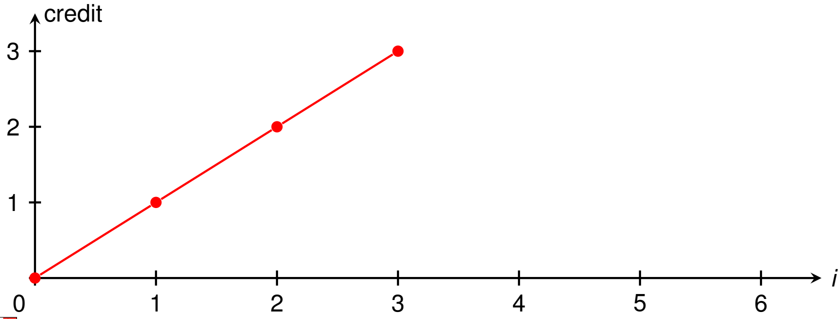

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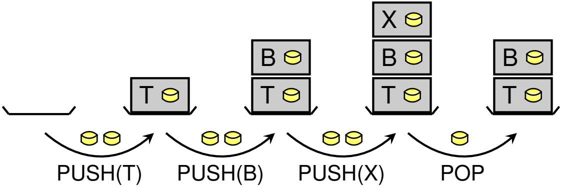
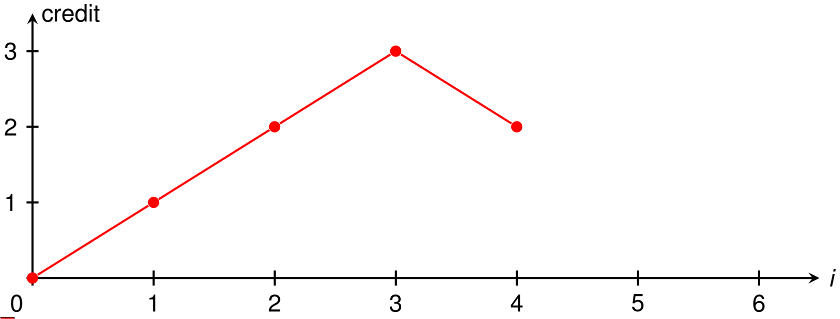
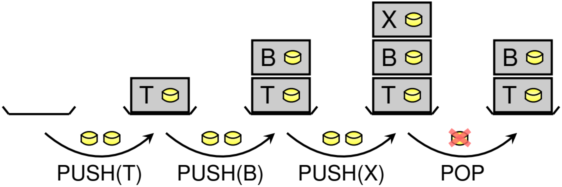
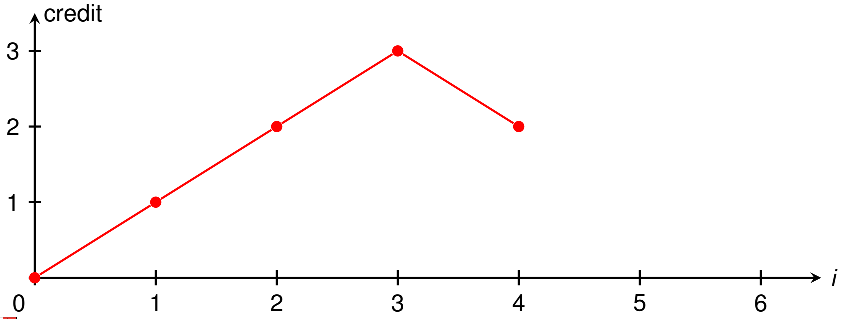




Stack and Coins





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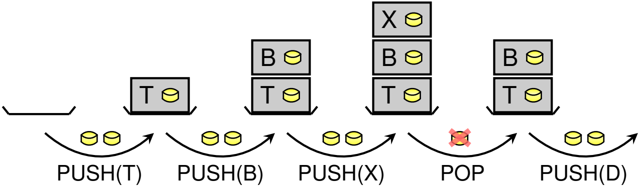

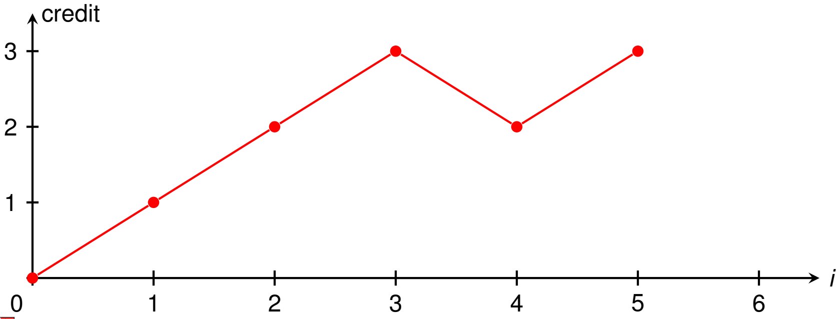

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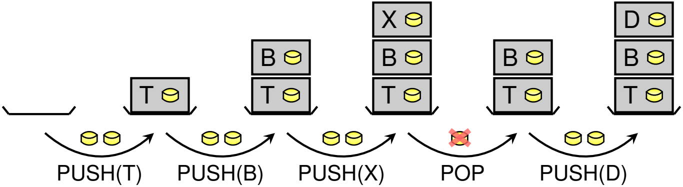

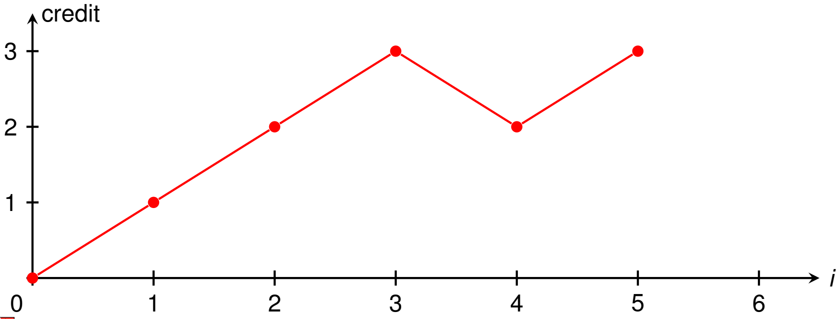

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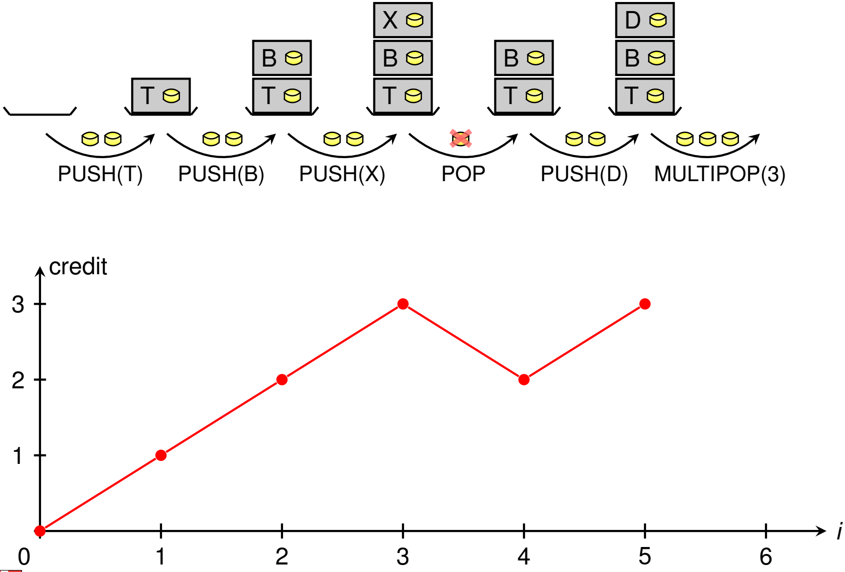

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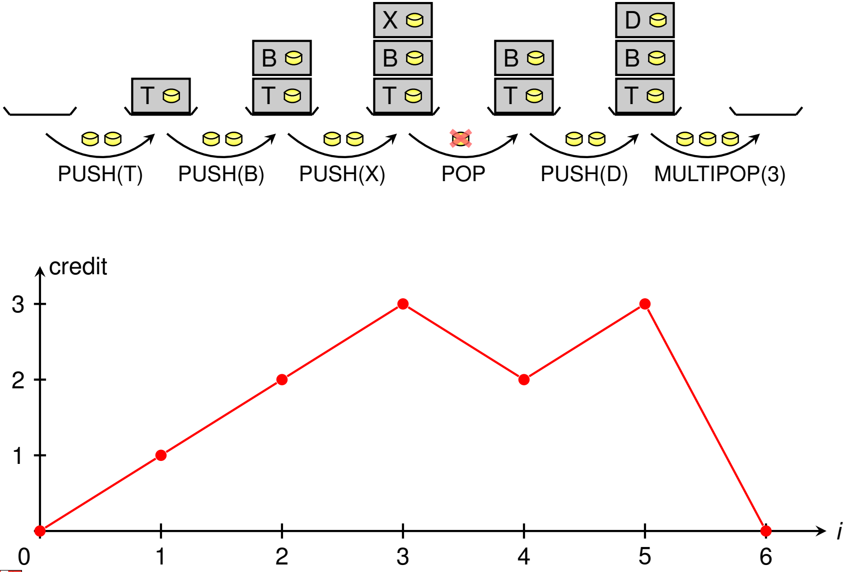

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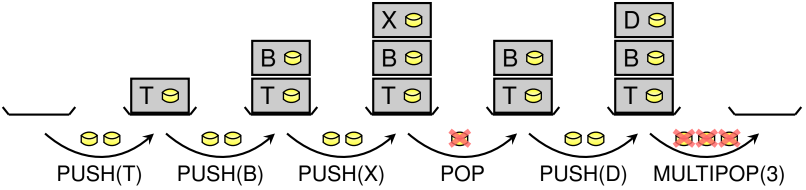

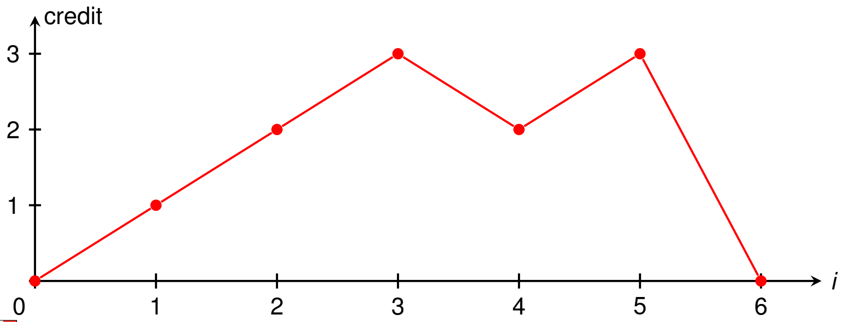

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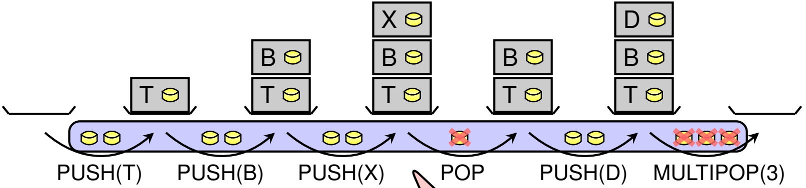
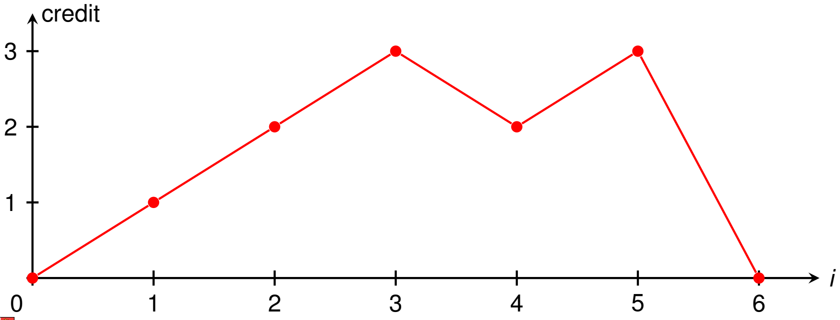


Every operation costs at most **two coins**!



Potential Method in Detail

- c_i is the actual cost of operation i

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$c_i < \widehat{c}_i$, $c_i = \widehat{c}_i$ or $c_i > \widehat{c}_i$ are all possible!

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Function that maps states of the data structure to some value

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$$\widehat{c}_i = c_i + (\Phi_i - \Phi_{i-1})$$

- PUSH(): $c_i = 1$
- POP: $c_i = 1$

- PUSH(): $\Phi_i - \Phi_{i-1} = 1$
- POP: $\Phi_i - \Phi_{i-1} = -1$

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$$\sum_{i=1}^{n} \widehat{c}_i = \sum_{i=1}^{n} \left(c_i + \Phi_i - \Phi_{i-1}\right) =$$

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$$\sum_{i=1}^{n} \widehat{c}_i = \sum_{i=1}^{n} (c_i + \Phi_i - \Phi_{i-1}) = \sum_{i=1}^{n} c_i + \Phi_n - \Phi_0$$

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$$\sum_{i=1}^{n} \widehat{c}_i = \sum_{i=1}^{n} (c_i + \Phi_i - \Phi_{i-1}) = \sum_{i=1}^{n} c_i + \Phi_n$$

If $\Phi_n \geq 0$ for all n, sum of amortized costs is an upper bound for the sum of actual costs!

Stack: Analysis via Potential Method

$\Phi_i =$

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Φ_i = # objects in the stack after ith operation (= # coins)

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$\Phi_i = \#$ objects in the stack after ith operation $(= \#$ coins$)$

PUSH

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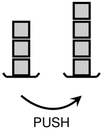
- actual cost: $c_i = 1$



Stack: Analysis via Potential Method

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PUSH

- actual cost: $c_i = 1$
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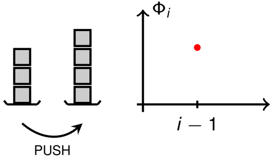

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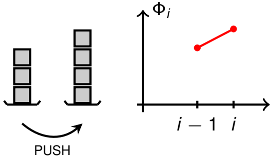

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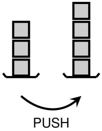

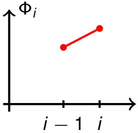

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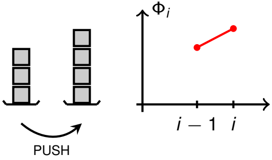

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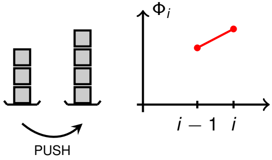

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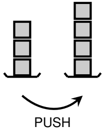

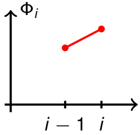

POP

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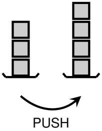
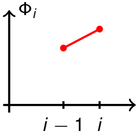
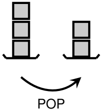




POP

- $c_i = 1$



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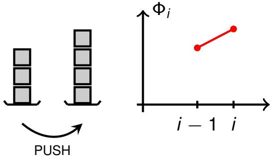

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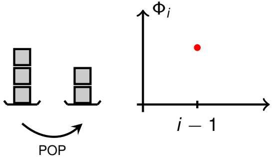

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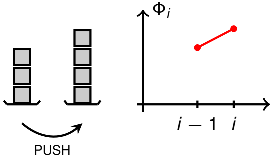

POP

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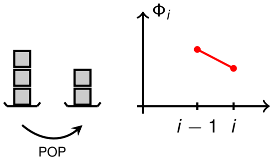

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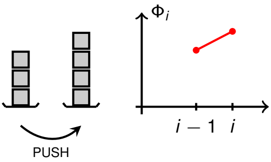

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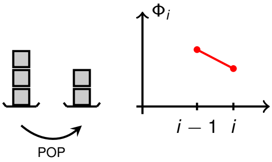

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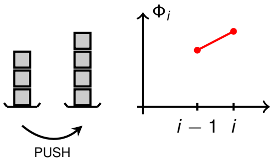

POP

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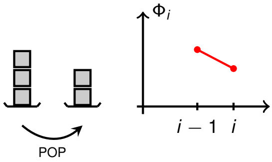

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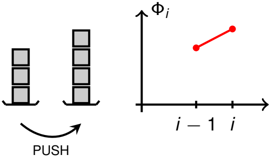

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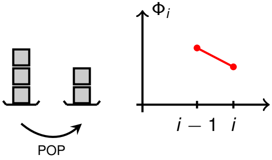
Stack is non-empty!



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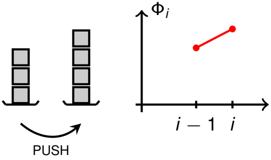

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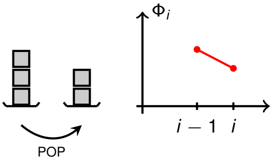

MULTIPOP(k)

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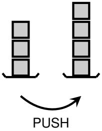

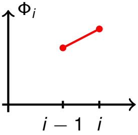

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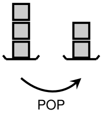

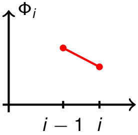

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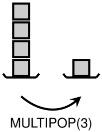

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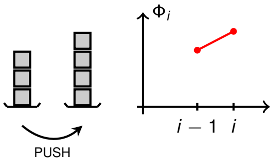

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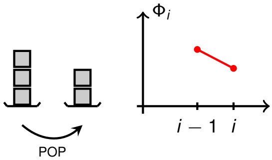

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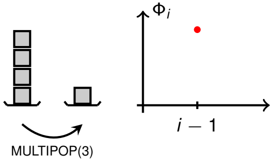

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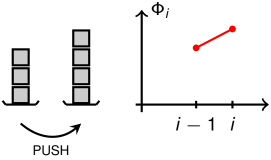

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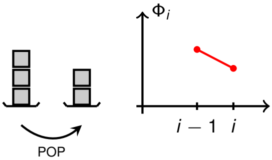

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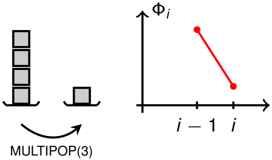

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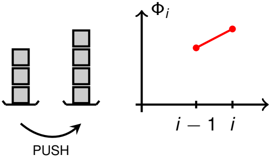

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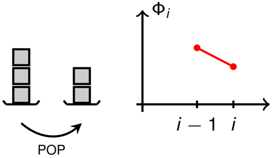

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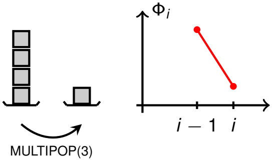

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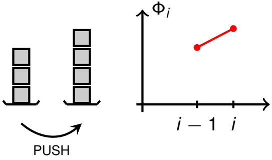

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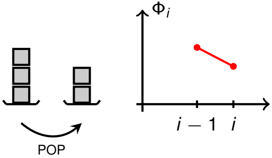

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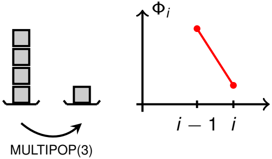

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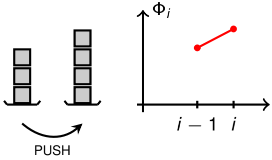

Amortized Cost $\leq 2 \Rightarrow T(n) \leq 2n$

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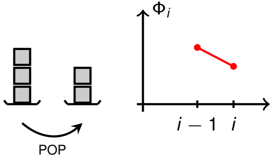

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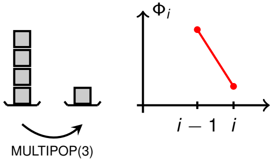

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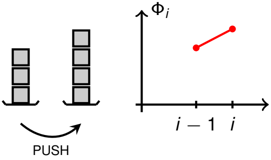

Amortized Cost $\leq 2 \Rightarrow T(n) \leq 2n$

$n/2$ PUSH, $n/2$ POP $\Rightarrow T(n) \leq n$

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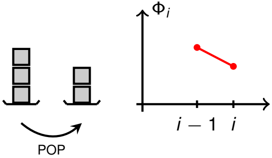

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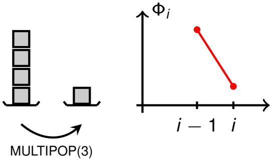

Second Example: Binary Counter

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- Array $A[k-1], A[k-2], \ldots, A[0]$ of k bits
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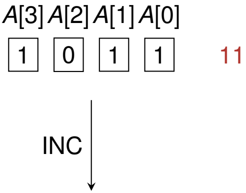

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INC(A)
i = 0
while i < k and A[i]==1
    A[i] = 0
    i = i + 1
A[i] = 1
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$$\begin{array}{cccccc} A[3] & A[2] & A[1] & A[0] & & \\ \boxed{1} & \boxed{0} & \boxed{1} & \boxed{1} & & 11 \end{array}$$

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What is the total cost of a sequence of n INC operations?

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  - total cost: $\cancel{\leq k}$
    number of flips (smallest index of a zero)

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2	0	0	0	0	0	0	1	0	3

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3	0	0	0	0	0	0	1	1	4

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Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10
7	0	0	0	0	0	1	1	1	11

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10
7	0	0	0	0	0	1	1	1	11

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10
7	0	0	0	0	0	1	1	1	11
8	0	0	0	0	1	0	0	0	15

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10
7	0	0	0	0	0	1	1	1	11
8	0	0	0	0	1	0	0	0	15

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10
7	0	0	0	0	0	1	1	1	11
8	0	0	0	0	1	0	0	0	15
9	0	0	0	0	1	0	0	1	16

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10
7	0	0	0	0	0	1	1	1	11
8	0	0	0	0	1	0	0	0	15
9	0	0	0	0	1	0	0	1	16

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10
7	0	0	0	0	0	1	1	1	11
8	0	0	0	0	1	0	0	0	15
9	0	0	0	0	1	0	0	1	16
10	0	0	0	0	1	0	1	0	18

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10
7	0	0	0	0	0	1	1	1	11
8	0	0	0	0	1	0	0	0	15
9	0	0	0	0	1	0	0	1	16
10	0	0	0	0	1	0	1	0	18

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10
7	0	0	0	0	0	1	1	1	11
8	0	0	0	0	1	0	0	0	15
9	0	0	0	0	1	0	0	1	16
10	0	0	0	0	1	0	1	0	18
11	0	0	0	0	1	0	1	1	19

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10
7	0	0	0	0	0	1	1	1	11
8	0	0	0	0	1	0	0	0	15
9	0	0	0	0	1	0	0	1	16
10	0	0	0	0	1	0	1	0	18
11	0	0	0	0	1	0	1	1	19

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10
7	0	0	0	0	0	1	1	1	11
8	0	0	0	0	1	0	0	0	15
9	0	0	0	0	1	0	0	1	16
10	0	0	0	0	1	0	1	0	18
11	0	0	0	0	1	0	1	1	19
12	0	0	0	0	1	1	0	0	22

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10
7	0	0	0	0	0	1	1	1	11
8	0	0	0	0	1	0	0	0	15
9	0	0	0	0	1	0	0	1	16
10	0	0	0	0	1	0	1	0	18
11	0	0	0	0	1	0	1	1	19
12	0	0	0	0	1	1	0	0	22

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10
7	0	0	0	0	0	1	1	1	11
8	0	0	0	0	1	0	0	0	15
9	0	0	0	0	1	0	0	1	16
10	0	0	0	0	1	0	1	0	18
11	0	0	0	0	1	0	1	1	19
12	0	0	0	0	1	1	0	0	22
13	0	0	0	0	1	1	0	1	23

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10
7	0	0	0	0	0	1	1	1	11
8	0	0	0	0	1	0	0	0	15
9	0	0	0	0	1	0	0	1	16
10	0	0	0	0	1	0	1	0	18
11	0	0	0	0	1	0	1	1	19
12	0	0	0	0	1	1	0	0	22
13	0	0	0	0	1	1	0	1	23

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10
7	0	0	0	0	0	1	1	1	11
8	0	0	0	0	1	0	0	0	15
9	0	0	0	0	1	0	0	1	16
10	0	0	0	0	1	0	1	0	18
11	0	0	0	0	1	0	1	1	19
12	0	0	0	0	1	1	0	0	22
13	0	0	0	0	1	1	0	1	23
14	0	0	0	0	1	1	1	0	25

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10
7	0	0	0	0	0	1	1	1	11
8	0	0	0	0	1	0	0	0	15
9	0	0	0	0	1	0	0	1	16
10	0	0	0	0	1	0	1	0	18
11	0	0	0	0	1	0	1	1	19
12	0	0	0	0	1	1	0	0	22
13	0	0	0	0	1	1	0	1	23
14	0	0	0	0	1	1	1	0	25

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10
7	0	0	0	0	0	1	1	1	11
8	0	0	0	0	1	0	0	0	15
9	0	0	0	0	1	0	0	1	16
10	0	0	0	0	1	0	1	0	18
11	0	0	0	0	1	0	1	1	19
12	0	0	0	0	1	1	0	0	22
13	0	0	0	0	1	1	0	1	23
14	0	0	0	0	1	1	1	0	25
15	0	0	0	0	1	1	1	1	26

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10
7	0	0	0	0	0	1	1	1	11
8	0	0	0	0	1	0	0	0	15
9	0	0	0	0	1	0	0	1	16
10	0	0	0	0	1	0	1	0	18
11	0	0	0	0	1	0	1	1	19
12	0	0	0	0	1	1	0	0	22
13	0	0	0	0	1	1	0	1	23
14	0	0	0	0	1	1	1	0	25
15	0	0	0	0	1	1	1	1	26

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10
7	0	0	0	0	0	1	1	1	11
8	0	0	0	0	1	0	0	0	15
9	0	0	0	0	1	0	0	1	16
10	0	0	0	0	1	0	1	0	18
11	0	0	0	0	1	0	1	1	19
12	0	0	0	0	1	1	0	0	22
13	0	0	0	0	1	1	0	1	23
14	0	0	0	0	1	1	1	0	25
15	0	0	0	0	1	1	1	1	26
16	0	0	0	1	0	0	0	0	31

Incrementing a Binary Counter

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10
7	0	0	0	0	0	1	1	1	11
8	0	0	0	0	1	0	0	0	15
9	0	0	0	0	1	0	0	1	16
10	0	0	0	0	1	0	1	0	18
11	0	0	0	0	1	0	1	1	19
12	0	0	0	0	1	1	0	0	22
13	0	0	0	0	1	1	0	1	23
14	0	0	0	0	1	1	1	0	25
15	0	0	0	0	1	1	1	1	26
16	0	0	0	1	0	0	0	0	31

Incrementing a Binary Counter: Aggregate Analysis

Counter Value	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0
1	0	0	0	1	1
2	0	0	1	0	3
3	0	0	1	1	4
4	0	1	0	0	7
5	0	1	0	1	8
6	0	1	1	0	10
7	0	1	1	1	11

Incrementing a Binary Counter: Aggregate Analysis

Counter Value	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0
1	0	0	0	1	1
2	0	0	1	0	3
3	0	0	1	1	4
4	0	1	0	0	7
5	0	1	0	1	8
6	0	1	1	0	10
7	0	1	1	1	11

Incrementing a Binary Counter: Aggregate Analysis

Counter Value	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0
1	0	0	0	1	1
2	0	0	1	0	3
3	0	0	1	1	4
4	0	1	0	0	7
5	0	1	0	1	8
6	0	1	1	0	10
7	0	1	1	1	11

Incrementing a Binary Counter: Aggregate Analysis

Counter Value	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0
1	0	0	0	1	1
2	0	0	1	0	3
3	0	0	1	1	4
4	0	1	0	0	7
5	0	1	0	1	8
6	0	1	1	0	10
7	0	1	1	1	11

Incrementing a Binary Counter: Aggregate Analysis

Counter Value	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0
1	0	0	0	1	1
2	0	0	1	0	3
3	0	0	1	1	4
4	0	1	0	0	7
5	0	1	0	1	8
6	0	1	1	0	10
7	0	1	1	1	11

Incrementing a Binary Counter: Aggregate Analysis

Counter Value	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0
1	0	0	0	1	1
2	0	0	1	0	3
3	0	0	1	1	4
4	0	1	0	0	7
5	0	1	0	1	8
6	0	1	1	0	10
7	0	1	1	1	11

- Bit $A[i]$ is only flipped every 2^i increments

Incrementing a Binary Counter: Aggregate Analysis

Counter Value	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0
1	0	0	0	1	1
2	0	0	1	0	3
3	0	0	1	1	4
4	0	1	0	0	7
5	0	1	0	1	8
6	0	1	1	0	10
7	0	1	1	1	11

- Bit $A[i]$ is only flipped every 2^i increments
- In a sequence of n increments from 0, bit $A[i]$ is flipped $\lfloor \frac{n}{2^i} \rfloor$ times

Incrementing a Binary Counter: Aggregate Analysis

Counter Value	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0
1	0	0	0	1	1
2	0	0	1	0	3
3	0	0	1	1	4
4	0	1	0	0	7
5	0	1	0	1	8
6	0	1	1	0	10
7	0	1	1	1	11

- Bit $A[i]$ is only flipped every 2^i increments
- In a sequence of n increments from 0, bit $A[i]$ is flipped $\lfloor \frac{n}{2^i} \rfloor$ times

$$T(n) \leq$$

Incrementing a Binary Counter: Aggregate Analysis

Counter Value	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Total Cost
0	0	0	0	0	0
1	0	0	0	1	1
2	0	0	1	0	3
3	0	0	1	1	4
4	0	1	0	0	7
5	0	1	0	1	8
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$$T(n) \leq \sum_{i=0}^{k-1} \left\lfloor \frac{n}{2^i} \right\rfloor \leq \sum_{i=0}^{k-1} \frac{n}{2^i} = n \cdot \left(1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{k-1}} \right)$$

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Aggregate Analysis: The amortized cost per operation is $\frac{T(n)}{n} \leq 2$.

$$T(n) \leq \sum_{i=0}^{k-1} \left\lfloor \frac{n}{2^i} \right\rfloor \leq \sum_{i=0}^{k-1} \frac{n}{2^i} = n \cdot \left(1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{k-1}}\right) \leq 2 \cdot n.$$

Binary Counter: Analysis via Potential Function

$$\Phi_i =$$

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$\Phi_i = \#$ ones in the binary representation of i

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Increment without Carry-Over

- actual cost: $c_i = 1$

1 1 0 0

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Increment without Carry-Over

- actual cost: $c_i = 1$

$$1\ 1\ 0\ \boxed{0} \xrightarrow{\text{INC}} 1\ 1\ 0\ 1$$

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Increment without Carry-Over

- actual cost: $c_i = 1$
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1 1 0 0

$\big\downarrow$ INC

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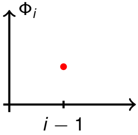

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$\Phi_0 = 0 \checkmark \quad \Phi_i \geq 0 \checkmark$

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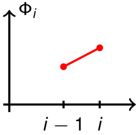

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- amortized cost: $\widehat{c_i} =$

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$\downarrow$ INC

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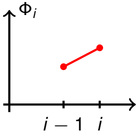

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1 1 0 0

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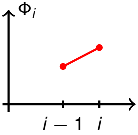

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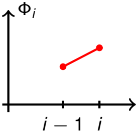
$$\begin{array}{c} 1\ 1\ 0\ \boxed{0} \\ \big\downarrow \text{INC} \\ 1\ 1\ 0\ 1 \end{array}$$



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1 1 0 0

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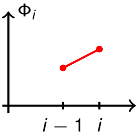

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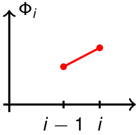
1 1 0 0

$\downarrow$ INC

1 1 0 1



0 1 1 1

Increment with Carry-Over

- $c_i = x + 1$, (x lowest index of a zero)

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1 1 0 0

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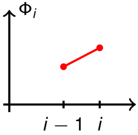

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1 1 0 0

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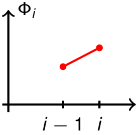
1 1 0 1



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0 1 1 1

$\downarrow$ INC

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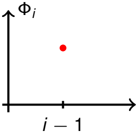

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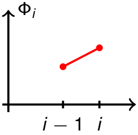
1 1 0 $\boxed{0}$

$\downarrow$ INC

1 1 0 1



Increment with Carry-Over

- $c_i = x + 1$, (x lowest index of a zero)
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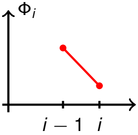
$\boxed{0\ 1\ 1\ 1}$

$\downarrow$ INC

1 0 0 0



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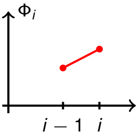
1 1 0 $\boxed{0}$

↓ INC

1 1 0 1



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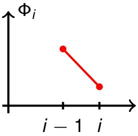
$\boxed{0\ 1\ 1\ 1}$

↓ INC

1 0 0 0



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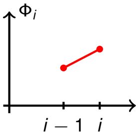
1 1 0 0

↓ INC

1 1 0 1



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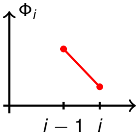
↓ INC

1 0 0 0



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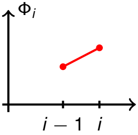
1 1 0 1



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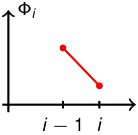
1 0 0 0



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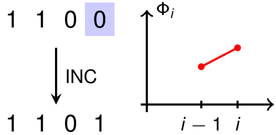

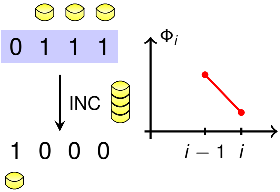

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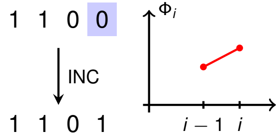

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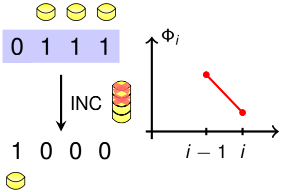

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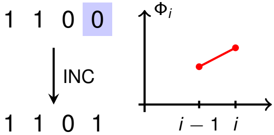

Amortized Cost $= 2 \Rightarrow T(n) \leq 2n$

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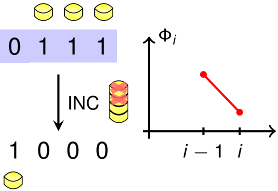

Summary

Amortized Analysis

- Average costs over a sequence of n operations
- overcharge cheap operations and undercharge expensive operations
- no probability/average case analysis involved!

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- Determine an absolute upper bound $T(n)$

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E.g. by bounding the number of expensive operations

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Aggregate Analysis

- Determine an absolute upper bound $T(n)$
- every operation has amortized cost $\frac{T(n)}{n}$

$T(n)$

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Potential Method

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- use savings from cheap operations to compensate for expensive ones

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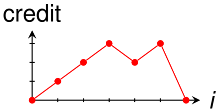

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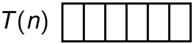
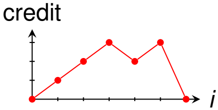
- every operation has amortized cost $\frac{T(n)}{n}$



Potential Method

- use savings from cheap operations to compensate for expensive ones
- operations may have different amortized cost



Summary

Amortized Analysis

- Average costs over a sequence of n operations
- overcharge cheap operations and undercharge expensive operations
- no probability/average case analysis involved!

Aggregate Analysis

- Determine an absolute upper bound $T(n)$
- every operation has amortized cost $\frac{T(n)}{n}$

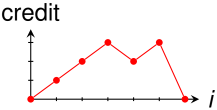
$T(n)$ 

Full power of this method will become clear later!

$T(n)$ 

Potential Method

- use savings from cheap operations to compensate for expensive ones
- operations may have different amortized cost



Next Lecture: Fibonacci Heap

Operation	Binomial heap worst-case cost
MAKE-HEAP	$\mathcal{O}(1)$
INSERT	$\mathcal{O}(\log n)$
MINIMUM	$\mathcal{O}(\log n)$
EXTRACT-MIN	$\mathcal{O}(\log n)$
UNION	$\mathcal{O}(\log n)$
DECREASE-KEY	$\mathcal{O}(\log n)$
DELETE	$\mathcal{O}(\log n)$

Next Lecture: Fibonacci Heap

Operation	Binomial heap worst-case cost	Fibonacci heap amortized cost
MAKE-HEAP	$\mathcal{O}(1)$	$\mathcal{O}(1)$
INSERT	$\mathcal{O}(\log n)$	$\mathcal{O}(1)$
MINIMUM	$\mathcal{O}(\log n)$	$\mathcal{O}(1)$
EXTRACT-MIN	$\mathcal{O}(\log n)$	$\mathcal{O}(\log n)$
UNION	$\mathcal{O}(\log n)$	$\mathcal{O}(1)$
DECREASE-KEY	$\mathcal{O}(\log n)$	$\mathcal{O}(1)$
DELETE	$\mathcal{O}(\log n)$	$\mathcal{O}(\log n)$

Crucial for many applications including shortest paths and minimum spanning trees!