small bug found in universal RM wde (p49) see web page for amechon

λ -Definable functions

Definition. $f \in \mathbb{N}^n \rightarrow \mathbb{N}$ is λ -definable if there is a closed λ -term F that represents it: for all $(x_1, \ldots, x_n) \in \mathbb{N}^n$ and $y \in \mathbb{N}$

- if $f(x_1, \ldots, x_n) = y$, then $F \underline{x_1} \cdots \underline{x_n} =_{\beta} y$
- if $f(x_1, \ldots, x_n)$, then $F \underline{x_1} \cdots \underline{x_n}$ has no β -nf.

This condition can make it quite tricky to find a λ -term representing a non-total function.

For now, we concentrate on total functions. First, let us see why the elements of **PRIM** (primitive recursive functions) are λ -definable.

If $f \in \mathbb{N}^n \to \mathbb{N}$ is represented by a λ -term F and $g \in \mathbb{N}^{n+2} \to \mathbb{N}$ is represented by a λ -term G, we want to show λ -definability of the unique $h \in \mathbb{N}^{n+1} \to \mathbb{N}$ satisfying

$$\begin{cases} h(\vec{a},0) = f(\vec{a}) \\ h(\vec{a},a+1) = g(\vec{a},a,h(\vec{a},a)) \end{cases}$$

or equivalently

$$h(\vec{a}, a) = if \ a = 0 \ then \ f(\vec{a})$$

else $g(\vec{a}, a - 1, h(\vec{a}, a - 1))$

If $f \in \mathbb{N}^n \to \mathbb{N}$ is represented by a λ -term F and $g \in \mathbb{N}^{n+2} \to \mathbb{N}$ is represented by a λ -term G, we want to show λ -definability of the unique $h \in \mathbb{N}^{n+1} \to \mathbb{N}$ satisfying $h = \Phi_{f,g}(h)$ where $\Phi_{f,g} \in (\mathbb{N}^{n+1} \to \mathbb{N}) \to (\mathbb{N}^{n+1} \to \mathbb{N})$ is given by $\Phi_{f,g}(h)(\vec{a},a) \triangleq if \ a = 0 \ then \ f(\vec{a})$ else $g(\vec{a}, a - 1, h(\vec{a}, a - 1))$

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 \blacktriangleright show that $\Phi_{f,g}$ is λ -definable;

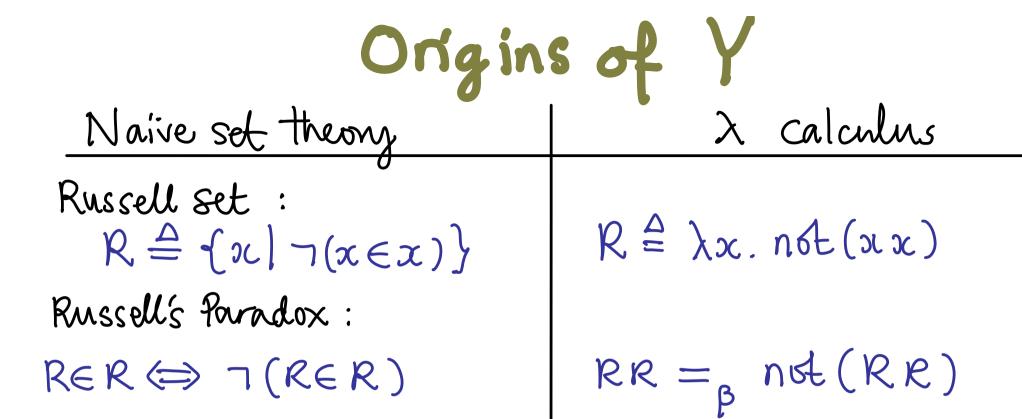
() ZZZX. If(Eqox)(FZ)(GZ(Predx)(ZZ(Predx)))

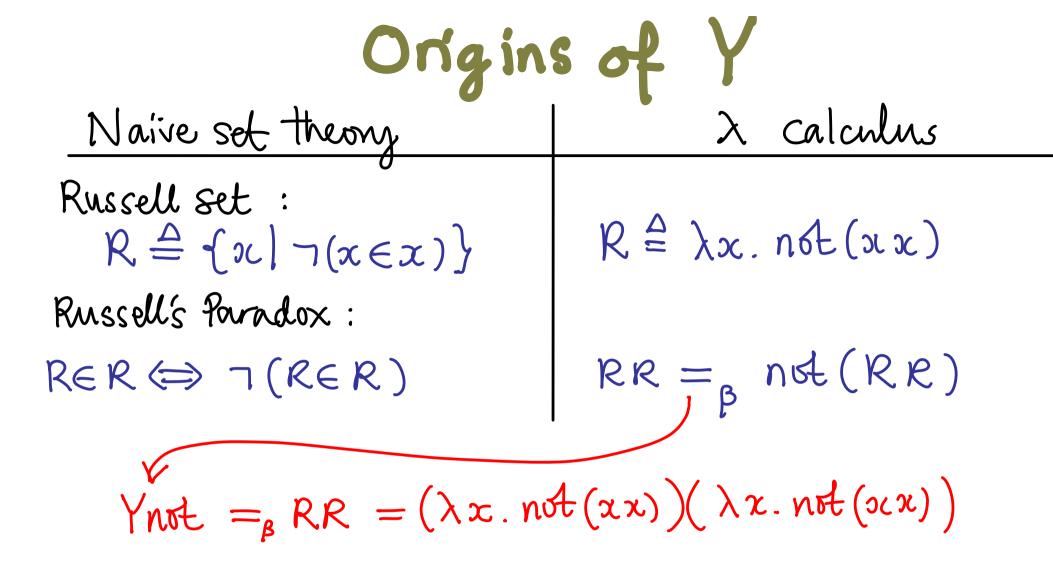
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• show that $\Phi_{f,g}$ is λ -definable;

• show that we can solve fixed point equations X = M X up to β -conversion in the λ -calculus.

Origins of YNaive set theory
$$\lambda$$
 calculusRussell set : $\mathcal{R} \triangleq \{x \in x\}$ $\mathcal{R} \triangleq \{x \in x\}$ $\mathcal{R} \triangleq \lambda x. not(x x)$ $\Lambda of \triangleq \lambda b. If b False True$





Origins of YNaive set theory
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Ynot =
$$_{\beta} RR = (\lambda x. not(xx))(\lambda x. not(xx))$$

Yf = $(\lambda x. f(xx))(\lambda x. f(xx))$
Y = $\lambda f. (\lambda x. f(xx))(\lambda x. f(xx))$

Curry's fixed point combinator **Y** $\mathbf{Y} \triangleq \lambda f. (\lambda x. f(xx))(\lambda x. f(xx))$

satisfies $\mathbf{Y}M \rightarrow (\lambda x.M(xx))(\lambda x.M(xx))$

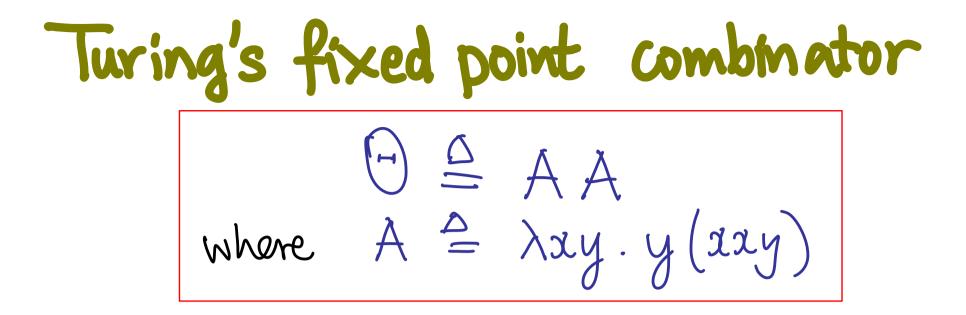
Curry's fixed point combinator **Y** $\mathbf{Y} \triangleq \lambda f. (\lambda x. f(x x)) (\lambda x. f(x x))$

satisfies $\mathbf{Y}M \rightarrow (\lambda x. M(x x))(\lambda x. M(x x))$ $\rightarrow M((\lambda x. M(x x))(\lambda x. M(x x)))$

hence $\mathbf{Y} M \rightarrow M((\lambda x. M(x x))(\lambda x. M(x x))) \leftarrow M(\mathbf{Y} M).$

So for all λ -terms M we have

 $\mathbf{Y}M =_{\beta} M(\mathbf{Y}M)$



Turing's fixed point combinator

$$\Theta \cong AA$$

where $A \cong \lambda x y \cdot y(x x y)$

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If $f \in \mathbb{N}^n \to \mathbb{N}$ is represented by a λ -term F and $g \in \mathbb{N}^{n+2} \to \mathbb{N}$ is represented by a λ -term G, we want to show λ -definability of the unique $h \in \mathbb{N}^{n+1} \to \mathbb{N}$ satisfying $h = \Phi_{f,g}(h)$ where $\Phi_{f,g} \in (\mathbb{N}^{n+1} \to \mathbb{N}) \to (\mathbb{N}^{n+1} \to \mathbb{N})$ is given by $\Phi_{f,g}(h)(\vec{a},a) \triangleq if \ a = 0 \ then \ f(\vec{a})$ else $g(\vec{a}, a - 1, h(\vec{a}, a - 1))$

We now know that h can be represented by

 $Y(\lambda z \vec{x} x. \operatorname{lf}(\operatorname{Eq}_0 x)(F \vec{x})(G \vec{x} (\operatorname{Pred} x)(z \vec{x} (\operatorname{Pred} x)))).$

Example

Factorial function fact $\in \mathbb{N} \to \mathbb{N}$ satisfies fact (n) = if n = other 1 else n.(fact(n-1))

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Factorial function fact $\in \mathbb{N} \to \mathbb{N}$ satisfies fact(n) = if n = other 1 else n.(fact(n-i))and is λ -definable — it's represented by $fact \stackrel{a}{=} Y(\lambda f x. If (Eq. x) 1 (Mult x (f (Pred x))))$ (where Mult $\triangleq \lambda x_1 x_2 f x$. $x_1(x_2 f) x$ represents multiplication).

Recall that the class **PRIM** of primitive recursive functions is the smallest collection of (total) functions containing the basic functions and closed under the operations of composition and primitive recursion.

Combining the results about λ -definability so far, we have: every $f \in PRIM$ is λ -definable.

So for λ -definability of all recursive functions, we just have to consider how to represent minimization. Recall...

Minimization

Given a partial function $f \in \mathbb{N}^{n+1} \to \mathbb{N}$, define $\mu^n f \in \mathbb{N}^n \to \mathbb{N}$ by $\mu^n f(\vec{x}) \triangleq$ least x such that $f(\vec{x}, x) = 0$ and for each $i = 0, \dots, x - 1$, $f(\vec{x}, i)$ is defined and > 0(undefined if there is no such x)

Can express $\mu^n f$ in terms of a fixed point equation: $\mu^n f(\vec{x}) \equiv g(\vec{x}, 0)$ where g satisfies $g = \Psi_f(g)$ with $\Psi_f \in (\mathbb{N}^{n+1} \to \mathbb{N}) \to (\mathbb{N}^{n+1} \to \mathbb{N})$ defined by

 $\Psi_f(g)(\vec{x},x) \equiv if f(\vec{x},x) = 0$ then x else $g(\vec{x},x+1)$

Representing minimization

Suppose $f \in \mathbb{N}^{n+1} \to \mathbb{N}$ (totally defined function) satisfies $\forall \vec{a} \exists a \ (f(\vec{a}, a) = 0)$, so that $\mu^n f \in \mathbb{N}^n \to \mathbb{N}$ is totally defined.

Thus for all $\vec{a} \in \mathbb{N}^n$, $\mu^n f(\vec{a}) = g(\vec{a}, 0)$ with $g = \Psi_f(g)$ and $\Psi_f(g)(\vec{a}, a)$ given by *if* $(f(\vec{a}, a) = 0)$ *then a else* $g(\vec{a}, a + 1)$. So if f is represented by a λ -term F, then $\mu^n f$ is represented by

 $\lambda \vec{x} \cdot \mathbf{Y}(\lambda z \, \vec{x} \, x \cdot \mathbf{If}(\mathbf{Eq}_0(F \, \vec{x} \, x)) \, x \, (z \, \vec{x} \, (\mathbf{Succ} \, x))) \, \vec{x} \, \underline{0}$

Recursive implies λ -definable

Fact: every partial recursive $f \in \mathbb{N}^n \rightarrow \mathbb{N}$ can be expressed in a standard form as $f = g \circ (\mu^n h)$ for some $g, h \in \text{PRIM}$. (Follows from the proof that computable = partial-recursive.)

Hence every (total) recursive function is λ -definable.

More generally, every partial recursive function is λ -definable, but matching up \uparrow with $\Xi\beta$ -nf makes the representations more complicated than for total functions: see [Hindley, J.R. & Seldin, J.P. (CUP, 2008), chapter 4.]

Computable = λ -definable

Theorem. A partial function is computable if and only if it is λ -definable.

We already know that computable = partial recursive $\Rightarrow \lambda$ -definable. So it just remains to see that λ -definable functions are RM computable. To show this one can

 code λ-terms as numbers (ensuring that operations for constructing and deconstructing terms are given by RM computable functions on codes)

• write a RM interpreter for (normal order) β -reduction.

The details are straightforward, if tedious.

Numerical coding of
$$\lambda$$
-terms
Fix an emuration $x_0, x_1, x_2, ...$ of the set of variables.
For each λ -term M, define $\lceil M \rceil \in \mathbb{N}$ by

$$\lceil x_i^2 \rceil = \lceil [0, \hat{z}]^7$$

$$\lceil \lambda x_i . M^2 \rceil = \lceil [1, \hat{z}, \lceil M^2 \rceil]^2$$

$$\lceil M N^2 \rceil = \lceil [2, \lceil M^2, \lceil N^2 \rceil]^2$$
(where $\lceil n_0, n_1, ..., n_k \rceil^2$ is the numerical using of lists
of numbers from $p \neq 3$).

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