

Recall: λ -Terms, M

are built up from a given, countable collection of

- ▶ variables x, y, z, \dots

by two operations for forming λ -terms:

- ▶ λ -abstraction: $(\lambda x.M)$
(where x is a variable and M is a λ -term)
- ▶ application: $(M M')$
(where M and M' are λ -terms).

Some random examples of λ -terms:

$x \quad (\lambda x.x) \quad ((\lambda y.(x y))x) \quad (\lambda y.((\lambda y.(x y))x))$

Substitution $N[M/x]$

$$\begin{aligned}x[M/x] &= M \\y[M/x] &= y \quad \text{if } y \neq x \\(\lambda y.N)[M/x] &= \lambda y.N[M/x] \quad \text{if } y \# (M x) \\(N_1 N_2)[M/x] &= N_1[M/x] N_2[M/x]\end{aligned}$$

Side-condition $y \# (M x)$ (y does not occur in M and $y \neq x$) makes substitution “capture-avoiding”.

E.g. if $x \neq y$

$$(\lambda y.x)[y/x] \neq \lambda y.y$$

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$N[M/x]$ = result of replacing all free occurrences of x in N with M , avoiding "capture" of free variables in M by λ -binders in N

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Can always satisfy this up to α -equivalence

E.g. if $x \neq y \neq z \neq x$

$$(\lambda y.x)[y/x] =_{\alpha} (\lambda z.x)[y/x] = \lambda z.y$$

In fact $N \mapsto N[M/x]$ induces a totally defined function from the set of α -equivalence classes of λ -terms to itself.

=

$$\lambda x. (\lambda z. z) y x \left[\lambda z. y / y \right]$$

=

$$\lambda x. (\lambda z. z) y x \quad [\quad \lambda z. y / y \quad]$$

no possible
capture

$$\begin{aligned}
 & \lambda x. (\lambda z. z) y x \left[\lambda z. y / y \right] \\
 = & \lambda x. (\lambda z. z) (\lambda z. y) x
 \end{aligned}$$

$$\begin{aligned}
 & \lambda x. (\lambda u. u) x y \left[\lambda y. x / y \right] \\
 = &
 \end{aligned}$$

$$\begin{aligned}
 & \lambda x. (\lambda z. z) y x \ [\ \lambda x. y / y \] \\
 = & \lambda x. (\lambda z. z) (\lambda x. y) x
 \end{aligned}$$

$$\begin{aligned}
 & \lambda x. (\lambda u. u) x y \ [\ \lambda y. x / y \] \quad \text{possible capture} \\
 = &
 \end{aligned}$$

$$\lambda x. (\lambda z. z) y x \ [\ \lambda x. y / y \]$$

$$= \lambda x. (\lambda z. z) (\lambda x. y) x$$

$$\lambda x. (\lambda u. u) x y \ [\ \lambda y. x / y \] \quad \text{possible capture...}$$

$$=_{\alpha} \lambda z. (\lambda u. u) z y \ [\ \lambda y. x / y \] \quad \text{...}\alpha\text{-convert to avoid}$$

$$\lambda x. (\lambda z. z) y x \left[\lambda x. y / y \right]$$

$$= \lambda x. (\lambda z. z) (\lambda x. y) x$$

$$\lambda x. (\lambda u. u) x y \left[\lambda y. x / y \right] \quad \text{possible capture...}$$

$$\equiv_{\alpha} \lambda z. (\lambda u. u) z y \left[\lambda y. x / y \right] \quad \text{...}\alpha\text{-convert to avoid}$$

$$= \lambda z. (\lambda u. u) z (\lambda y. x)$$

β -Reduction

Recall that $\lambda x.M$ is intended to represent the function f such that $f(x) = M$ for all x . We can regard $\lambda x.M$ as a function on λ -terms via substitution: map each N to $M[N/x]$.

So the natural notion of computation for λ -terms is given by stepping from a

β -redex $(\lambda x.M)N$

to the corresponding

β -reduct $M[N/x]$

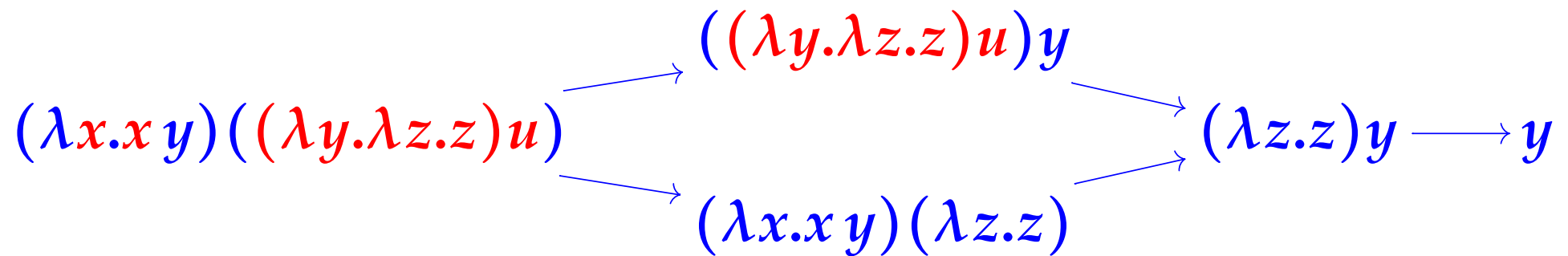
β -Reduction

One-step β -reduction, $M \rightarrow M'$:

$$\begin{array}{c} \frac{}{(\lambda x.M)N \rightarrow M[N/x]} \qquad \frac{M \rightarrow M'}{\lambda x.M \rightarrow \lambda x.M'} \\[2ex] \frac{M \rightarrow M'}{MN \rightarrow M'N} \qquad \frac{M \rightarrow M'}{NM \rightarrow NM'} \\[2ex] \frac{N =_{\alpha} M \quad M \rightarrow M' \quad M' =_{\alpha} N'}{N \rightarrow N'} \end{array}$$

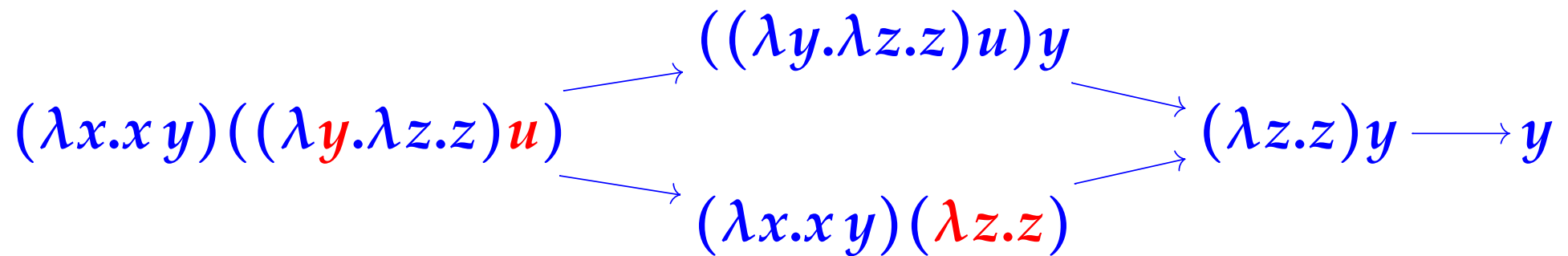
β -Reduction

E.g.



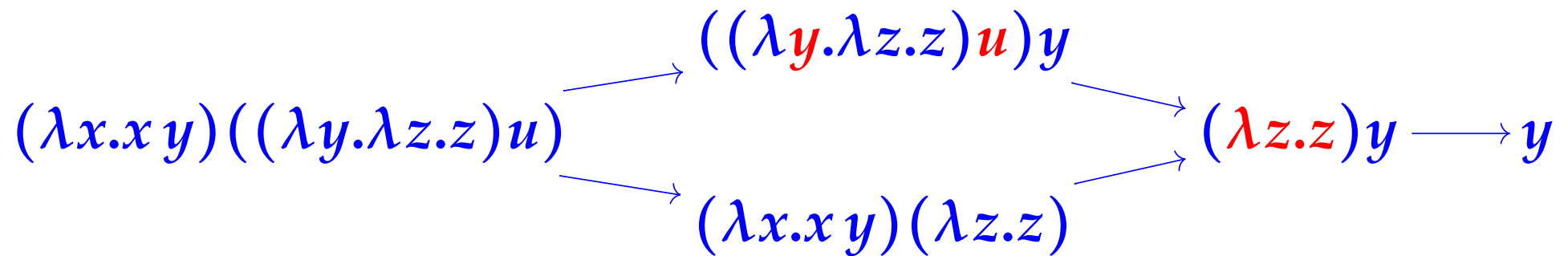
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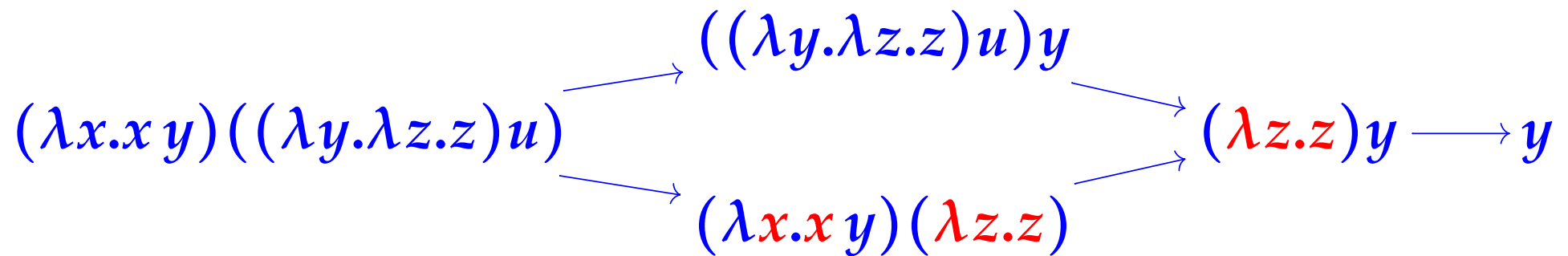
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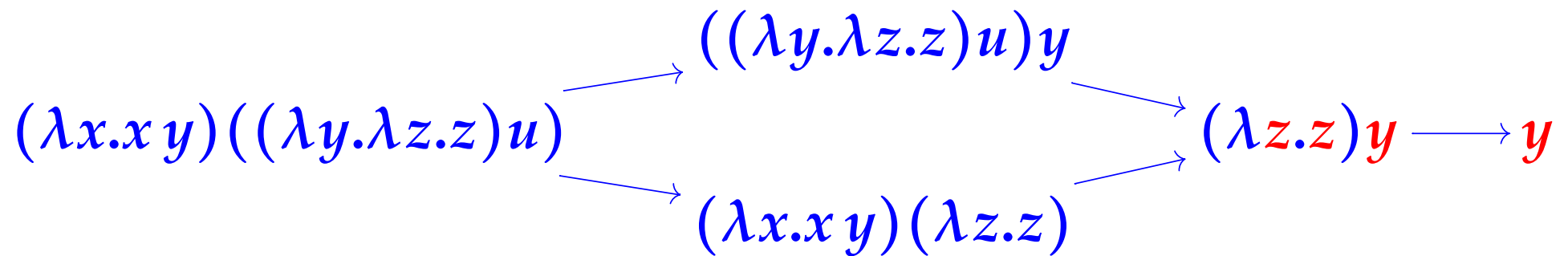
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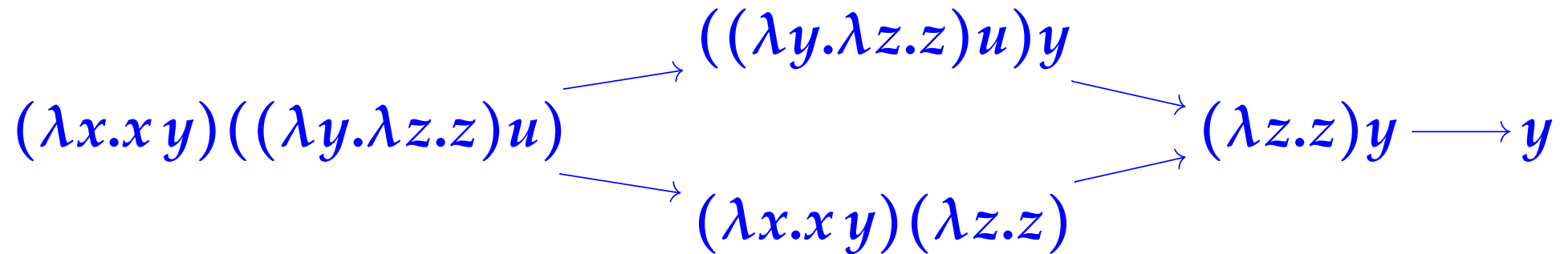
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E.g. of “up to α -equivalence” aspect of reduction:

$$(\lambda x. \lambda y. x)y =_{\alpha} (\lambda x. \lambda z. x)y \rightarrow \lambda z. y$$

Many-step β -reduction, $M \rightarrow\!\!\rightarrow M'$:

$$\frac{M =_{\alpha} M'}{M \rightarrow\!\!\rightarrow M'}$$

(no steps)

$$\frac{M \rightarrow M'}{M \rightarrow\!\!\rightarrow M'}$$

(1 step)

$$\frac{M \rightarrow\!\!\rightarrow M' \quad M' \rightarrow M''}{M \rightarrow\!\!\rightarrow M''}$$

(1 more step)

E.g.

$$(\lambda x.x y)((\lambda y z.z)u) \rightarrow\!\!\rightarrow y$$

$$(\lambda x.\lambda y.x)y \rightarrow\!\!\rightarrow \lambda z.y$$

β -Conversion $M =_{\beta} N$

Informally: $M =_{\beta} N$ holds if N can be obtained from M by performing zero or more steps of α -equivalence, β -reduction, or β -expansion (= inverse of a reduction).

E.g. $u((\lambda x y. v x) y) =_{\beta} (\lambda x. u x)(\lambda x. v y)$

because $(\lambda x. u x)(\lambda x. v y) \rightarrow u(\lambda x. v y)$

and so we have

$$\begin{aligned} u((\lambda x y. v x) y) &=_{\alpha} u((\lambda x y'. v x) y) \\ &\rightarrow u(\lambda y'. v y) && \text{reduction} \\ &=_{\alpha} u(\lambda x. v y) \\ &\leftarrow (\lambda x. u x)(\lambda x. v y) && \text{expansion} \end{aligned}$$

β -Conversion $M =_{\beta} N$

is the binary relation inductively generated by the rules:

$$\frac{M =_{\alpha} M'}{M =_{\beta} M'}$$

$$\frac{M \rightarrow M'}{M =_{\beta} M'}$$

$$\frac{M =_{\beta} M'}{M' =_{\beta} M}$$

$$\frac{M =_{\beta} M' \quad M' =_{\beta} M''}{M =_{\beta} M''}$$

$$\frac{M =_{\beta} M'}{\lambda x.M =_{\beta} \lambda x.M'}$$

$$\frac{M =_{\beta} M' \quad N =_{\beta} N'}{M N =_{\beta} M' N'}$$

Church-Rosser Theorem

Theorem. \rightarrow is **confluent**, that is, if $M_1 \leftarrow M \rightarrow M_2$, then there exists M' such that $M_1 \rightarrow M' \leftarrow M_2$.

[Proof omitted.]

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Corollary. To show that two terms are β -convertible, it suffices to show that they both reduce to the same term. More precisely: $M_1 =_{\beta} M_2$ iff $\exists M (M_1 \rightarrow M \leftarrow M_2)$.

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Corollary. $M_1 =_\beta M_2$ iff $\exists M (M_1 \rightarrow M \leftarrow M_2)$.

Proof. $=_\beta$ satisfies the rules generating \rightarrow ; so $M \rightarrow M'$ implies $M =_\beta M'$. Thus if $M_1 \rightarrow M \leftarrow M_2$, then $M_1 =_\beta M =_\beta M_2$ and so $M_1 =_\beta M_2$.

Conversely, the relation $\{(M_1, M_2) \mid \exists M (M_1 \rightarrow M \leftarrow M_2)\}$ satisfies the rules generating $=_\beta$: the only difficult case is closure of the relation under transitivity and for this we use the Church-Rosser theorem: $M_1 \rightarrow M \leftarrow M_2 \rightarrow M' \leftarrow M_3$

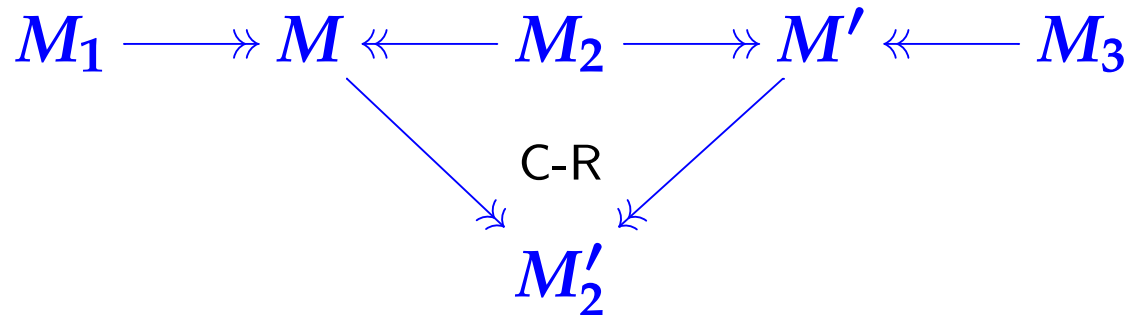
Church-Rosser Theorem

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β -Normal Forms

Definition. A λ -term N is in β -normal form (nf) if it contains no β -redexes (no sub-terms of the form $(\lambda x.M)M'$). M has β -nf N if $M =_{\beta} N$ with N a β -nf.

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Note that if N is a β -nf and $N \rightarrow N'$, then it must be that $N =_{\alpha} N'$ (why?).

Hence if $N_1 =_{\beta} N_2$ with N_1 and N_2 both β -nfs, then $N_1 =_{\alpha} N_2$. (For if $N_1 =_{\beta} N_2$, then by Church-Rosser $N_1 \rightarrow M' \leftarrow N_2$ for some M' , so $N_1 =_{\alpha} M' =_{\alpha} N_2$.)

So the β -nf of M is unique up to α -equivalence if it exists.

Non-termination

Some λ terms have no β -nf.

E.g. $\Omega \triangleq (\lambda x. x x)(\lambda x. x x)$ satisfies

- ▶ $\Omega \rightarrow (x x)[(\lambda x. x x)/x] = \Omega,$
- ▶ $\Omega \rightarrow M$ implies $\Omega =_{\alpha} M.$

So there is no β -nf N such that $\Omega =_{\beta} N.$

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- ▶ $\Omega \rightarrow M$ implies $\Omega =_{\alpha} M$.

So there is no β -nf N such that $\Omega =_{\beta} N$.

A term can possess both a β -nf and infinite chains of reduction from it.

E.g. $(\lambda x.y)\Omega \rightarrow y$, but also $(\lambda x.y)\Omega \rightarrow (\lambda x.y)\Omega \rightarrow \dots$.

Non-termination

Normal-order reduction is a deterministic strategy for reducing λ -terms: reduce the “left-most, outer-most” redex first.

- ▶ left-most: reduce M before N in $M N$, and then
- ▶ outer-most: reduce $(\lambda x.M)N$ rather than either of M or N .

(cf. call-by-name evaluation).

Fact: normal-order reduction of M always reaches the β -nf of M if it possesses one.

$$\frac{M_1 =_{\alpha} M'_1 \quad M'_1 \rightarrow_{\text{NOR}} M'_2 \quad M'_2 =_{\alpha} M_2}{M_1 \rightarrow_{\text{NOR}} M_2}$$

$$\frac{M \rightarrow_{\text{NOR}} M'}{\lambda x. M \rightarrow_{\text{NOR}} \lambda x. M'}$$

$$\frac{M_1 \rightarrow_{\text{NOR}} M'_1}{M_1 M_2 \rightarrow_{\text{NOR}} M'_1 M_2}$$

$$\frac{}{(\lambda x. M) M' \rightarrow_{\text{NOR}} M[M'/x]}$$

$$\frac{M \rightarrow_{\text{NOR}} M'}{UM \rightarrow_{\text{NOR}} UM'}$$

where $\begin{cases} U ::= x \mid UN \\ N ::= \lambda x. N \mid U \end{cases}$

β -normal forms

"neutral" forms