Lemma *Given an NFA M, for each subset $Q \subseteq States_M$ and each pair of states $q, q' \in States_M$, there is a regular expression $r^Q_{q,q'}$ satisfying*

$$L(r^Q_{q,q'}) = \{u \in (\Sigma_M)^* \mid q \xrightarrow{u}{}^{*} q' \text{ in } M \text{ with all intermediate states of the sequence in } Q\}.$$

Hence $L(M) = L(r)$, where $r = r_1 | \cdots | r_k$ and

$k =$ number of accepting states,

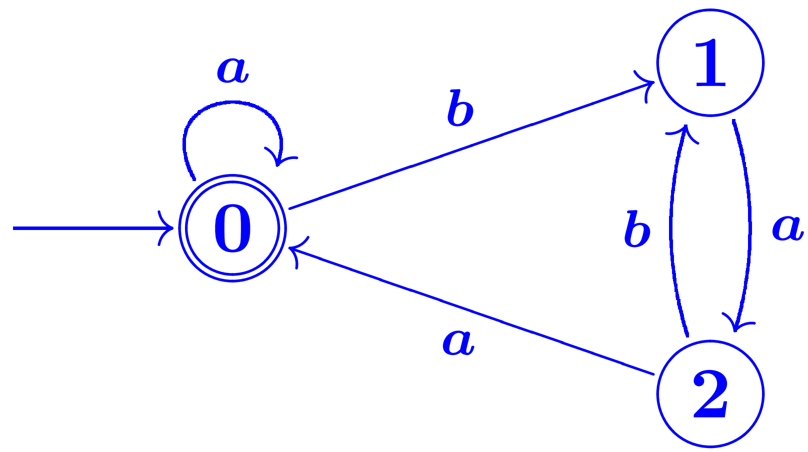
$r_i = r^Q_{s,q_i}$ with $Q = States_M$,

$s =$ start state,

$q_i = i$th accepting state.

(In case $k = 0$, take r to be the regular expression $\emptyset$.)

Example



Direct inspection yields:

$r_{i,j}^{\{0\}}$	0	1	2
0			
1	$\emptyset$	ε	a
2	aa^*	a^*b	ε

$r_{i,j}^{\{0,2\}}$	0	1	2
0	a^*	a^*b	
1			
2			

$Not(M)$

- $States_{Not(M)} \stackrel{\text{def}}{=} States_M$
- $\Sigma_{Not(M)} \stackrel{\text{def}}{=} \Sigma_M$
- transitions of $Not(M)$ = transitions of M
- start state of $Not(M)$ = start state of M
- $Accept_{Not(M)} = \{q \in States_M \mid q \notin Accept_M\}$.

Provided M is a *deterministic* finite automaton, then u is accepted by $Not(M)$ iff it is not accepted by M:

$$L(Not(M)) = \{u \in \Sigma^* \mid u \notin L(M)\}.$$

$And(M_1, M_2)$

- states of $And(M_1, M_2)$ are all ordered pairs (q_1, q_2) with $q_1 \in States_{M_1}$ and $q_2 \in States_{M_2}$
- alphabet of $And(M_1, M_2)$ is the common alphabet of M_1 and M_2
- $(q_1, q_2) \xrightarrow{a} (q_1', q_2')$ in $And(M_1, M_2)$ iff $q_1 \xrightarrow{a} q_1'$ in M_1 and $q_2 \xrightarrow{a} q_2'$ in M_2
- start state of $And(M_1, M_2)$ is (s_{M_1}, s_{M_2})
- (q_1, q_2) accepting in $And(M_1, M_2)$ iff q_1 accepting in M_1 and q_2 accepting in M_2.