A set $S \subseteq U \times U$ is $\underline{TC}(R)$-closed iff
(1) $\forall (a,b) \in R.\ (a,b) \in S$ equivalently $R \subseteq S$
(2) $\forall (a,b),(b,c) \in S.\ (a,c) \in S$ equivalently S is transitive.

Transitive Closure

The set of rule instances $\mathbf{TC}(R)$ for the transitive closure of a relation $R \subseteq U \times U$ is given by

$$\frac{}{(a,b)}\ (a,b) \in R \qquad\qquad \frac{(a,b) \quad (b,c)}{(a,c)}$$

Claim: $I_{\mathbf{TC}(R)} = R^+$ $\qquad R^+ =_{\text{def}} \bigcup_{n \in \mathbb{N}} R^n$

the least $\underline{T}C(R)$-closed set.

Claim $I_{TC(R)} = R^+$.

We show (i) $I_{TC(R)} \subseteq R^+$ and (ii) $R^+ \subseteq I_{TC(R)}$.

(i) $I_{TC(R)} \subseteq R^+ =^{def} \bigcup_{n \in \mathbb{N}} R^n$

METHOD 1 Since $I_{TC(R)}$ is the least $TC(R)$-closed set.

it is enough to show that R^+ is $TC(R)$-closed

Equivalently that (1) $R \subseteq R^+$ and (2) R^+ is transitive

For (1): $R \subseteq R \cup R^2 \cup \dots \cup R^n \cup \dots = R^+$ and we are done.

For (2):

We need show:

$$(x,y) \in R^+,\ (y,z) \in R^+ \overset{?}{\Longrightarrow} (x,z) \in R^+$$

Relates two elements whenever there is a path of length n between them.

$$\overbrace{R^n}$$

Suppose $(x,y), (y,z) \in R^+ = \bigcup_{n \in \mathbb{N}} R^n$

Then $(x,y) \in R^k$ for some $k \in \mathbb{N}$

and $(y,z) \in R^l$ for some $l \in \mathbb{N}$

Hence $(x,z) \in R^{k+l} \subseteq R^+$

So $(x,z) \in R^+$

$$R^{n+1} = R^n \circ R$$

$$R^2 = R \circ R$$

$$= \left\{ (a,c) \mid \exists b. \begin{array}{l} aRb \\ bRc \end{array} \right\}$$

$$R^3 = R^2 \circ R$$

$$= \left\{ (a,d) \mid \exists c \begin{array}{l} aR^2c \\ cRd \end{array} \right\}$$

$$= \left\{ (a,d) \mid \exists c \exists b \begin{array}{l} aRb \\ bRc \\ cRd \end{array} \right\}$$

METHOD 2 : $I_{\underline{TC}(R)} \subseteq R^+$

iff $\forall (a,b) \in I_{\underline{TC}(R)}.\ \underbrace{(a,b) \in R^+}_{P(a,b)}$

Show $P(a,b)$ for all $(a,b) \in I_{\underline{TC}(R)}$ by rule induction:

(i) $\forall (a,b) \in R.\ P(a,b)$

(ii) $\forall (a,b),(b,c) \in I_{\underline{TC}(R)}.\ P(a,b)\ \&\ P(b,c) \Rightarrow P(a,c)$

This is shown as before.

Show: $R^+ \subseteq I_{TC(R)}$.

equivalently $\forall (a,b) \in R^+ .\ (a,b) \in I_{TC(R)}$

$R^+ = \bigcup_{n \in \mathbb{N}} R^n$

Lemma: $\bigcup_{i \in I} A_i \subseteq B$ iff $\forall i \in I.\ A_i \subseteq B$

equivalently $\bigcup_{n \in \mathbb{N}} R^n \subseteq I_{TC(R)}$

iff $\forall n \in \mathbb{N}.\ R^n \subseteq I_{TC(R)}$ $(*)$

We show $(*)$ by induction on $n \in \mathbb{N}$.

Base case: $R^1 \subseteq I_{\underline{TC}(R)}$.

$R^1 = R$ and the axiom rule instances

$$\frac{}{(a,b)}\ (a,b) \in R$$

imply $R \subseteq I_{\underline{TC}(R)}$.

Inductive step: Assuming $R^n \subseteq I_{\underline{TC}(R)}$ for $n \in \mathbb{N}$

We need show $R^{n+1} \subseteq I_{\underline{TC}(R)}$.

Induction hypothesis: $R^n \subseteq I_{\underline{TC}(R)}$.

$$R^{n+1} = R^n \circ R$$

$$\subseteq I_{\underline{TC}(R)} \circ R$$

by inductive hyp. & Lemma

$$\subseteq I_{\underline{TC}(R)} \circ I_{\underline{TC}(R)}$$

because $R \subseteq I_{\underline{TC}(R)}$ & Lemma

$$\subseteq I_{\underline{TC}(R)}$$

by $I_{\underline{TC}(R)}$ being $\underline{TC}(R)$-closed and Lemma'.

Lemma If $S_1 \subseteq T_1$ $S_2 \subseteq T_2$ Then $S_1 \circ S_2 \subseteq T_1 \circ T_2$

Lemma' A relation S is transitive iff $S \circ S \subseteq S$

$\boxtimes$

Another proof for $R^{n+1} \subseteq I_{\underline{TC}(R)}$

$\forall (a,b) \in R^{n+1}.\ (a,b) \in I_{\underline{TC}(R)}$.

Let $(a,b) \in R^{n+1}$. So $(a,c) \in R^n$ and $(c,b) \in R$

for some c. So by induction hypothesis $(a,c) \in I_{\underline{TC}(R)}$

and by the axioms we also have $(c,b) \in I_{\underline{TC}(R)}$.

Then by the rule instance

$$\frac{(a,c) \quad (c,b)}{(a,b)}$$

we have $(a,b) \in I_{\underline{TC}(R)}$

$\boxtimes$

Exercise:

1. Write the closure property of the inductively defined set $I_{\mathbf{TC}(R)}$.

2. Write the principle of induction for the inductively defined set $I_{\mathbf{TC}(R)}$.

3. Use the above to show that

   $\forall z \in U.\, (y, z) \in I_{\mathbf{TC}(R)} \Rightarrow (x, z) \in I_{\mathbf{TC}(R)}$ for all $(x, y) \in I_{\mathbf{TC}(R)}$

   That is, that $I_{\mathbf{TC}(R)}$ is transitive.

Derivation Trees

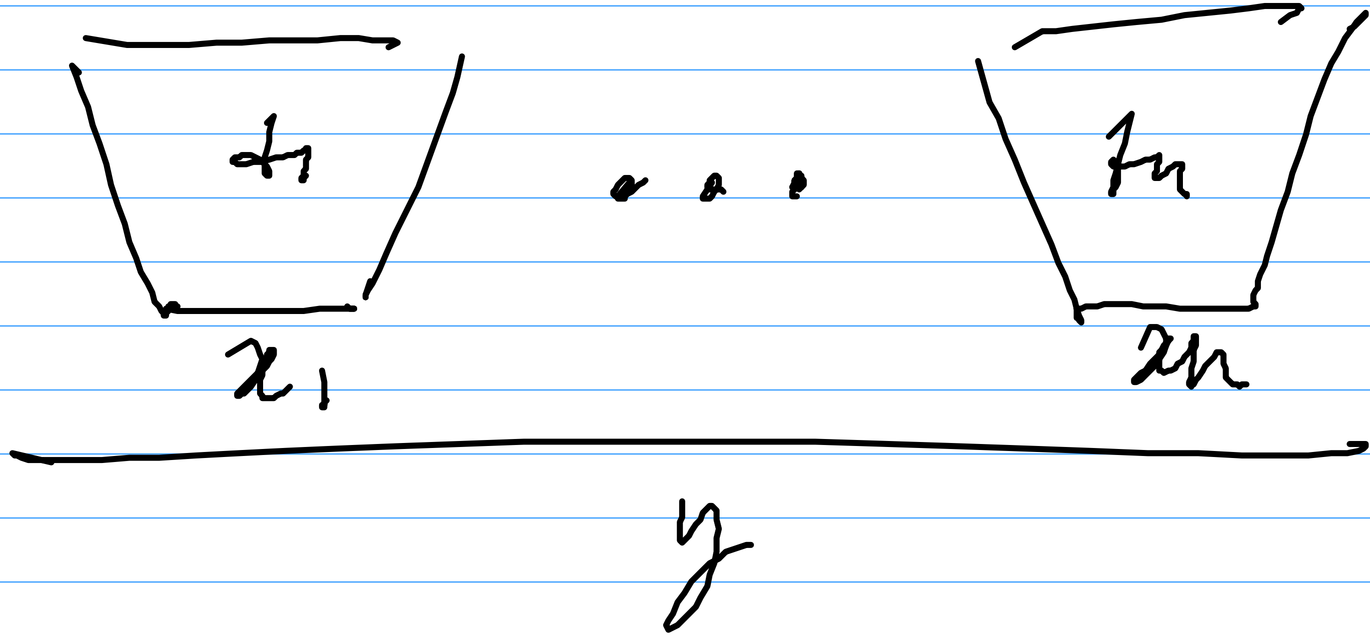
Example:

$$\cfrac{\cfrac{}{(a,c)} \qquad (a,c)\in R \qquad \cfrac{}{(c,b)}}{(a,b)}$$

Formally: (1) For all axioms $(\emptyset, y)$ we have $\dfrac{}{y}$ a derivation tree

(2) For all rules $(\{x_1, \ldots, x_n\}/y)$, given derivation trees $\dfrac{t_1}{x_1} \; \ldots \; \dfrac{t_n}{x_n}$ we have

The induced derivation tree



Induction on Derivations

Let P be a property of derivations.

If

P holds for all axioms $(\emptyset/y)$

and

for each rule instance $(\{x_1, \ldots, x_n\}/y)$,

for all derivations d_i of x_i for $1 \leq i \leq n$, P holding

for $d_1, \ldots, d_n$ implies that P holds for $(\{d_1, \ldots, d_n\}/y)$

then

P holds for all derivations

Fundamental Property:

An element is in I_R iff there is an R-derivation of it.