The powerset construction increases cardinality

There cannot be a bijection

$$D \cong \mathcal{P}(D)$$

In fact, there cannot be a surjection

$$D \to \mathcal{P}(D)$$

Unlimited Size

Diagonalisation Theorem: For sets N and X, if there exists a surjective function $\mathsf{N} \to (\mathsf{N} \Rightarrow \mathsf{X})$ then every function in $(\mathsf{X} \Rightarrow \mathsf{X})$ has a fixed point. Hence, $\mathsf{X} \cong [1]$.

a fixed point of a function say $f : X \to X$ is an element $x \in X$ such that $f(x) = x$.

? Which sets X are such that every function on them have a fixed point?

! $X \neq \emptyset$, X could be a singleton, $|X| \neq 2$,
Suppose $|X| \geq 2$ let $x_1 \neq x_2$ be elements of X
Consider the function $f : X \to X$ given by $f(x) = \begin{cases} x_1 & \text{if } x = x_2 \\ x_2 & \text{otw} \end{cases}$
It has no fixed point!

Let $e : N \to (N \Rightarrow X)$ be a surjection.

Ie for every function $h : N \to X$ there exists $k \in N$ such that $e(k) = h$

Let f be a function $X \to X$. Define $g : N \to X$ given by $g(n) \stackrel{\text{def}}{=} f(e\,n\,n)$. Then there exists $k \in N$ such that

$$e(k) = g$$

In particular

$$e(k)\,k = g(k) = f(e\,k\,k)$$

And so $e\,k\,k$ is a fixed point of f. $\boxtimes$

Two corollaries: Let D be a set.

- There is no surjection $D \to \mathcal{P}(D)$.
- There exists a surjection $D \to (D \Rightarrow D)$ iff $D \cong [1]$.

($\Leftarrow$) $([1] \Rightarrow [1]) \cong [1^1] = [1]$

($\Rightarrow$) A surjection $D \to (D \Rightarrow D)$ forces D to have fixed points for all functions $D \to D$ and hence to be a singleton.

$\mathcal{P}(D) \cong (D \Rightarrow [2])$

A surjection $D \to \mathcal{P}(D)$ would give a surjection $D \to (D \Rightarrow [2])$. But this is impossible because not every function on $[2]$ has a fixed point.

NB: In ML, we have the non-trivial datatype

$$D = \text{fun of } (D \to D)$$

!

Chapter 5

Reading list:

5.1 Sets defined by rules—examples

5.2 Inductively defined sets

5.3 Rule induction

5.4 Derivation trees

Suggested exercises: 5.1, 5.5, 5.6, 5.7, 5.8, 5.9, 5.10, 5.11, 5.12

Inductive Definitions

— idea —

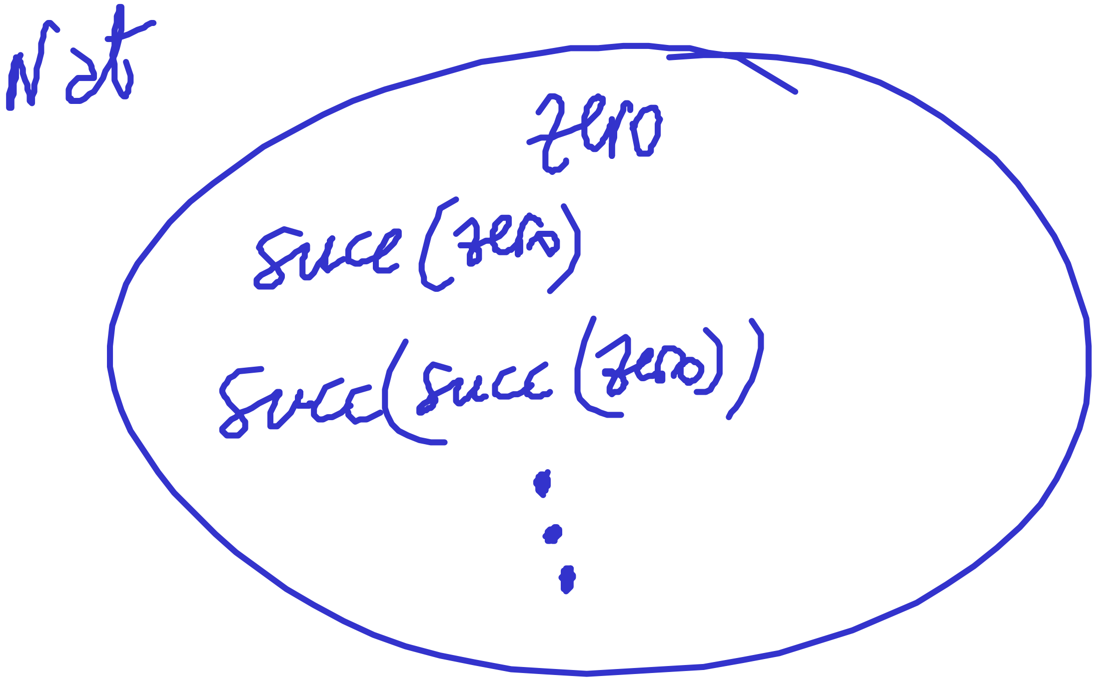
Sets described by stipulating that:

1. certain elements are in the set;
2. new elements are specified as generated from old ones;
3. no other elements are in the set.

Examples:

- Natural numbers in ML

```
datatype
  Nat = zero | succ of Nat
```



String

empty

prefix('a', empty)

prefix('b', prefix('a', empty))

⋮

Examples:

- Natural numbers in ML

```
datatype
  Nat = zero | succ of Nat
```

- Strings in ML

```
datatype
  String = empty | prefix of char * String
```

$a \quad b \quad c \; \ldots$

$\mathtt{T} \quad \mathtt{F}$

$a \wedge \mathtt{T} \qquad \mathtt{T} \vee \mathtt{F}$

$\neg(\mathtt{T} \vee \mathtt{F}) \qquad \ldots$

- Grammar of Boolean propositions

$$
\begin{array}{rlll}
A, B, \ldots & \text{(Boolean propositions)} & & \\
::= & a, b, \ldots & & \text{(propositional variables)} \\
\mid & \mathtt{T} \mid \mathtt{F} & & \text{(propositional constants)} \\
\mid & A \wedge B & & \text{(conjunction)} \\
\mid & A \vee B & & \text{(disjunction)} \\
\mid & \neg A & & \text{(negation)}
\end{array}
$$

$$\frac{\text{Premises}}{\text{Conclusion}}$$

could be empty in which case the rule is called an AXIOM

Rules

— an ubiquitous formalism —

Examples:

- The rules **Nat** for natural numbers

$$\frac{}{\mathtt{zero}} \qquad \frac{n}{\mathtt{succ}(n)}$$

- The rules **String** for strings

$$\frac{}{\varepsilon} \qquad \frac{s}{a.s}\ a \text{ a character}$$

- The rules **BoolProp** for Boolean propositions

$$\frac{}{a}\ a \text{ a propositional variable}$$

$$\frac{}{\mathtt{T}} \qquad \frac{}{\mathtt{F}}$$

another notation

$$\frac{A \quad B}{A \wedge B} \qquad \frac{A \quad B}{A \vee B} \qquad \frac{A}{\neg A}$$

$(\{A, B\} / A \wedge B)$

Rule Instances

- *Rule instances* are pairs

$$(X/y)$$

where X is a set of *premises* and y is a *conclusion*.

Rule Instances

Closure

- *Rule instances* are pairs

$$(X/y)$$

where X is a set of *premises* and y is a *conclusion*.

- A set of rule instances R specifies a way to build a set:

> Each rule instance (X/y) in R, stipulates that
>
> if all the elements of X are in the set then so is y.

- There is a least set with the above property! We denote it I_R and called it the set inductively defined by the rule instances R.

Closed Sets

For a set of rule instances R, a set Q is said to be R-closed whenever

for all (X/y) in R,

if $X \subseteq Q$ then $y \in Q$

Consider

$$\{Q \mid Q \text{ is } R\text{-closed}\}$$

EXAMPLE

Wrt. $\mathbb{N}_0$ consider

$$\left.\begin{array}{ccc} \dfrac{\quad}{0} & \qquad & \dfrac{n}{n+4} \end{array}\right\} R$$

$I_R \subseteq \mathbb{N}_0$

The set inductively defined by the rules R

$\parallel$

multiples of 4

$S \subseteq \mathbb{N}_0$ is R-closed

iff

- $0 \in S$

&

- $\forall n \in S.\ n+4 \in S$

Examples: $\mathbb{N}_0$, Even, Mult_4, $\mathrm{Mult}_4 \cup \mathrm{Odd}$, ...

Mult_4: multiples of 4