NB: If $x_0 = x_1$ then $\{x_0, x_1\} = \{x_0\}$

Unordered pairs:

For every two objects x_0 and x_1, we can form the set

$$\{\, x_0, x_1 \,\}$$

Example:

$$\{\, 0 \,, \{\, 0, 1 \,\} \,\}$$

Unordered pairs:

For every two objects x_0 and x_1, we can form the set

$$\{ x_0, x_1 \}$$

Example:

$$\{ 0, \{ 0, 1 \} \}$$

Indexed sets:

For every collection of objects x_i for i ranging over a set I, we can form the set

$$\{ x_i \mid i \in I \}$$

Example

$R \subseteq U \times U$

$R^n \subseteq U \times U \; (n \in \mathbb{N})$

$\{ R^n \mid n \in \mathbb{N} \}$ is a set.

defining property: for all Z, $X \cup Y \subseteq Z$ iff $X \subseteq Z$ and $Y \subseteq Z$

Union:

For every two sets X and Y, we can form the set

$$X \cup Y =_{\text{def}} \{\, a \mid a \in X \text{ or } a \in Y \,\}$$

consisting of their union.

Union:

For every two sets X and Y, we can form the set

$$X \cup Y =_{\text{def}} \{ a \mid a \in X \text{ or } a \in Y \}$$

consisting of their union.

If I is a set and X_i $(i \in I)$ are sets. Then $\{ X_i \mid i \in I \}$ is a set and so is $\bigcup \{ X_i \mid i \in I \}$

Big union:

For every set of sets $\mathcal{S}$, we can form the set

$$\bigcup \mathcal{S} =_{\text{def}} \{ a \mid a \in X \text{ for some } X \in \mathcal{S} \}$$

Example: The notation $\bigcup_{i \in I} X_i$ stands for $\bigcup \{ X_i \mid i \in I \}$.

Example $R^+ = \bigcup_{n \in N} R^n$

~~defining~~ property for all z,
$z \subseteq X \cap Y$ iff $z \subseteq X$ and $z \subseteq Y$

Intersection:

For every two sets X and Y, we have the set

$$X \cap Y =_{\text{def}} \{ a \in X \mid a \in Y \}$$

Intersection:

For every two sets X and Y, we have the set

$$X \cap Y =_{\text{def}} \{\, a \in X \mid a \in Y \,\}$$

Big intersection:

For every non-empty set of sets $\mathcal{S}$, we have the set

$$\bigcap \mathcal{S} =_{\text{def}} \left\{\, a \in \bigcup \mathcal{S} \;\middle|\; \forall X \in \mathcal{S}.\, a \in X \,\right\}$$

Example: The notation $\bigcap_{i \in I} X_i$ stands for $\bigcap \{\, X_i \mid i \in I \,\}$.

play an important role in considering INDUCTIVE DEFINITIONS

Product:

For every two sets X and Y, we have the set

$$X \times Y =_{\text{def}} \{\, (a, b) \mid a \in X \text{ and } b \in Y \,\}$$

where $(a, b) =_{\text{def}} \{a, \{a, b\}\} \in \mathcal{P}(\mathcal{P}(X \cup Y))$.

This is a good definition because with it

$(a, b) = (x, y)$ iff $a = x$ and $b = y$

To prove this consider two cases (1) $a = b$ and (2) $a \neq b$.

Product:

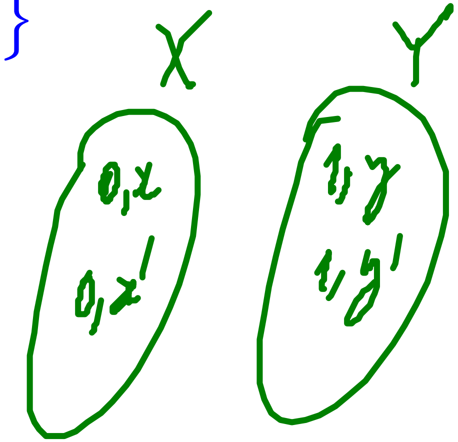
For every two sets X and Y, we have the set

$$X \times Y =_{\text{def}} \{\, (a,b) \mid a \in X \text{ and } b \in Y \,\}$$

where $(a,b) =_{\text{def}} \{a, \{a,b\}\} \in \mathcal{P}(\mathcal{P}(X \cup Y))$.

idea

$X + Y =$



Sums or (disjoint union):

For every two sets X and Y, we have the set $= \{(1,y) \mid y \in Y\}$

$$X + Y =_{\text{def}} (\{0\} \times X) \cup (\{1\} \times Y)$$

$\|$

$\{(0,x) \mid x \in X\}$

Example: $[m] + [n]$

Exercise $\rightsquigarrow$ $\cong$ $[m+n]$

Relations:

For every pair of sets X and Y, we have the set

$$\mathcal{P}(X \times Y)$$

of relations from X to Y.

Example: $\mathcal{P}([m] \times [n]) \cong [2^{m \cdot n}]$ (EXERCISE)

Relations:

For every pair of sets X and Y, we have the set

$$\mathcal{P}(X \times Y)$$

of relations from X to Y.

Example: $\mathcal{P}\big([m] \times [n]\big)$

Partial functions:

For every pair of sets X and Y, we have the set

$$(X \rightharpoonup Y) =_{\mathrm{def}} \{\, f \in \mathcal{P}(X \times Y) \mid f \text{ is a partial function} \,\}$$

of partial functions from X to Y.

Example: $([m] \rightharpoonup [n]) \cong \big([m] \Rightarrow [n] + [1]\big)$ (EXERCISE)

$\# \; (n+1)^m$

Functions:

For every pair of sets X and Y, we have the set

$$(X \Rightarrow Y) =_{\text{def}} \{ f \in \mathcal{P}(X \times Y) \mid f \text{ is a function} \}$$

of functions from X to Y.

Example: $([m] \Rightarrow [n]) \cong [n^m]$ (EXERCISE)

Functions:

For every pair of sets X and Y, we have the set

$$(X \Rightarrow Y) =_{\mathrm{def}} \{ f \in \mathcal{P}(X \times Y) \mid f \text{ is a function} \}$$

of functions from X to Y.

Example: $([m] \Rightarrow [n])$ Intuitively $\prod_{i \in \mathbb{N}} X_i$

Indexed product: $= X_1 \times X_2 \times \cdots \times X_n \times \cdots$

For every collection of sets X_i ranging over a set I, we have the indexed product set

$$\prod_{i \in I} X_i =_{\mathrm{def}} \{ f \in (I \Rightarrow \bigcup_{i \in I} X_i) \mid \forall i \in I.\, f(i) \in X_i \}$$

Example for $I = [2]$, $\prod_{i \in I} X_i \cong X_0 \times X_1$

Example $I = [2]$, $\sum_{i \in I} X_i \cong X_0 + X_1$

Infinitely $I = \mathbb{N}$, $\sum_{i \in I} X_i = X_1 + X_2 + \cdots + X_n + \cdots$

Indexed sum:

For every collection of sets X_i ranging over a set I, we have the indexed sum set

$$\sum_{i \in I} X_i =_{\text{def}} \bigcup_{i \in I} \left(\{i\} \times X_i \right)$$

Indexed sum:

For every collection of sets X_i ranging over a set I, we have the indexed sum set

$$\sum_{i \in I} X_i =_{\text{def}} \bigcup_{i \in I} \left(\{i\} \times X_i \right)$$

$X^0 =_{\text{def}} [1]$

Finite sequences (or strings):

$X^{n+1} =_{\text{def}} X \times X^n$

For a set X, we have the set of finite sequences

$$X^\star =_{\text{def}} \sum_{n \in \mathbb{N}_0} X^n$$

NB: If A is countable then so is $A^\star$.

$\mathbb{N}^* = \sum_{n\in\mathbb{N}_0} \mathbb{N}^n$ is countable

Let us define a surjection

$$\mathbb{N} \to \mathbb{N}^*$$

as follows

$$\mathbb{N} \xrightarrow{\cong} \mathbb{N}_0 \times \mathbb{N} \to \sum_{n\in\mathbb{N}_0} \mathbb{N}^n$$

$$\underbrace{(n, i) \xmapsto{def} (n, e_n(i))}_{\text{Bijective}}$$

$$\frac{A, B \text{ countable}}{A\times B \text{ countable}}$$

There is a bijection

$$\mathbb{N} \to \mathbb{N}\times\mathbb{N}$$

There is a bijection

$$\mathbb{N} \xrightarrow{e_n} \mathbb{N}^n$$

$\boxtimes$

Calculus of Bijections

$$[1] \times A \cong A \quad , \quad (A \times B) \times C \cong A \times (B \times C) \quad , \quad A \times B \cong B \times A$$

Calculus of Bijections

$$[1] \times A \cong A \quad , \quad (A \times B) \times C \cong A \times (B \times C) \quad , \quad A \times B \cong B \times A$$

$$[0] + A \cong A \quad , \quad (A + B) + C \cong A + (B + C) \quad , \quad A + B \cong B + A$$

$\sum_{i \in \emptyset} A_i = \emptyset = [0]$

Calculus of Bijections

$$[1] \times A \cong A \quad , \quad (A \times B) \times C \cong A \times (B \times C) \quad , \quad A \times B \cong B \times A$$

$$[0] + A \cong A \quad , \quad (A + B) + C \cong A + (B + C) \quad , \quad A + B \cong B + A$$

$$\left(\sum_{i \in I} A_i \right) \times B \cong \sum_{i \in I} (A_i \times B) \quad , \quad [0] \times B \cong [0]$$

$\Rightarrow$ is like exponentiation $\quad (\prod_i b_i)^a = \prod_i (b_i)^a$

Calculus of Bijections

$$[1] \times A \cong A \;,\; (A \times B) \times C \cong A \times (B \times C) \;,\; A \times B \cong B \times A$$

$$[0] + A \cong A \;,\; (A + B) + C \cong A + (B + C) \;,\; A + B \cong B + A$$

$$\left(\sum_{i \in I} A_i\right) \times B \cong \sum_{i \in I} (A_i \times B) \;,\; [0] \times B \cong [0]$$

$$\left(A \Rightarrow \left(\prod_{i \in I} B_i\right)\right) \cong \prod_{i \in I} \left(A \Rightarrow B_i\right) \;,\; A \Rightarrow [1] \cong [1]$$

$$(b)^{\sum_i a_i} = \prod_i (b^{a_i})$$

Calculus of Bijections

$$[1] \times A \cong A \quad , \quad (A \times B) \times C \cong A \times (B \times C) \quad , \quad A \times B \cong B \times A$$

$$[0] + A \cong A \quad , \quad (A + B) + C \cong A + (B + C) \quad , \quad A + B \cong B + A$$

$$\left(\sum_{i \in I} A_i\right) \times B \cong \sum_{i \in I} (A_i \times B) \quad , \quad [0] \times B \cong [0]$$

$$\left(A \Rightarrow \left(\prod_{i \in I} B_i\right)\right) \cong \prod_{i \in I} (A \Rightarrow B_i) \quad , \quad A \Rightarrow [1] \cong [1]$$

$$\left(\left(\sum_{i \in I} A_i\right) \Rightarrow B\right) \cong \prod_{i \in I} (A_i \Rightarrow B) \quad , \quad [0] \Rightarrow B \cong [1]$$

$$A \cong \sum_{a \in A} [\mathbf{1}]$$

$$A \cong \sum_{a \in A} [1]$$

$$([1] \Rightarrow A) \cong A \quad , \quad (A \Rightarrow B) \cong \prod_{a \in A} B$$

$$A \cong \sum_{a \in A} [1]$$

$$([1] \Rightarrow A) \cong A \quad , \quad (A \Rightarrow B) \cong \prod_{a \in A} B$$

Uncurry

$$((A \times B) \Rightarrow C) \cong (A \Rightarrow (B \Rightarrow C)) \qquad c^{a \cdot b} = (c^a)^b$$

Curry

$$A \cong \sum_{a \in A} [1]$$

$$([1] \Rightarrow A) \cong A \quad , \quad (A \Rightarrow B) \cong \prod_{a \in A} B$$

$$((A \times B) \Rightarrow C) \cong (A \Rightarrow (B \Rightarrow C))$$

$$\mathcal{P}(A) \cong (A \Rightarrow [2]) \quad , \quad (A \Rightarrow \mathcal{P}(B)) \cong \mathcal{P}(A \times B)$$

$$(A \Rightarrow \mathcal{P}B) \cong A \Rightarrow (B \Rightarrow [2])$$

$$\cong (A \times B) \Rightarrow [2]$$

$$\cong \mathcal{P}(A \times B)$$

$$A \cong \sum_{a\in A}[1]$$

$$([1] \Rightarrow A) \cong A \quad , \quad (A \Rightarrow B) \cong \prod_{a\in A} B$$

$$((A \times B) \Rightarrow C) \cong (A \Rightarrow (B \Rightarrow C))$$

$$\mathcal{P}(A) \cong (A \Rightarrow [2]) \quad , \quad (A \Rightarrow \mathcal{P}(B)) \cong \mathcal{P}(A \times B)$$

$$\mathcal{P}\Big(\sum_{i\in I} A_i\Big) \cong \prod_{i\in I} \mathcal{P}(A_i)$$

$$A \cong \textstyle\sum_{a \in A} [1]$$

$$([1] \Rightarrow A) \cong A \quad , \quad (A \Rightarrow B) \cong \textstyle\prod_{a \in A} B$$

$$((A \times B) \Rightarrow C) \cong (A \Rightarrow (B \Rightarrow C))$$

$$\mathcal{P}(A) \cong (A \Rightarrow [2]) \quad , \quad (A \Rightarrow \mathcal{P}(B)) \cong \mathcal{P}(A \times B)$$

$$\mathcal{P}\big(\textstyle\sum_{i \in I} A_i\big) \cong \textstyle\prod_{i \in I} \mathcal{P}(A_i)$$

$$(A \rightharpoonup B) \cong (A \Rightarrow (B + [1])) \quad , \quad (A \rightharpoonup [1]) \cong \mathcal{P}(A)$$

$$A \cong \sum_{a \in A} [1]$$

$$([1] \Rightarrow A) \cong A \quad , \quad (A \Rightarrow B) \cong \prod_{a \in A} B$$

$$((A \times B) \Rightarrow C) \cong (A \Rightarrow (B \Rightarrow C))$$

$$\mathcal{P}(A) \cong (A \Rightarrow [2]) \quad , \quad (A \Rightarrow \mathcal{P}(B)) \cong \mathcal{P}(A \times B)$$

$$\mathcal{P}\left(\sum_{i \in I} A_i\right) \cong \prod_{i \in I} \mathcal{P}(A_i)$$

$$(A \rightharpoonup B) \cong (A \Rightarrow (B + [1])) \quad , \quad (A \rightharpoonup [1]) \cong \mathcal{P}(A)$$

$$((A \times B) \rightharpoonup C) \cong (A \Rightarrow (B \rightharpoonup C))$$

$\cong$ is an equivalence relation

More examples:

- $(A \Rightarrow (B \Rightarrow C)) \overset{?}{\cong} (B \Rightarrow (A \Rightarrow C))$

$$\begin{array}{c} \cong\!\!\!\! \\ (A \times B) \Rightarrow C \cong (B \times A) \Rightarrow C \cong B \Rightarrow (A \Rightarrow C) \end{array}$$

NB: If $A \cong X$, $B \cong Y$

Then

$$A \times B \cong X \times Y$$
$$A \Rightarrow B \cong X \Rightarrow Y$$
$$A + B \cong X + Y$$
$$\mathcal{P}(A) \cong \mathcal{P}(X)$$
$$A \Rightarrow B \cong X \Rightarrow Y$$

- $\mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{N}) \cong \mathcal{P}(\mathbb{N})$

$$\mathcal{P}(\mathbb{N}) \cong (\mathbb{N} \Rightarrow [2])$$

$$
\begin{aligned}
\mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{N}) &\cong (\mathbb{N} \Rightarrow [2]) \times (\mathbb{N} \Rightarrow [2]) \\
&\cong \mathbb{N} \Rightarrow ([2] \times [2]) \\
&\cong \mathbb{N} \Rightarrow ([2] \Rightarrow [2]) \\
&\cong (\mathbb{N} \times [2]) \Rightarrow [2] \\
&\cong \mathbb{N} \Rightarrow [2] \\
&\cong \mathcal{P}(\mathbb{N})
\end{aligned}
$$

$(A \Rightarrow B) \times (A \Rightarrow C) \cong A \Rightarrow (B \times C)$

$[2] \times [2] \cong [2] \Rightarrow [2]$

Curry/Uncurry

$\mathbb{N} \times [2] \cong \mathbb{N}$

$$\mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{N}) \cong (\mathbb{N} \Rightarrow [2]) \times (\mathbb{N} \Rightarrow [2])$$

$$(A \Rightarrow C) \times (B \Rightarrow C) \cong (A+B) \Rightarrow C$$

$$\cong (\mathbb{N} + \mathbb{N}) \Rightarrow [2]$$

$$\mathbb{N} + \mathbb{N} \cong \mathbb{N}$$

$$\cong \mathbb{N} \Rightarrow [2]$$

$$\cong \mathcal{P}(\mathbb{N})$$