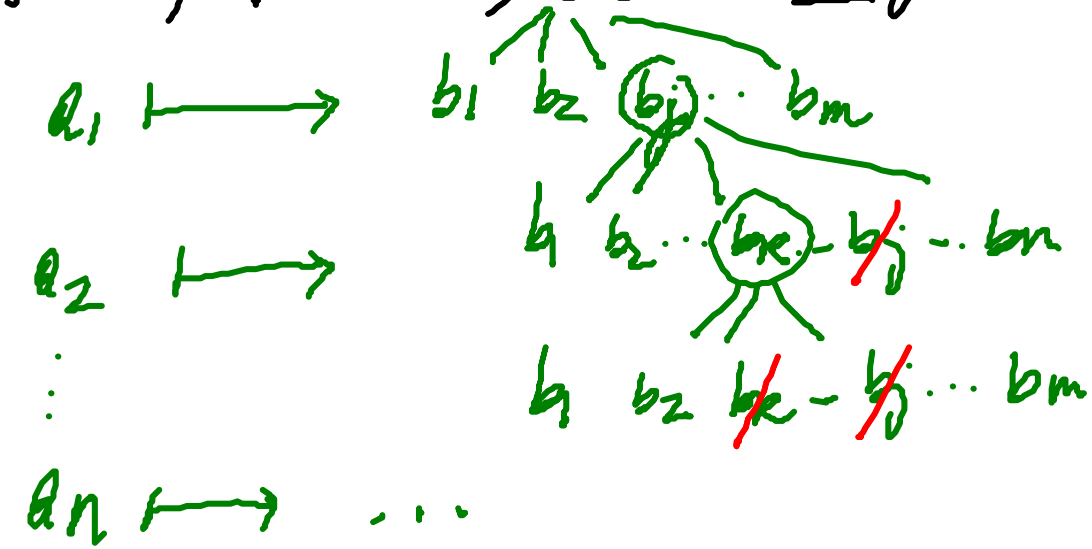
Injective functions

Def A function $f: A \to B$ is injective whenever for all $a \neq a'$ in A, we have $f(a) \neq f(a')$

The set of all injections from A to B

Say $\#A = n$, $\#B = m$, $\boxed{?}$ $\#\underline{\mathrm{Inj}}(A,B)$?



m
$\times\; m-1$
$\times\; m-2$
$\times \ldots$
$m-n+1$

Examples: ...

Notation $\quad m^{\underline{n}} =_{def} m \times (m-1) \times \cdots \times (m-n+1)$

$= \binom{m}{n} \times n!$

$$\begin{tikzcd}
A \arrow[rr, "f \text{ injection}"] \arrow[dr, "f'"'] & & B \\
& f(A) \arrow[ur, phantom, "\subseteq"] &
\end{tikzcd}$$

f' defined for $a \in A$ as

$f'(a) = f(a)$

$f(A)$ image of f defined $\{ f(a) \mid a \in A \} \subseteq B$

subsets of B of size n

$\binom{m}{n}$

is bijective

bijections is $n!$

$\# \underline{\mathrm{Inj}}(A, B) = \binom{m}{n} \times n! = m^{\underline{n}}$

Let $f : A \to B$ be a function

$\#A = n$

$\#B = m$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "f'"'] & B \\ & f(A) \arrow[u, phantom, sloped, "\subseteq"] \end{tikzcd}$$

$f'(a) \overset{\text{def}}{=} f(a)$

$f(A) \overset{\text{def}}{=} \{ b \in B \mid \exists a \in A.\ f(a) = b \}$

f' is a surjection

$$\#(A \to B) = m^n$$

$$= \sum_{K \subseteq B} \#\mathrm{Sur}(A, K) = \sum_k \#\mathrm{Sur}(n,k) \times \binom{m}{k}$$

$$= \sum_k S(n,k) \times k! \binom{m}{k}$$

$$= \sum_k S(n,k)\, m^{\underline{k}}$$

An important symbolic identity

$$\boxed{x^n = \sum_k S(n,k) \times x^{\underline{k}}}$$

Images

For a relation $R \subseteq A \times B$, one defines the ...

Direct image:

$$R[\cdot] : \mathcal{P}(A) \to \mathcal{P}(B)$$

$R[S] = \{ b \in B \mid \exists a \in S.\ aRb \}$ $S \subseteq A$

$\mathcal{P}(A \times B) \longrightarrow (\mathcal{P}(A) \Rightarrow \mathcal{P}(B))$

$\# \mathcal{P}(A \times B) = 2^{nm}$

$\# (\mathcal{P}(A) \Rightarrow \mathcal{P}(B)) = 2^{m \times 2^n}$

EXERCISE Show that it is an injection.

Images

For a relation $R \subseteq A \times B$, one defines the ...

Direct image:

$$R[\cdot] : \mathcal{P}(A) \to \mathcal{P}(B)$$

Inverse image: = direct image of the inverse.

$$R^{-1}[\cdot] : \mathcal{P}(B) \to \mathcal{P}(A)$$

Finite Sets

$n \in \mathbb{N}_0$

$[n] = \{0, 1, 2, \ldots, n-1\}$

$\#[n] = n$

terminology: ORDINALS

The prototypical finite sets:

$$\begin{cases} [0] & =_{\text{def}} \emptyset \\ [n+1] & =_{\text{def}} [n] \cup \{n\} \quad (n \in \mathbb{N}_0) \end{cases}$$

Def A set is finite if it is in bijection with some $[n]$ for $n \in \mathbb{N}_0$.

EXERCISES

Cardinality

$\#$

1. $[n] \times [m] \cong [n.m]$ — $n.m$
2. $\mathcal{P}([n]) \cong [2^n]$ $\mathcal{P}[n] \cong ([n] \Rightarrow [2])$ — 2^n

   $\mathcal{P}[n] \cong ([n] \rightharpoonup [1])$
3. $([n] \rightharpoonup [m])$ — $(m+1)^n$
4. $([n] \Rightarrow [m])$ — m^n
5. $\mathrm{Bij}([n],[m])$ — $\begin{cases} 0 & \text{if } n \neq m \\ n! & \text{otherwise.} \end{cases}$
6. $\mathrm{Inj}([n],[m])$ — $m^{\underline{n}}$
7. $\mathrm{Sur}([n],[m])$ — $S(n,m)\, m!$
8. $[n] \cup [m] \cong [\max(n,m)]$ — $\max(n,m)$
9. $[n] \cap [m] \cong [\min(n,m)]$ — $\min(n,m)$

$\# A = n \qquad \# B = m$

EXERCISE

$$\mathcal{P}[n] \cong ([n] \to [1])$$

To give

$$A \xrightarrow{f} B \qquad \text{partial function}$$

is to give

a subset of A, say D (namely the domain of definition of f) and a (total) function from D to B

$$\#(A \rightharpoonup B) = \sum_k \binom{n}{k} \times m^k = (m+1)^n.$$

Countable Sets

The prototypical $\boxed{\text{countable}}$ sets: $\mathbb{N}_0$, $\mathbb{N}$.

Countable Sets

The prototypical countable sets: $\mathbb{N}_0$, $\mathbb{N}$.

For a set S, the following are equivalent:

1. S is finite or there is a bijection $\mathbb{N} \to S$.
2. There is a bijection $A \to S$ for $A \subseteq \mathbb{N}$.
3. There is an injection $S \to \mathbb{N}$.
4. S is empty or there is a surjection $\mathbb{N} \to S$.

Countable sets are those for which the above hold.

(1) ⇒ (2)

(2) ⇒ (3)

$A \xrightarrow[f]{\cong} S$ for $A \subseteq \mathbb{N}$

Consider

$S \xrightarrow[\cong]{f^{-1}} A \subseteq \mathbb{N}$

is an injection

(3) ⇒ (4)

If there an injection $S \to \mathbb{N}$ then either $S = \emptyset$ or there is a surjection $\mathbb{N} \xrightarrow{e} S$.

enumeration

$e(0), e(1), \dots, e(n), \dots$

Assume S is non-empty

That is there is an element $s \in S$.

Let $f : S \to \mathbb{N}$ be an injection.

$\Downarrow$ for all $n \in \mathbb{N}$ either there is no $x \in S$ such that $f(x) = n$ (1) or (2) there is exactly one, call it $\varepsilon_n \in S$.

Define

$$e(n) \overset{\text{def}}{=} \begin{cases} s & \text{in case (1)} \\ \varepsilon_n & \text{in case (2)} \end{cases}$$