An answer to Exercise 4.9

> For any two sets X and Y, with Y containing at least two elements, there cannot be an injection from the set of functions $(X \to Y)$ to the set X.

(i) For every set X, there is no injection $\mathcal{P}(X) \to X$.

Proof. Aiming at a contradiction, let $f : \mathcal{P}(X) \to X$ be an injective function (that is, such that $f(A) = f(B)$ implies $A = B$ for all $A, B \in \mathcal{P}(X)$).

Then, for

$$W =_{\text{def}} \{\, x \in X \mid \exists\, Z \in \mathcal{P}(X).\ x = f(Z) \text{ and } f(Z) \notin Z \,\} \ \in \mathcal{P}(X)$$

we have the following contradiction

$$\begin{aligned} & f(W) \in W \\ & \quad \iff \exists\, Z \in \mathcal{P}(X).\ f(W) = f(Z) \text{ and } f(Z) \notin Z \\ & \quad \iff f(W) \notin W \end{aligned}$$

where the first equivalence follows from the definition of W and the second one from the injectivity of f.

(ii) If a set Y has at least two distinct elements then there is an injection $\mathcal{P}(X) \to (X \to Y)$.

Proof. For y_0 and y_1 two distinct elements of Y, define first the function

$$t : (X \to \{\mathtt{T}, \mathtt{F}\}) \to (X \to Y)$$

given by

$$t =_{\text{def}} \lambda f \in (X \to \{\mathtt{T}, \mathtt{F}\}).\lambda x \in X.\ \text{if } f(x) \text{ then } y_1 \text{ else } y_0$$

and check that it is injective (for which you will need to use that $y_0 \neq y_1$).

Consider then the composite function

$$t \circ \chi : \mathcal{P}(X) \to (X \to Y)\ ,$$

for $\chi : \mathcal{P}(X) \to (X \to \{\mathtt{T}, \mathtt{F}\})$ the bijection of Exercise 4.8.

(iii) There is no injection from $(X \to Y)$ to X when Y has at least two distinct elements.

Proof. If there were an injective function $f : (X \to Y) \to X$ then the composite function $f \circ t \circ \chi : \mathcal{P}(X) \to X$ would be injective, contradicting (i).