Denotational Semantics of PCF

[Chapter 6, p 69]

Denotational semantics of PCF types

types τ are mapped to domains $[\![\tau]\!]$:

$$[\![nat]\!] \stackrel{\text{def}}{=} \mathbb{N}_\perp$$ (flat domain)

$$[\![bool]\!] \stackrel{\text{def}}{=} \mathbb{B}_\perp$$ (flat domain)

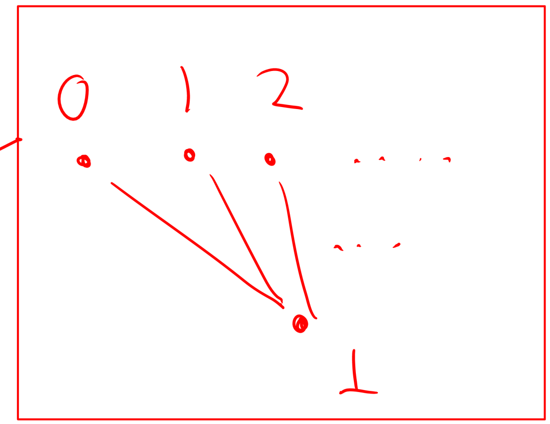
where $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{B} = \{true, false\}$.

Denotational semantics of PCF types

types τ are mapped to domains $[\![\tau]\!]$:

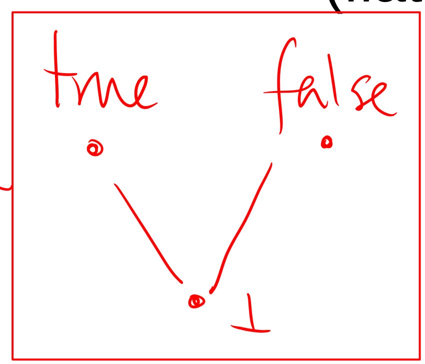
$$[\![nat]\!] \stackrel{\text{def}}{=} \mathbb{N}_\perp$$

(flat domain)



$$[\![bool]\!] \stackrel{\text{def}}{=} \mathbb{B}_\perp$$

(flat domain)



where $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{B} = \{true, false\}$.

We need $\perp$ to give a meaning to terms like

$fix(\mathbf{fn}\ x : nat.\ succ(x))$

Denotational semantics of PCF types

$$[\![nat]\!] \stackrel{\text{def}}{=} \mathbb{N}_\perp$$ (flat domain)

$$[\![bool]\!] \stackrel{\text{def}}{=} \mathbb{B}_\perp$$ (flat domain)

$$[\![\tau \to \tau']\!] \stackrel{\text{def}}{=} [\![\tau]\!] \to [\![\tau']\!]$$ (function domain).

all continuous functions from domain $[\![\tau]\!]$ to domain $[\![\tau']\!]$

By only using continuous functions, we can give a meaning to fix(M) terms via Tarski's Thm.

Denotational semantics of PCF type environments

$$[\![\Gamma]\!] \stackrel{\text{def}}{=} \prod_{x \in dom(\Gamma)} [\![\Gamma(x)]\!]$$ product of domains

Denotational semantics of PCF type environments

$$[\![\Gamma]\!] \stackrel{\text{def}}{=} \prod_{x \in dom(\Gamma)} [\![\Gamma(x)]\!]$$ product of domains

If $\Gamma = \{x_1 : \tau_1, \ldots, x_n : \tau_n\}$

(i.e. $dom\,\Gamma = \{x_1, \ldots, x_n\}$ & $\Gamma(x_i) = \tau_i$)

then

$$[\![\Gamma]\!] \cong [\![\tau_1]\!] \times \cdots \times [\![\tau_n]\!]$$

$$= \{(d_1, \ldots, d_n) \mid \forall i = 1..n.\ d_i \in [\![\tau_i]\!]\}$$

with partial order

$(d_1, \ldots, d_n) \sqsubseteq (d_1', \ldots, d_n') \iff \forall i = 1..n.\ d_i \sqsubseteq d_i'$ in $[\![\tau_i]\!]$

Denotational semantics of PCF type environments

$$[\![\Gamma]\!] \stackrel{\text{def}}{=} \prod_{x \in dom(\Gamma)} [\![\Gamma(x)]\!]$$ (Γ-environments)

$=$ the domain of partial functions ρ from variables to domains such that $dom(\rho) = dom(\Gamma)$ and $\rho(x) \in [\![\Gamma(x)]\!]$ for all $x \in dom(\Gamma)$

partial order:

$\underset{\text{in } [\![\Gamma]\!]}{\rho \sqsubseteq \rho'} \iff \forall x \in dom(\Gamma) \,.\, \underset{\text{in } [\![\Gamma(x)]\!]}{\rho(x) \sqsubseteq \rho'(x)}$

Denotational semantics of PCF type environments

$$[\![\Gamma]\!] \stackrel{\text{def}}{=} \prod_{x \in dom(\Gamma)} [\![\Gamma(x)]\!] \qquad \text{(Γ-environments)}$$

$=$ the domain of partial functions ρ from variables to domains such that $dom(\rho) = dom(\Gamma)$ and $\rho(x) \in [\![\Gamma(x)]\!]$ for all $x \in dom(\Gamma)$

For the empty type environment $\emptyset$,

$$[\![\emptyset]\!] = \{\perp\}$$

domain with one element (necessarily least)

where $\perp$ denotes the unique partial function with $dom(\perp) = \emptyset$.

Denotational semantics of PCF

To every typing judgement

$$\Gamma \vdash M : \tau$$

we associate a continuous function

$$[\![\Gamma \vdash M]\!] : [\![\Gamma]\!] \to [\![\tau]\!]$$

between domains.

For each $\rho \in [\![\Gamma]\!]$, we give an element

$$[\![\Gamma \vdash M]\!](\rho) \in [\![\tau]\!]$$

which is <u>continuous in ρ</u>

For example

$$\{x : nat,\ y : nat \to nat,\ z : nat\} \vdash \text{if } zero(x) \text{ then } y\,x \text{ else } z : nat$$

For example

$$\underbrace{\{x : nat,\ y : nat \to nat,\ z : nat\}}_{\Gamma} \vdash \underbrace{\text{if zero}(x) \text{ then } y\,x \text{ else } z}_{M} : \underbrace{nat}_{\tau}$$

denotation is a continuous function

$$\underbrace{\mathbb{N}_\perp \times (\mathbb{N}_\perp \to \mathbb{N}_\perp) \times \mathbb{N}_\perp}_{[\![\Gamma]\!]} \longrightarrow \underbrace{\mathbb{N}_\perp}_{[\![\tau]\!]}$$

For example

$$\underbrace{\{x : nat,\ y : nat \to nat,\ z : nat\}}_{\Gamma} \vdash \underbrace{\text{if zero}(x) \text{ then } y\,x \text{ else } z}_{M} : \underbrace{nat}_{\tau}$$

denotation is a continuous function

$$\underbrace{\mathbb{N}_\perp \times (\mathbb{N}_\perp \to \mathbb{N}_\perp) \times \mathbb{N}_\perp}_{[\![\Gamma]\!]} \longrightarrow \underbrace{\mathbb{N}_\perp}_{[\![\tau]\!]}$$

namely

$$(d_1, d_2, d_3) \longmapsto \begin{cases} \perp & \text{if } d_1 = \perp \\ d_2(d_1) & \text{if } d_1 = 0 \\ d_3 & \text{if } d_1 = 1, 2, 3, \ldots \end{cases}$$

Denotational semantics of PCF

To every typing judgement

$$\Gamma \vdash M : \tau$$

we associate a continuous function

$$[\![\Gamma \vdash M]\!] : [\![\Gamma]\!] \to [\![\tau]\!]$$

between domains.

Definition is by induction on the structure of M, or equivalently, on the derivation of $\Gamma \vdash M : \tau$ from the typing rules (p. 56)

Denotational semantics of PCF terms, I

$$[\![\Gamma \vdash \mathbf{0}]\!](\rho) \stackrel{\text{def}}{=} 0 \in [\![nat]\!]$$

$$[\![\Gamma \vdash \mathbf{true}]\!](\rho) \stackrel{\text{def}}{=} true \in [\![bool]\!]$$

$$[\![\Gamma \vdash \mathbf{false}]\!](\rho) \stackrel{\text{def}}{=} false \in [\![bool]\!]$$

Functions $f : D \to E$ that are
<u>constant</u> $(\forall d, d' \in D.\ f(d) = f(d'))$
are continuous.

Denotational semantics of PCF terms, I

$$[\![\Gamma \vdash \mathbf{0}]\!](\rho) \stackrel{\text{def}}{=} 0 \in [\![nat]\!]$$

$$[\![\Gamma \vdash \mathbf{true}]\!](\rho) \stackrel{\text{def}}{=} true \in [\![bool]\!]$$

$$[\![\Gamma \vdash \mathbf{false}]\!](\rho) \stackrel{\text{def}}{=} false \in [\![bool]\!]$$

$$[\![\Gamma \vdash x]\!](\rho) \stackrel{\text{def}}{=} \rho(x) \in [\![\Gamma(x)]\!] \qquad \big(x \in dom(\Gamma)\big)$$

The projection functions $(d_1, \ldots, d_n) \mapsto d_i$ are continuous.

Denotational semantics of PCF terms, II

$$[\![\Gamma \vdash \mathbf{succ}(M)]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M]\!](\rho) + 1 & \text{if } [\![\Gamma \vdash M]\!](\rho) \neq \bot \\ \bot & \text{if } [\![\Gamma \vdash M]\!](\rho) = \bot \end{cases}$$

Thus $[\![\Gamma \vdash \mathrm{succ}(M)]\!] = S_\bot \circ \underbrace{[\![\Gamma \vdash M]\!]}$

continuous, by induction

$S_\bot : \mathbb{N}_\bot \to \mathbb{N}_\bot$
is the continuous function

$$S_\bot(d) \stackrel{\text{def}}{=} \begin{cases} \bot & \text{if } d = \bot \\ d+1 & \text{if } d \neq \bot \end{cases}$$

Composition of continuous functions is continuous

Denotational semantics of PCF terms, II

$$[\![\Gamma \vdash \mathbf{succ}(M)]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M]\!](\rho) + 1 & \text{if } [\![\Gamma \vdash M]\!](\rho) \neq \bot \\ \bot & \text{if } [\![\Gamma \vdash M]\!](\rho) = \bot \end{cases}$$

$$[\![\Gamma \vdash \mathbf{pred}(M)]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M]\!](\rho) - 1 & \text{if } [\![\Gamma \vdash M]\!](\rho) > 0 \\ \bot & \text{if } [\![\Gamma \vdash M]\!](\rho) = 0, \bot \end{cases}$$

So

$$[\![\Gamma \vdash pred(M)]\!] = p_\bot \circ [\![\Gamma \vdash M]\!]$$

$p_\bot : \mathbb{N}_\bot \to \mathbb{N}_\bot$ is the cts function

$$p_\bot(d) \stackrel{\text{def}}{=} \begin{cases} \bot & \text{if } d = \bot, 0 \\ d - 1 & \text{if } d > 0 \end{cases}$$

Denotational semantics of PCF terms, II

$$[\![\Gamma \vdash \mathbf{succ}(M)]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M]\!](\rho) + 1 & \text{if } [\![\Gamma \vdash M]\!](\rho) \neq \perp \\ \perp & \text{if } [\![\Gamma \vdash M]\!](\rho) = \perp \end{cases}$$

$$[\![\Gamma \vdash \mathbf{pred}(M)]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M]\!](\rho) - 1 & \text{if } [\![\Gamma \vdash M]\!](\rho) > 0 \\ \perp & \text{if } [\![\Gamma \vdash M]\!](\rho) = 0, \perp \end{cases}$$

$$[\![\Gamma \vdash \mathbf{zero}(M)]\!](\rho) \stackrel{\text{def}}{=} \begin{cases} \mathit{true} & \text{if } [\![\Gamma \vdash M]\!](\rho) = 0 \\ \mathit{false} & \text{if } [\![\Gamma \vdash M]\!](\rho) > 0 \\ \perp & \text{if } [\![\Gamma \vdash M]\!](\rho) = \perp \end{cases}$$

So

$[\![\Gamma \vdash \text{zero}(M)]\!] = z_\perp \circ [\![\Gamma \vdash M]\!]$ where $z_\perp : \mathbb{N}_\perp \to \mathbb{B}_\perp$ is....

Denotational semantics of PCF terms, III

$$[\![\Gamma \vdash \mathbf{if}\ M_1\ \mathbf{then}\ M_2\ \mathbf{else}\ M_3]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M_2]\!](\rho) & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \mathit{true} \\ [\![\Gamma \vdash M_3]\!](\rho) & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \mathit{false} \\ \perp & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \perp \end{cases}$$

So

$$[\![\sim]\!] = if \circ \langle [\![\Gamma \vdash M_1]\!], [\![\Gamma \vdash M_2]\!], [\![\Gamma \vdash M_3]\!] \rangle$$

[Proposition 3.2.1, p 35]

If D & $D_1, \ldots, D_n$ are domains,
$f_1 : D \to D_1, \ldots, f_n : D \to D_n$ are continuous fns,
then

$$\langle f_1, \ldots, f_n \rangle : D \longrightarrow D_1 \times \cdots \times D_n$$
$$d \longmapsto (f_1(d), \ldots, f_n(d))$$

is also a continuous function.

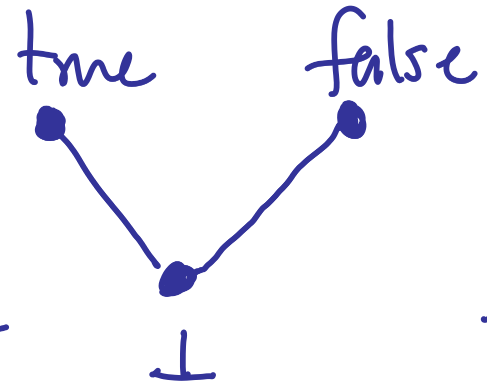
[Proposition 3.2.2, p 35]

For each domain D, the function

$$\mathrm{if} : \mathbb{B}_\perp \times D \times D \longrightarrow D$$

$$(d_1, d_2, d_3) \mapsto \begin{cases} d_2 & \text{if } d_1 = \mathrm{true} \\ d_3 & \text{if } d_1 = \mathrm{false} \\ \perp & \text{if } d_1 = \perp \end{cases}$$

is continuous.

(Recall: $\mathbb{B}_\perp = \{$ 

$\}$.)

Denotational semantics of PCF terms, III

$$[\![\Gamma \vdash \mathbf{if}\ M_1\ \mathbf{then}\ M_2\ \mathbf{else}\ M_3]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M_2]\!](\rho) & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \mathit{true} \\ [\![\Gamma \vdash M_3]\!](\rho) & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \mathit{false} \\ \bot & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \bot \end{cases}$$

$$\boxed{[\![\Gamma \vdash M_1\ M_2]\!](\rho) \stackrel{\text{def}}{=} \big([\![\Gamma \vdash M_1]\!](\rho)\big)\ \big([\![\Gamma \vdash M_2]\!](\rho)\big)}$$

[Proposition 3.3.1, p 39]

For all domains D & E, the evaluation function

$$ev : (D \to E) \times D \longrightarrow E$$

$$ev(f, d) = f(d)$$

is continuous.

$$\frac{\Gamma \vdash M_1 : \tau \to \tau' \qquad \Gamma \vdash M_2 : \tau}{\Gamma \vdash M_1\, M_2 : \tau'} \; (:app)$$

$$\boxed{\dfrac{\Gamma \vdash M_1 : \tau \to \tau' \qquad \Gamma \vdash M_2 : \tau}{\Gamma \vdash M_1 M_2 : \tau'} \quad (:\mathrm{app})}$$

$$[\![\Gamma \vdash M_1]\!] : [\![\Gamma]\!] \to \left([\![\tau]\!] \to [\![\tau']\!]\right)$$

$$[\![\Gamma \vdash M_2]\!] : [\![\Gamma]\!] \to [\![\tau]\!]$$

$$\boxed{\dfrac{\Gamma \vdash M_1 : \tau \to \tau' \qquad \Gamma \vdash M_2 : \tau}{\Gamma \vdash M_1 M_2 : \tau'} \; (:app)}$$

$$[\![\Gamma \vdash M_1]\!] : [\![\Gamma]\!] \longrightarrow ([\![\tau]\!] \to [\![\tau']\!])$$

$$[\![\Gamma \vdash M_2]\!] : [\![\Gamma]\!] \longrightarrow [\![\tau]\!]$$

$$[\![\Gamma]\!] \xrightarrow{\langle [\![\Gamma \vdash M_1]\!], [\![\Gamma \vdash M_2]\!] \rangle} ([\![\tau]\!] \to [\![\tau']\!]) \times [\![\tau]\!]$$

$$\boxed{\dfrac{\Gamma \vdash M_1 : \tau \to \tau' \qquad \Gamma \vdash M_2 : \tau}{\Gamma \vdash M_1 M_2 : \tau'} \; (:\mathrm{app})}$$

$$[\![\Gamma \vdash M_1]\!] : [\![\Gamma]\!] \longrightarrow \left([\![\tau]\!] \to [\![\tau']\!]\right)$$

$$[\![\Gamma \vdash M_2]\!] : [\![\Gamma]\!] \longrightarrow [\![\tau]\!]$$

$$\begin{array}{ccc} [\![\Gamma]\!] & \xrightarrow{\langle [\![\Gamma \vdash M_1]\!],\, [\![\Gamma \vdash M_2]\!] \rangle} & \left([\![\tau]\!] \to [\![\tau']\!]\right) \times [\![\tau]\!] \\ & & \Big\downarrow \mathrm{ev} \\ \rho & \longmapsto \; [\![\Gamma \vdash M_1]\!](\rho)\left([\![\Gamma \vdash M_2]\!](\rho)\right) & [\![\tau']\!] \end{array}$$

Denotational semantics of PCF terms, III

$$[\![\Gamma \vdash \mathbf{if}\ M_1\ \mathbf{then}\ M_2\ \mathbf{else}\ M_3]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M_2]\!](\rho) & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \mathit{true} \\ [\![\Gamma \vdash M_3]\!](\rho) & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \mathit{false} \\ \perp & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \perp \end{cases}$$

$$[\![\Gamma \vdash M_1\ M_2]\!](\rho) \stackrel{\text{def}}{=} \big([\![\Gamma \vdash M_1]\!](\rho)\big)\,\big([\![\Gamma \vdash M_2]\!](\rho)\big)$$

So $[\![\Gamma \vdash M_1\,M_2]\!] = ev \circ \langle [\![\Gamma \vdash M_1]\!], [\![\Gamma \vdash M_2]\!] \rangle$

Denotational semantics of PCF terms, IV

$$\llbracket \Gamma \vdash \mathbf{fn}\, x : \tau \,.\, M \rrbracket(\rho) \stackrel{\text{def}}{=} \lambda d \in \llbracket \tau \rrbracket \,.\, \llbracket \Gamma[x \mapsto \tau] \vdash M \rrbracket(\rho[x \mapsto d]) \qquad \big(x \notin dom(\Gamma)\big)$$

NB: $\rho[x \mapsto d] \in \llbracket \Gamma[x \mapsto \tau] \rrbracket$ is the function mapping x to $d \in \llbracket \tau \rrbracket$ and otherwise acting like ρ.

[Proposition 3.3.1, p 39,40]

For all domains D', D & E,

if $f : D' \times D \longrightarrow E$ is continuous,

then so is

$$cur(f) : D' \longrightarrow (D \to E)$$

$$cur(f)(d') \stackrel{def}{=} \lambda d \in D . f(d', d)$$

$$(:\mathrm{fn}) \quad \frac{\Gamma[x \mapsto \tau] \vdash M : \tau'}{\Gamma \vdash \mathrm{fn}\, x : \tau.\, M : \tau \to \tau'} \text{ if } x \notin \mathrm{dom}(\Gamma)$$

$$(:fn)\quad \frac{\Gamma[x \mapsto \tau] \vdash M : \tau'}{\Gamma \vdash \mathbf{fn}\, x : \tau.\, M : \tau \to \tau'} \text{ if } x \notin dom(\Gamma)$$

$$[\![\Gamma[x \mapsto \tau] \vdash M]\!] : [\![\Gamma[x \mapsto \tau]]\!] \longrightarrow [\![\tau']\!]$$

$$
(:\mathrm{fn})\quad \frac{\Gamma[x \mapsto \tau] \vdash M : \tau'}{\Gamma \vdash \mathbf{fn}\ x : \tau.\ M : \tau \to \tau'} \quad \text{if } x \notin \mathrm{dom}(\Gamma)
$$

$$
\begin{tikzcd}
\rho[x \mapsto d] & {[\![\Gamma[x \mapsto \tau]]\!]} \arrow[rr, "{[\![\Gamma[x \mapsto \tau] \vdash M]\!]}"] & & {[\![\tau']\!]} \\
(\rho, d) \arrow[u, mapsto] & {[\![\Gamma]\!] \times [\![\tau]\!]} \arrow[u, "\cong"'] \arrow[urr, "\text{compose}"'] & &
\end{tikzcd}
$$

$$\boxed{(:fn)\quad \frac{\Gamma[x \mapsto \tau] \vdash M : \tau'}{\Gamma \vdash \mathbf{fn}\, x : \tau.\, M : \tau \to \tau'} \quad \text{if } x \notin dom(\Gamma)}$$

$$\begin{tikzcd}
\rho[x \mapsto d] & {[\![\Gamma[x \mapsto \tau]]\!]} \arrow[rr, "{[\![\Gamma[x \mapsto \tau] \vdash M]\!]}"] & & {[\![\tau']\!]} \\
(\rho, d) \arrow[u, mapsto] & {[\![\Gamma]\!] \times [\![\tau]\!]} \arrow[u, "\cong"'] \arrow[urr] & &
\end{tikzcd}$$

cur of this continuous function gives

$$\begin{aligned}
[\![\Gamma]\!] &\longrightarrow ([\![\tau]\!] \to [\![\tau']\!]) \\
\rho &\longmapsto \lambda d \in [\![\tau]\!].\ [\![\Gamma[x \mapsto \tau] \vdash M]\!](\rho[x \mapsto d])
\end{aligned}$$

Denotational semantics of PCF terms, IV

$$\begin{aligned} & [\![\Gamma \vdash \mathbf{fn}\, x : \tau \,.\, M]\!](\rho) \\ & \stackrel{\text{def}}{=} \lambda d \in [\![\tau]\!] \,.\, [\![\Gamma[x \mapsto \tau] \vdash M]\!](\rho[x \mapsto d]) \end{aligned} \qquad \big(x \notin dom(\Gamma)\big)$$

NB: $\rho[x \mapsto d] \in [\![\Gamma[x \mapsto \tau]]\!]$ is the function mapping x to $d \in [\![\tau]\!]$ and otherwise acting like ρ.

Denotational semantics of PCF terms, V

$$[\![\Gamma \vdash \mathbf{fix}(M)]\!](\rho) \stackrel{\text{def}}{=} \mathit{fix}([\![\Gamma \vdash M]\!](\rho))$$

So $[\![\Gamma \vdash \mathrm{fix}(M)]\!] = \mathrm{fix} \circ [\![\Gamma \vdash M]\!]$

Recall that fix is the function assigning least fixed points to continuous functions.

Recall (p41):

Continuity of the fixpoint operator

Let D be a domain.

By Tarski's Fixed Point Theorem we know that each continuous function $f \in (D \to D)$ possesses a least fixed point, $\mathit{fix}(f) \in D$.

Proposition. *The function*

$$\mathit{fix} : (D \to D) \to D$$

is continuous.