MPhil Advanced Computer Science
Topics in Logic and Complexity

Lent 2011 — Anuj Dawar

Exercise Sheet 1

1. Show that, for every nondeterministic machine M which uses $O(\log n)$ work space, there is a machine R with three tapes (`input`, `work` and `output`) which works as follows. On input x, R produces on its output tape a description of the configuration graph for M, x, and R uses $O(\log |x|)$ space on its work tape.

   Explain why this means that if $\mathsf{Reachability}$ is in L, then $\mathsf{L} = \mathsf{NL}$.

2. Show that a language L is in $\mathsf{co\text{-}NP}$ if, and only if, there is a nondeterministic Turing machine M and a polynomial p such that M halts in time $p(n)$ for all inputs of length x, and L is exactly the set of strings x such that *all* computations of M on input x end in an accepting state.

3. Define a *strong* nondeterministic Turing machine as one where each computation has three possible outcomes: accept, reject or maybe. If M is such a machine, we say that it accepts L, if for every $x \in L$, every computation path of M on x ends in either accept or maybe, with at least one accept *and* for $x \notin L$, every computation path of M on x ends in reject or maybe, with at least one reject.

   Show that if L is decided by a strong nondeterministic Turing machine running in polynomial time, then $L \in \mathsf{NP} \cap \mathsf{co\text{-}NP}$.

4. *Geography* and *HEX* are examples of two-player games played on graphs for which the problem of deciding which of the two players has a winning strategy is PSpace-complete (see slide 49 of the notes). The games are defined as follows.

   Geography We are given a directed graph $G = (V, E)$ with a distinguished start vertex $s \in V$. At the beginning of the game, s is *marked*. The players mover alternately. The player whose turn it is marks a previouasly unmarked vertex v such that there is an edge from u to v, where u is the vertex marked most recently by the other player. A player who gets stuck (i.e. the vertex most recently marked is u and all edges leaving u go to marked vertices) loses the game.

   HEX We are given a directed graph $G = (V, E)$ with two distinguished vertices $a, b \in V$. There are two players (*red* and *blue*) who take alternate turns. In each turn, the player chooses a vertex not previously coloured and colours it with its own colour (player *red* colours it red or player *blue* colours it blue). The game ends when all nodes have been coloured. If there is a path from a to b consisting entirely of red vertices, then player *red* has won, otherwise *blue* has won.

Explain why both these problems are in PSpace. Prove, by means of suitable reductions, that they are PSpace-complete.

5. A second-order Horn sentence (SO-Horn sentence, for short) is one of the form
$$Q_1R_1 \ldots Q_pR_p(\forall \mathbf{x} \bigwedge_i C_i)$$
where, each Q_i is either $\exists$ or $\forall$, each R_i is a relational variable and each C_i is a *Horn* clause, which is defined for our purposes as a disjunction of atomic and negated atomic formulas such that it contains at most one positive occurrence of a relational variable. A sentence is said to be ESO-Horn if it is as above, and all Q_i are $\exists$.

   (a) Show that any ESO-Horn sentence in a relational signature defines a class of structures decidable in polynomial time.

   (b) Show that, if K is an isomorphism-closed class of structures in a relational signature including $<$, such that each structure in K interprets $<$ as a linear order and
$$\{[\mathcal{A}]_< \mid \mathcal{A} \in K\}$$
   is decidable in polynomial time, then there is an ESO-Horn sentence that defines K.

   (c) Show that any SO-Horn sentence is equivalent to an ESO-Horn sentence.

6. Recall that a Boolean formula is in *conjunctive normal form* if it is the conjunction of a collection of *clauses*, each of which is the disjunction of a set of *literals*. Each literal is either a propositional variable or the negation of a propositional variable. We say that a formula is in *3-CNF* if it is in conjunctive normal form and each clause contains exactly 3 literals. It is in *2-CNF* if it is in conjunctive normal form and each clause contains exactly 2 literals.

   The problem of deciding whether a given formula in 3-CNF is satisfiable is known to be NP-complete. Here, the aim is to show that the problem of deciding whether a given formula in 2-CNF is satisfiable is in NL.

   (a) Show that every clause containing 2 literals can be written as an implication in exactly two ways.

   For any formula ϕ in 2-CNF, define the directed graph G_ϕ to be the graph whose set of vertices is the set of all literals that occur in ϕ, and in which there is an edge from literal x to literal y if, and only if, the implication $(x \to y)$ is equivalent to one of the clauses in ϕ.

   (b) Show that ϕ is *unsatisfiable* if, and only if, there is a literal x such that there is a path in G_ϕ from x to $\neg x$ *and* a path from $\neg x$ to x.

(c) Explain why it follows that the problem of determining whether a formula in 2-CNF is satisfiable is in NL.

7. Show that *Cook's theorem*—that the problem SAT (see slide 47) is NP-complete—can be obtained as a consequence of Fagin's theorem.

8. A graph $G = (V, E)$ is said to be *Hamiltonian* if it contains a cycle which visits every vertex exactly once. The problem of determining whether a graph is Hamiltonian is known to be NP-complete. Write down a sentence of ESO that defines this property.

9. We have seen a sentence of ESO that defines the structures with an even number of elements (slide 72). Can you define the property in USO?

10. We have seen a sentence of ESO that defines the 3-colourable graphs (slide 7). We can, of course, write a similar sentence to define the 2-colourable graphs. However, the property of being 2-colourable is in P, since a graph is 2-colourable if, and only if, it has no cycles of odd length. Can you write a USO sentence that defines the 2-colourable graphs?

11. Recall that a graph is *planar* if it can be drawn in the plane without any crossing edges. It is decidable in polynomial time whether a given graph is planar. Can you write a USO sentence that defines the planar graphs? How about an ESO sentence?

12. Show that the levels of the polynomial hierarchy are closed under polynomial time reductions. That is to say, if L_1 is a decision problem in Σ_n (or Π_n) for some n and $L_2 \leq_P L_1$ then L_2 is also in Σ_n (or Π_n respectively).

13. Recall the definition of *quantified Boolean formulas* (slide 62). We now define the following restricted classes of formulas.

    - A quantified Boolean formula is said to be Σ_1 if it consists of a sequence of existential quantifiers followed by a Boolean formula without quantifiers.
    - A quantified Boolean formula is said to be Π_1 if it consists of a sequence of universal quantifiers followed by a Boolean formula without quantifiers.
    - A quantified Boolean formula is said to be Σ_{n+1} if it consists of a sequence of existential quantifiers followed by a Π_n formula.
    - A quantified Boolean formula is said to be Π_{n+1} if it consists of a sequence of universal quantifiers followed by a Σ_n formula.

    For each n define Σ_n-*QBF* to be the problem of determining, given a Σ_n formula without free variables, whether or not it evaluates to true. Π_n-*QBF* is defined similarly for Π_n formulas.

    Prove that Σ_n-*QBF* is complete for the complexity class Σ_n^1 (i.e. the nth existential level of the polynomial hierarchy), and that Π_n-*QBF* is complete for the complexity class Π_n^1.

14. If σ is a relational signature (i.e. it contains no function or constant symbols), and $\mathbb{A}$ and $\mathbb{B}$ are σ-structures, write $\mathbb{A} + \mathbb{B}$ for the structure whose universe is the disjoint union of the universes of $\mathbb{A}$ and $\mathbb{B}$ and where each relation symbol R of σ is interpreted by the corresponding union of its interpretations in $\mathbb{A}$ and $\mathbb{B}$. Similarly, write $n\mathbb{A}$ for the disjoint union of n copies of $\mathbb{A}$.

    (a) Show that, if $\mathbb{A} \equiv_q \mathbb{A}'$ and $\mathbb{B} \equiv_q \mathbb{B}'$, then $\mathbb{A} + \mathbb{B} \equiv_q \mathbb{A}' + \mathbb{B}'$.

    (b) Show that, for $n, m \geq q$, $n\mathbb{A} \equiv_q m\mathbb{A}$.

15. A *clique* in a graph $G = (V, E)$ is a set $X \subseteq V$ of vertices such that for any $u, v \in X$ if $u \neq v$ then (u, v) is an edge in E. The decision problem CLIQUE is the problem of deciding, given a graph G and a positive integer k whether or not G contains a clique with k or more elements. This problem is known to be NP-complete.

    We will represent this problem as a class of structures as follows. The vocabulary consists of two binary relations E and $<$ and one constant k. Consider structures $\mathcal{G} = (V, E, <, k)$ in this vocabulary where (V, E) is a graph, $<$ is a linear order on V and k is some element of V. We say that $\mathcal{G}$ is in CLIQUE if there is a set $X \subseteq V$ of vertices which forms a clique in the graph (V, E) and so that the number of elements in X is larger than the number of elements in $\{v \in V \mid v < k\}$, i.e. the number of elements before k in the linear order.

    (a) Give a sentence of existential second-order logic that defines the class of structures CLIQUE.

    (b) Prove that there is no sentence of first-order logic that defines this class of structures.