§1: Fourier and related methods - Problem Sheet

1. Given a complex linear space, V, define the notion of an *inner product* and in the case of $V = \mathbb{C}^n$ show that for any two vectors $x, y \in \mathbb{C}^n$

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y_i}$$

where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ defines an inner product.

2. Suppose that V is a complex inner product space. Show the *Cauchy-Schwarz inequality*, namely, that for all $u, v \in V$

$$|\langle u, v \rangle|^2 \leq \langle u, u \rangle \langle v, v \rangle \,.$$

Define the notion of a *norm* for V and show that

$$||v|| = +\sqrt{\langle v, v \rangle}$$

is a norm.

3. Suppose that V is an inner product space and let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal system for V and let $W = \text{span}\{e_1, e_2, \ldots, e_n\}$. Using $\tilde{u} = \sum_{k=1}^{n} \langle u, e_k \rangle e_k$ for the *orthogonal projection* of $u \in V$ on W show that

$$||\tilde{u}||^2 = \sum_{k=1}^{n} |\langle u, e_k \rangle|^2 \leq ||u||^2 \,.$$

Now, consider the case of an infinite orthonormal system $\{e_1, e_2, \ldots\}$ and show that the limit

$$\lim_{n \to \infty} \sum_{k=1}^{n} |\langle u, e_k \rangle|^2 = \sum_{k=1}^{\infty} |\langle u, e_k \rangle|^2$$

exists and that the limit is bounded above by

$$\sum_{k=1}^{\infty} |\langle u, e_k \rangle|^2 \leq ||u||^2 \,.$$

Hence deduce that

$$\lim_{k \to \infty} \langle u, e_k \rangle = 0 \,.$$

4. Calculate the Fourier series of the function $f(x)$ ($x \in [-\pi, \pi]$) defined by

$$f(x) = \begin{cases} 1 & 0 \leq x < \pi \\ 0 & -\pi \leq x < 0 \,. \end{cases}$$

Find also the complex Fourier series for $f(x)$.

5. Suppose that $f(x)$ is a 2π-periodic function with complex Fourier series

$$\sum_{n=-\infty}^{\infty} c_n e^{inx} .$$

Now consider the shifted version of $f(x)$ given by

$$g(x) = f(x - x_0)$$

where x_0 is a constant. Find the relationship between the complex Fourier coefficients of $g(x)$ in terms of those of $f(x)$. How do the magnitudes of the corresponding coefficients compare?

6. Suppose that $f(x)$ and $g(x)$ are two functions defined for real x and that they have Fourier transforms $F(\omega)$ and $G(\omega)$, respectively. Show that

$$\int_{-\infty}^{\infty} f(x)G(x)dx = \int_{-\infty}^{\infty} F(x)g(x)dx .$$

You may assume that the above integrals exist and that you may change the order of integration in your calculations.

7. Consider the functions $f_b(x)$ and $g(x)$ defined by

$$f_b(x) = \begin{cases} 0 & x > b \\ 1 & -b < x \le b \\ 0 & x \le -b \end{cases}$$

where $b > 0$ is a constant and

$$g(x) = \begin{cases} 0 & x > 4 \\ 1 & 3 < x \le 4 \\ 1.5 & 2 < x \le 3 \\ 1 & 1 < x \le 2 \\ 0 & x \le 1 . \end{cases}$$

Use the Fourier transform of $f_b(x)$ (derived in lectures) together with properties of Fourier transforms (which you should state carefully) to construct the Fourier transform of $g(x)$.

8. Suppose that the N-point DFT of the sequence $f[n]$ is given by $F[k]$ where $f(n)$ is itself a N-periodic sequence, that is $f(n+N) = f(n)$ for $n = 0, 1, \dots, N-1$. Show that the shifted sequence $f[n-m]$ has DFT

$$e^{-2\pi imk/N} F[k]$$

where m is a constant integer. Show also that $\overline{f[n]}$, the complex conjugate of $f[n]$, has DFT $\overline{F[-k]}$. Suppose that $f[-2] = -1, f[-1] = -2, f[0] = 0, f[1] = 2, f[2] = 1$. Find the 5-point DFT of $f[n]$. Can you explain why it is purely imaginary?